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A note on Krein's theorem

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Abstract. In this paper we give a new proof of Krein's theorem.

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Resumen. Se da una nueva demostración del teorema de Krein.

The Theorem of Hahn-Banach, among other applications, is used to prove the classical Theorem of Krein. The aim of this note is to give another proof of this result using the Theorem of Riesz.

An ordered vector space is a real vector space $\mathbf{X}$ equipped with a transitive, reflexive, antisymmetric relation $\leq$ satisfying the following conditions:

(I) If x, y, z are elements of $\mathbf{X}$ and $x \leq y$, then $x + z \leq y + z$.
(II) If x, y are elements of $\mathbf{X}$ and α is a positive real number, then $x \leq y$ implies $\alpha x \leq \alpha y$.

The positive cone (or simply the cone) $\mathbf{C}$ is an ordered vector space $\mathbf{X}$ is defined by $\mathbf{C} = \{x \in \mathbf{X} : x \geq \theta\}$, where θ denotes the zero element in $\mathbf{X}$. The cone has the following "geometric" properties:

(III) $\mathbf{C} + \mathbf{C} \subset \mathbf{C}$
(IV) $\alpha\mathbf{C} \subset \mathbf{C}$ for each positive real number α (V) $\mathbf{C} \cap (-\mathbf{C}) = \{\theta\}$.

In particular, it follows from (III) and (IV) that $\mathbf{C}$ is a convex set in $\mathbf{X}$. On the other hand, if $\mathbf{C}$ is a subset of a real vector space $\mathbf{X}$ satisfying (III), (IV) and (V), then

$$x \leq y \quad \text{if} \quad y - x \in \mathbf{C}$$

defines an order relation $\leq$ on $\mathbf{X}$ with respect to which $\mathbf{X}$ is an ordered vector space with positive cone $\mathbf{C}$. Thus, for a given real vector space $\mathbf{X}$, there is a canonical one-to-one correspondence between the collection of order relations with properties (I) and (II) and the collection of all subsets of $\mathbf{X}$ with properties (III), (IV), and (V). Some authors called *wedge* a subset of $\mathbf{X}$ containing θ and satisfying (III) and (IV).

If $(\mathbf{X}, \mathbf{C})$ is an ordered vector space, and $\phi : \mathbf{X} \to \Re$ is a linear functional, we say that ϕ is a positive functional if $\phi(\mathbf{C}) \subset [0, +\infty)$.

If $\mathbf{Y} \subset \mathbf{X}$ is a vector subspace, and ϕ is a linear functional on $\mathbf{Y}$, we say that ϕ is positive if it is a positive functional relative to the induced order on $\mathbf{Y}$, i.e., $\phi(\mathbf{C} \cap \mathbf{Y}) \subset [0, +\infty)$.

For completeness, we give proofs of Theorem 1 and Corollary 1.

Theorem 1. [RIESZ] *Let $(\mathbf{X}, \mathbf{C})$ be an ordered vector space, $\mathbf{Y}$ a linear subspace of $\mathbf{X}$, and $\phi_0 : \mathbf{Y} \to \Re$ a positive linear functional on $\mathbf{Y}$. Assume that for every $x \in \mathbf{X}$ there exists $y \in \mathbf{Y}$ such that $x \leq y$. Then there exists a positive linear functional $\phi : \mathbf{X} \to \Re$ such that $\phi | \mathbf{Y} = \phi_0$.*

Proof. We will show that ϕ_0 can be extended to a positive linear functional on a space of the form $\mathbf{Y} + \Re x_0$, where $x_0 \in \mathbf{X} \setminus \mathbf{Y}$. The proof can be then completed by an standard application of transfinite induction. Fix, therefore, $x_0 \in \mathbf{X} \setminus \mathbf{Y}$, and observe that the sets $\mathbf{A} = \{y \in \mathbf{Y} : y \leq x_0\}$ and $\mathbf{B} = \{y \in \mathbf{Y} : y \geq x_0\}$ are not empty, and $\phi_0(y') \leq \phi_0(y'')$ whenever $y' \in \mathbf{A}$ and $y'' \in \mathbf{B}$. There is, therefore, a real number α satisfying

$$\sup\{\phi_0(y') : y' \in \mathbf{A}\} \leq \alpha \leq \inf\{\phi_0(y'') : y'' \in \mathbf{B}\}.$$

Define a functional ϕ' on $\mathbf{Y} + \Re x_0$ by setting

$$\phi'(y + tx_0) = \phi_0(y) + t\alpha, \; y \in \mathbf{Y}, t \in \Re.$$

It suffices to show that ϕ' is positive. Assume, indeed, that $y + tx_0 \geq 0$ or, equivalently, $tx_0 \geq -y$, if $t > 0$. Then $-y/t \in \mathbf{A}$, hence $\phi_0(-y/t) \leq \alpha$ and $\phi'(y + tx_0) \geq 0$. The case when $t < 0$ is treated analogously, using the fact that $y/|t| \in \mathbf{B}$. ☑

Corollary 1. *The extension ϕ in Theorem 1 is unique if and only if for all $x_0 \in \mathbf{X}$ and $\mathbf{A}, \mathbf{B}$ as defined in the proof above, one has*

$$\sup\{\phi_0(y') : y' \in \mathbf{A}\} = \inf\{\phi_0(y'') : y'' \in \mathbf{B}\}.$$

Proof. **Necessity**. By contrapositive. First of all, let us define the following sets $\mathbf{A}_{x_0} = \{y' \in \mathbf{Y} : y' \leq x_0\}$ and $\mathbf{B}_{x_0} = \{y" \in \mathbf{Y} : y" \geq x_0\}$. Next suppose that for some $x_0 \in \mathbf{X}$

$$\sup\{\phi_0(y') : y' \in \mathbf{A}_{x_0}\} < \inf\{\phi_0(y'') : y'' \in \mathbf{B}_{x_0}\}.$$

Taking α_1, α_2 in between, with $\alpha_1 \neq \alpha_2$, by the proof of Theorem 1 there exist ϕ_1 and ϕ_2 such that $\phi_1(x_0) = \alpha_1$ and $\phi_2(x_0) = \alpha_2$; therefore, $\phi_1 \neq \phi_2$.
Sufficiency. Let $\phi : \mathbf{X} \to \Re$ be the extension of ϕ_0 and let $x_0 \in \mathbf{X} \setminus \mathbf{Y}$. We want to prove that

$$\phi(x_0) \geq \sup\{\phi_0(y') : y' \in \mathbf{A}_{x_0}\},$$
$$\phi(x_0) \leq \inf\{\phi_0(y''): y'' \in \mathbf{B}_{x_0}\}.$$

Then for all $y' \in \mathbf{A}_{x_0}$ we have $\phi(y') \leq \phi(x_0)$, and $\sup\{\phi_0(y') : y' \in \mathbf{A}_{x_0}\} \leq \phi_0(x_0)$. Similarly, we get

$$\phi(x_0) \leq \inf\{\phi_0(y'') : y'' \in \mathbf{B}_{x_0}\} \ .$$

But

$$\sup\{\phi_0(y') : y' \in \mathbf{A}_{x_0}\} = \inf\{\phi_0(y'') : y'' \in \mathbf{B}_{x_0}\} \ ,$$

so that

$$\phi(x_0) = \sup\{\phi_0(y') : y' \in \mathbf{A}_{x_0}\}$$

and

$$\phi(x_0) = \inf\{\phi_0(y'') : y'' \in \mathbf{B}_{x_0}\}.$$

Thus ϕ must be unique. ☑

Let $\mathbf{C}$ be a subset of the ordered vector space $\mathbf{X}$. A point $u \in \mathbf{C}$ is called internal of $\mathbf{C}$ if for every $z \in \mathbf{X}$ there exists $z_1 \in (u, z]$ such that $[u, z_1] \subset \mathbf{C}$.

Lemma 1. *Let* $(\mathbf{X}, \mathbf{C})$ *be an ordered vector space and let* $\mathbf{C} \neq \mathbf{X}$ *have an internal point* u. *Let* $\mathbf{Y} = \mathbf{span}\{u\} = \{\lambda u : \lambda \in \Re\}$ *then*

$$\mathbf{C} \cap \mathbf{Y} = \{\lambda u : \lambda \geq 0\}.$$

Proof. Let $y \in \{\lambda u : \lambda \geq 0\}$, then $y = \lambda u$ with $\lambda \geq 0$ implies $y \in \mathbf{Y}$, since $u \in \mathbf{C}$ we get $\lambda u \in \mathbf{C}$ so $y \in \mathbf{C}$ therefore $\{\lambda u : \lambda \geq 0\} \subset \mathbf{C} \cap \mathbf{Y}$. On the other hand we want to show that

$$\mathbf{Y} \cap \mathbf{C} \subset \{\lambda u : \lambda \geq 0\}.$$

To see this, assume that $\mathbf{C} \cap \mathbf{Y} \not\subset \{\lambda u : \lambda \geq 0\}$. Then there exists $y = \beta u \in \mathbf{C} \cap \mathbf{Y}$ with $\beta < 0$ and $y = \dfrac{\beta u}{|\beta|} \in \mathbf{C} \cap \mathbf{Y}$; this means that $-u \in \mathbf{C}$, so we have $\{u, -u\} \subset \mathbf{C}$. Now let $x \neq 0 \in \mathbf{X}$ and choose $z = u + x$, then there exists z_1 such that

$$z_1 = u + \alpha x \quad \text{with} \quad 0 < \alpha \leq 1 \quad \text{and} \quad [u, z_1] \subset \mathbf{C}.$$

So $z_1 - u = \alpha x \in \mathbf{C}$ then $\frac{1}{\alpha}(\alpha x) = x \in \mathbf{C}$ therefore $\mathbf{X} = \mathbf{C}$, a contradiction. ☑

Main Result

Theorem 2. [Krein] *Let* $(\mathbf{X}, \mathbf{C})$ *be an ordered vector space and let* $\mathbf{C} \neq \mathbf{X}$. *If* $\mathbf{C}$ *contains an internal point, then there exists* $f \in \mathbf{X}^*, f \neq 0$, *such that* $f \geq 0$ *on* $\mathbf{C}$.

Proof. Let $\mathbf{Y} = \mathbf{span}\{u\} = \{\lambda u : \lambda \in \Re\}$, where u is an internal point of $\mathbf{C}$, and define $f_0 : \mathbf{Y} \to \Re$ such that

$$f_0(\lambda u) = \lambda.$$

Thus

$$f_0(\lambda u + \beta u) = f_0\left((\lambda + \beta)u\right) = \lambda + \beta = f_0(\lambda u) + f_0(\beta u)$$

and

$$f_0(\beta \lambda u) = \beta \lambda = \beta f(\lambda u).$$

This means that f_0 is linear functional. By Lemma 1 we get

$$\mathbf{C} \cap \mathbf{Y} = \{\lambda u : \lambda \geq 0\}$$

and

$$f_0(\mathbf{C} \cap \mathbf{Y}) = \{\lambda : \lambda \geq 0\} \subset [0, \infty),$$

so f_0 is positive. Now given $x \in \mathbf{X}$, we want to prove that there exists $y \in \mathbf{Y}$ such that $x \leq y$. Using the definition of internal point we have that for $-x$ there exists $t > 0$ such that $t(-x) = c - u$, for some $c \in \mathbf{C}$ and thus $\dfrac{1}{t}\, u - x = c$ taking $y = \dfrac{1}{t}\, u \in \mathbf{Y}$. Then $x \leq y$. Now by Theorem 1 there exists a positive linear functional $f : \mathbf{X} \to \Re$ such that $f \neq 0$ because $f(u) = 1 = f_0(u)$, and $f \geq 0$ on $\mathbf{C}$. ☑

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