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Norms that Locally Depend on Countably Many Linear Functionals

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Introduction

Norms and bump functions on a Banach space X that locally depend on finitely many elements of the dual space X^* (sometimes called coordinates) are useful in renorming theory (cf. e.g. [3, Ch. V], [16]) or in the area of polyhedral spaces (cf. e.g. [11] and references therein). Such norms share some properties of norms on finite dimensional spaces and can be used in situations, where there are difficulties even with Hilbertian norms (cf. the results of Toruńczyk and others on smooth partitions of unity in nonseparable Banach spaces, (cf. e.g. [3, Ch. VIII]). Norms that locally depend on countably many coordinates are in turn closely related to countable tightness of the weak* topology of dual balls and can often substitute for Gâteaux differentiable norms. We show here that they can be used in questions on projectional resolutions of the identity, Valdivia compacts and biorthogonal systems, especially in spaces of continuous functions on scattered compacts.

We will work in real Banach spaces and keep the standard notation ([21], [3], [12], [6], [8]). In particular, $B_X = \{x \in X : \|x\| \leq 1\}$ is the unit ball of a Banach space X and $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X.

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The symbol χ_T denotes the characteristic function of a set T. If $A \subset X^*$, then $A_\perp = \{x \in X : f(x) = 0 \text{ for all } f \in A\}$. A norm closed subspace Y of X^* is a 1-*norming subspace* of X^*, if $\|x\| = \sup\{f(x) : f \in B_Y\}$ for every $x \in X$. If W is a subspace of a Banach space X and $x \in X$, then $\hat{x}$ denotes the coset of x in X/W and the norm $\|\hat{x}\|$ is the canonical norm of $\hat{x}$ in X/W, i.e. $\|\hat{x}\| = \inf_{x\in\hat{x}} \|x\|$. If $A \subset X^*$, then the weak* closed linear hull of A in X^* is denoted by $\overline{\operatorname{span}}^{w^*} A$. The ordinal ω_0 (ω_1) is the least infinite (uncountable) ordinal and $\aleph_0$ ($\aleph_1$) is the least infinite (uncountable) cardinal. The symbol ℓ_∞ (respectively c_0) denotes the space $\ell_\infty(\mathbb{N})$ (respectively $c_0(\mathbb{N})$) and $\ell_1(c)$ denotes the space $\ell_1(\Gamma)$, where Γ is a set of cardinality of the continuum c. The space ℓ_∞^n is the space $\mathbb{R}^n$ with the usual maximum norm. The *density character* or *density* of a Banach space X ($\operatorname{dens} X$) is the minimal cardinality of a norm dense set in X. The dual space X^* is called *weak* sequentially separable* if there is a countable set $D \subset X^*$ such that for each $f \in X^*$ there is a sequence $\{f_n\}$ in D such that $f_n \to f$ in the weak* topology. By a subspace in a Banach space we mean a norm closed subspace and by a norm on a Banach space X we mean an equivalent norm on X. We say that $f \in X^*$ *attains its norm* if there is $x \in S_X$ such that $f(x) = \|f\|$. The Bishop-Phelps theorem asserts that for every Banach space X, the set of all norm attaining functionals is norm dense in X^* (cf. e.g. [3, p. 13]). The Čech-Stone compactification of the positive integers is denoted by $\beta(\mathbb{N})$. A *Corson compact* is a compact space K that is homeomorphic to a set S in some $[-1,1]^\Gamma$ in its pointwise topology such that each point of S is countably supported, i.e. $\{\gamma \in \Gamma : x(\gamma) \neq 0\}$ is countable for every $x \in S$. A compact space K is called a *Valdivia compact* if K is homeomorphic to a set S in some $[-1,1]^\Gamma$ in its pointwise topology such that the set of all elements of S that are countably supported is dense in S. A topological space T has *countable tightness* if whenever $S \subset T$ and $a \in \overline{S}$, then there is a countable set $C \subset S$ such that $a \in \overline{C}$. A Banach space X is *weakly Lindelöf determined* (*WLD*) if B_{X^*} in its weak* topology is a Corson compact (cf. e.g. [6, p. 131], [8, Ch. II]). A *bump function* on a Banach space X is a real valued function on X with bounded nonempty support.

Definition 1. Let X be a Banach space and G be a subspace of X^*. We say that a continuous real valued function φ on X *locally depends on countably many* (*finitely many*) *elements of* G if for every $x \in X$, there are a neighborhood U of x in X, countably many elements $\{f_i\} \subset B_G$ (finitely many elements $\{f_1, \dots, f_n\} \subset B_G$), and a continuous function ψ on ℓ_∞ (on ℓ_∞^n) such that $\varphi(z) = \psi(f_1(z), f_2(z), \dots)$ for each $z \in U$. If $G = X^*$, we say that φ *locally depends on countably many coordinates*. If we speak of standard coordinate functionals in some spaces we say that φ *locally depends on countably many standard coordinates*.

If $G = X^*$ and, moreover, for every point $x \in X$, the functionals $\{f_i\} \subset B_{X^*}$ above can be chosen so that $X/\{f_i\}_\perp$ is separable, we say that φ *locally factors through separable quotients* or that φ is an *LFS function*.

We say that the norm $\|\cdot\|$ is an *LFS norm* on X if the function $\varphi = \|\cdot\|$ is an LFS function on $X \setminus \{0\}$.

If a function φ on a Banach space X locally depends on finitely many coordinates, then φ locally factors through finite dimensional quotients as $X/(\{f_i\}_{i=1}^n)_\perp$ has finite dimensional dual, isomorphic to $\operatorname{span}\{f_i\}_{i=1}^n$. If $\|\cdot\|$ is an LFS norm on a Banach space X, then by a composition of $\|\cdot\|$ with a suitable real valued function on the real line we can construct an LFS bump function on X

We will frequently use the following proposition.

Proposition 2. *The norm $\|\cdot\|$ of a Banach space X is an LFS norm if and only if for every $x \in S_X$, there are a neighborhood U of x in X and a subspace $W \subset X$ such that X/W is separable and $\|z\| = \|\hat{z}\|$ for all $z \in U$, where $\hat{z}$ is the coset of z in X/W. This happens if for each $x \in S_X$ there are a neighborhood U of x in X and a norm 1 linear projection $P : X \to X$, with PX separable and containing x, and such that $\|u\| = \|Pu\|$ for every $u \in U$.*

The norm $\|\cdot\|$ of X locally depends on finitely many coordinates if and only if given $x \in S_X$, there are a neighborhood U of x and a finite codimensional subspace W of X such that $\|z\| = \|\hat{z}\|$ for every $z \in U$.

Proof. Assume that $\|\cdot\|$ is an LFS norm. Given $x \in S_X$, let U, $C^x = \{f_i\}$ and ψ be from Definition 1 for the point x. Put $S^x = \overline{\operatorname{span}}^{w^*} C^x$. Since $X/C^x_\perp$ is separable, B_{S^x} is weak* metrizable and separable (cf. e.g. [12, p. 45]). Let $z \in U$ and let $f \in B_{X^*}$ be such that $f(z) = \|z\|$. Then for $h \in C^x_\perp$ with $\|h\|$ sufficiently small, $f(z \pm h) \le \|z \pm h\| = \|z\| = f(z)$ and hence $f(h) = 0$. Therefore $f \in B_{S^x}$ and we can put $W = S^x_\perp = C^x_\perp$. Indeed, if $z \in U$ and $\hat{z} \in X/W$, then $\|z\| = f(z) \le \sup_{g \in B_{S^x}} g(z) = \|\hat{z}\| \le \|z\|$.

If the condition holds, given $x \in S_X$, choose a weak* dense sequence $\{f_i\}$ in $B_{W^\perp}$. Then for $z \in U$, $\|z\| = \|\hat{z}\| = \sup\{f_i(z)\}$.

In order to observe the second part of the statement, put $W = P^{-1}(0)$ and note that $\|Py\| = \|\hat{y}\|$ for every $y \in X$.

The third part can be proved as the first part. ∎

Note that if $\|\cdot\|$ is an LFS norm on X and Y is a subspace of X, then the restriction of $\|\cdot\|$ to Y is an LFS norm on Y. Indeed, if $y \in Y$, U and W are from Proposition 2 for y, and for $z \in Y$, the coset of z in $Y/(Y \cap W)$ is denoted by $\hat{\hat{z}}$, then for $z \in Y \cap U$ we have $\|z\| = \|\hat{z}\| \leq \|\hat{\hat{z}}\| \leq \|z\|$.

Proposition 3. *If a real valued function φ on a WLD Banach space locally depends on countably many coordinates, then φ is an LFS function.*

Proof. If $\{f_i\} \subset X^*$, then there is a norm one projection P of X such that $P(X)$ is separable and $\{f_i\} \subset P^*(X^*)$ (cf. e.g. [6, Ch. 6, 8]). Then $X/\{f_i\}_\perp$ is separable as $X/P^{-1}(0)$ is separable, being isometric to $P(X)$. ∎

We will first discuss some examples.

Examples

First of all, *any equivalent norm on a separable Banach space X depends (globally) on countably many coordinates.* Indeed, if $\{f_i\}$ is a countable dense set in B_{X^*} in its weak* topology (cf. e.g. [12, p. 61]), then $\|x\| = \sup_i f_i(x)$ for any $x \in X$. On the other hand, if X^* is not weak* separable and the norm $\|\cdot\|$ of X depended globally on countably many $f_i \in X^*$, then choosing $z \in \cap_i f_i^{-1}(0)$, $z \neq 0$, we would have $\|z\| = 2\|z\|$, a contradiction.

The situation with local dependence of the norm on countably many coordinates is more involved. Before we start on it, we should note that *if Γ is uncountable, then the Hilbertian norm of $\ell_2(\Gamma)$ does not locally depend on countably many coordinates.* Indeed, if $x \in S_{\ell_2(\Gamma)}$ and, on a neighborhood of x, the norm of $\ell_2(\Gamma)$ depends on some countable collection $C^x \subset \ell_2(\Gamma)^*$ and $0 \neq y \in \cap_{f \in C^x} f^{-1}(0)$, then a small line segment centered at x in the direction y would lie in $S_{\ell_2(\Gamma)}$, a contradiction. The existence of such a nonzero y is guaranteed by the fact that $\ell_2(\Gamma)$ is not weakly separable (cf. e.g. [12, p. 61]).

1. The space ℓ_2 is an obvious example of the space whose norm is LSF. However, *ℓ_2 does not admit any continuous bump function that would locally depend on finitely many coordinates* as any such space necessarily contains an isomorphic copy of c_0 (cf. e.g. [3, p. 198]). An example of *a space that does not admit any continuous bump function that locally depends on countably many coordinates is $\ell_2(\Gamma)$, where Γ is uncountable.* Indeed, $\ell_2(\Gamma)$ does not contain any isomorphic copy of c_0 since it is reflexive and it is not isomorphic to any subspace of ℓ_∞ as its dual is not (weak*) separable. The statement thus follows from Theorem 4 below.

2. It is a basic fact that $\ell_1(c)$ is isometric to a subspace of $C[0,1]^*$ (cf. e.g. [12, p. 35]). The dual of any separable space is isometric to a subspace of ℓ_∞ (cf. e.g. [12, p. 71]). Thus $\ell_1(c)$ is isometric to a subspace of ℓ_∞. The dual ball of ℓ_∞^* is weak* separable by the Goldstine theorem (cf. e.g. [12, p. 46]). Hence the dual ball of $\ell_1(c)^*$ is weak* separable (use the restriction map). Therefore *the canonical norm of $\ell_1(c)$ locally depends on countably many coordinates.* Indeed, if $\{f_i\}$ is weak* dense in $B_{\ell_1(c)^*}$, then $\|x\| = \sup_i\{f_i(x)\}$ for $x \in \ell_1(c)$. The space $\ell_1(c)$ is not WLD as its dual ball does not have countable tightness in its weak* topology (use the fact that all elements of $c_0(c)$ have countable support, Goldstine's theorem and the fact that any Corson compact has countable tightness, cf. e.g. [12, p. 252]).

It follows from Theorem 11 below that $\ell_1(c)$ *does not admit any continuous bump function that would locally depend on countably many standard coordinates.*

By Theorem 9 below, *the space $\ell_1(c)$ does not admit any LFS norm* as $\operatorname{dens}\ell_1(c)^* = \operatorname{dens}\ell_\infty(c) = 2^c > c = \operatorname{card}\ell_1(c)$ (the elements of $\ell_1(c)$ have countable support). Also, *the space ℓ_∞ does not admit any LFS norm* as it contains a copy of $\ell_1(c)$ and thus $\operatorname{dens}\ell_\infty^* \geq \operatorname{dens}\ell_1(c)^* \geq 2^c > c = \operatorname{card}\ell_\infty$. On the other hand, by its definition, *the supremum norm on ℓ_∞ depends on countably many standard coordinates.*

3. For any infinite set Γ, *the supremum norm $\|\cdot\|$ on $c_0(\Gamma)$ depends locally on finitely many coordinates.* Indeed, let $x \in S_{c_0(\Gamma)}$. Find a finite set $F \subset \Gamma$ such that $\sup_{\gamma\in\Gamma\setminus F}|x(\gamma)| < \frac{1}{2}$. If $z \in c_0(\Gamma)$ is such that $\|z - x\| < \frac{1}{4}$, then

$$\sup_{\gamma\in\Gamma}|z(\gamma)| = \max\left\{\sup_{\gamma\in\Gamma\setminus F}|z(\gamma)|, \sup_{\gamma\in F}|z(\gamma)|\right\} = \sup_{\gamma\in F}|z(\gamma)| = \|Pz\|,$$

where P is the projection in $c_0(\Gamma)$ defined for $x \in c_0(\Gamma)$ by $Px = (\chi_F)x$.

Hájek showed in [14] that *every C^2-smooth function on $c_0(\Gamma)$ locally depends on countably many coordinates,* and hence, by Proposition 3, *it is an LFS function.*

It is proved in [3, p. 189] that for every infinite set Γ, $c_0(\Gamma)$ *admits an equivalent locally uniformly rotund norm that is a limit, uniform on bounded sets, of norms that locally depend on finitely many coordinates.* We recall that a norm $\|\cdot\|$ of X is *locally uniformly rotund* if $\|x_n - x\| \to 0$ whenever $x_n, x \in X$ and $2\|x_n\|^2 + 2\|x\|^2 - \|x + x_n\|^2 \to 0$. We do not know if every norm on $c_0(\Gamma)$ can be approximated by norms that locally depend on finitely many coordinates. If Γ is countable, then the answer is in the positive ([15]).

If Γ is uncountable, then equivalent norms on $c_0(\Gamma)$ that locally depend on countably many coordinates do not form a residual set in the metric space of all equivalent norms on $c_0(\Gamma)$ endowed with the metric of uniform convergence on the ball of $c_0(\Gamma)$. Indeed, otherwise, by the Baire category theorem, there would be a strictly convex norm on $c_0(\Gamma)$ that would locally depend on countably many coordinates (cf. e.g. [3, p. 52]), which is obviously not true (cf. e.g. the argument above in $\ell_2(\Gamma)$).

4. Let $C[0,\omega_1]$ denote the Banach space of all continuous functions on the ordinal segment $[0,\omega_1]$ in its supremum norm, where $[0,\omega_1]$ is considered in its usual order topology (cf. e.g. [5, p. 59]). The symbol $C_0[0,\omega_1]$ denotes the hyperplane in $C[0,\omega_1]$ formed by all elements in $C[0,\omega_1]$ that vanish at ω_1. Note that $C_0[0,\omega_1]$ is isomorphic to $C[0,\omega_1]$. Indeed, consider an operator $T : C[0,\omega_1] \to C_0[0,\omega_1]$ defined by $Tx = (x(\omega_1), x(0) - x(\omega_1), \dots, x(t) - x(\omega_1), \dots, x(\omega_1) - x(\omega_1))$. The space $c_0[0,\omega_1]^*$ is not weak* separable as every element of $c_0[0,\omega_1]^*$ has countable support on the standard unit vectors of $c_0[0,\omega_1]$. Indeed, assuming that $c_0[0,\omega_1]^*$ is weak* separable, the total support of all the elements of $c_0[0,\omega_1]^*$ would be countable. This is a contradiction as $[0,\omega_1]$ is uncountable. The space $c_0[0,\omega_1]$ is isomorphic to a subspace of $C_0[0,\omega_1]$. Indeed, define a mapping T from $c_0[0,\omega_1]$ into $C_0[0,\omega_1]$ by

$$Tx(\alpha) = \begin{cases} x(\beta) & \text{if there is } \beta \text{ such that } \beta + 1 = \alpha\,, \\ 0 & \text{if there is no such } \beta\,; \end{cases}$$

this is an isometry into. If $C_0[0,\omega_1]^*$ were weak* separable, so would be $c_0[0,\omega_1]^*$ (via the restriction map), which is not the case.

The supremum norm $\|\cdot\|$ of $C_0[0,\omega_1]$ is an LFS norm. Indeed, given $x \in S_X$, let $\alpha \in [0,\omega_1)$ be such that $x(\beta) = 0$ for all $\beta \geq \alpha$. Let $U = \{u \in X : \|u - x\|_\infty < \frac{1}{4}\}$ and let the projection P on $C_0[0,\omega_1]$ be defined by $Px = (\chi_{[0,\alpha]})x$ for $x \in C_0[0,\omega_1]$. Then for $u \in U$,

$$\begin{aligned} \frac{3}{4} \leq \|u\| &= \max\Big\{ \sup_{0\leq\beta\leq\alpha} |u(\beta)|, \sup_{\alpha+1\leq\beta\leq\omega_1} |u(\beta)| \Big\} \\ &\leq \max\Big\{ \sup_{0\leq\beta\leq\alpha} |u(\beta)|, \frac{1}{4} \Big\} = \sup_{0\leq\beta\leq\alpha} |u(\beta)| = \|Pu\|\,. \end{aligned}$$

Moreover, $P(C_0[0,\omega_1])$ is isometric to $C[0,\alpha]$ and is thus separable. Hence Proposition 2 applies.

$B_{C_0[0,\omega_1]^*}$ in its weak* topology is not a Valdivia compact ([19]). Indeed, otherwise, $C_0[0,\omega_1]$ would be WLD by Corollary 8 below. And this is not the

case, as $[0,\omega_1]$ is a subspace of $B_{C[0,\omega_1]^*}$ and this weak* compact is homeomorphic with the weak* compact $B_{C_0[0,\omega_1]^*}$ (since $C_0[0,\omega_1]$ is isomorphic to $C[0,\omega_1]$). Any Corson compact has countable tightness (cf. e.g. [12, p. 252]) and the segment $[0,\omega_1]$ does not have countable tightness as ω_1 is not in the closure of any countable set of ordinals strictly less than ω_1.

Recall that a *projectional resolution of the identity* (*PRI*) on a Banach space X with $\operatorname{dens} X = \aleph_1$ is a transfinite sequence of linear projections P_α, $0 \le \alpha \le \omega_1$, on X such that $P_0 = 0$, $P_{\omega_1} =$ Identity , $\|P_\alpha\| = 1$ for $\alpha > 0$, $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min\{\alpha,\beta\}}$, $P_\alpha(X)$ is separable for all $\alpha < \omega_1$ and the map $\alpha \to P_\alpha x$ is continuous from the ordinals into the norm topology of X for each $x \in X$. Since $B_{C_0[0,\omega_1]^*}$ is not a Valdivia compact in its weak* topology, the space $C_0[0,\omega_1]$ *does not admit any projectional resolution of the identity with respect to its supremum norm* (cf. e.g. [3, p. 251] or [7]). On the other hand, $B_{C[0,\omega_1]^*}$ *is a Valdivia compact in its weak* topology* (in order to see this, check the Dirac measures δ_α, $\alpha < \omega_1$, against the characteristic functions of $[\alpha+1,\omega_1]$, $\alpha < \omega_1$). Hence *the weak* to weak* continuous restriction map of $C[0,\omega_1]^*$ to $C_0[0,\omega_1]^*$ maps a Valdivia compact onto a compact set that is not a Valdivia compact.* The first example of a non Valdivia compact that is a continuous image of a Valdivia compact was found in [26].

The projections P_α of $C_0[0,\omega_1]$ defined by $P_\alpha(x) = \chi_{[0,\alpha]}x$ have all the properties of projectional resolution of identity but that one concerning the continuity of the map $\alpha \to P_\alpha x$ on ordinals for all x. In fact, if β is a limit ordinal less than ω_1, then the closure of $\cup_{\alpha<\beta} P_\alpha(C_0[0,\omega_1])$ is a hyperplane in $P_\beta(C_0[0,\omega_1])$ formed by all the functions in $P_\beta(C_0[0,\omega_1])$ that vanish at β. For $\beta = \omega_1$, this union itself is the whole space $C_0[0,\omega_1]$ as any function in $C_0[0,\omega_1]$ has a countable support. Moreover, $\cup_{\alpha<\omega_1} P^*_\alpha(C_0[0,\omega_1]^*) = C_0[0,\omega_1]^*$. Indeed, assume that $f \in S_{C_0[0,\omega_1]^*}$ and $x \in S_{C_0[0,\omega_1]}$ are such that $f(x) = 1$. Find $\alpha < \omega_1$ such that $x(\beta) = 0$ for all $\beta \ge \alpha$. If the projection P_α is defined for $\alpha \le \omega_1$ and $y \in C_0[0,\omega_1]$ by $P_\alpha y = \chi_{[0,\alpha]}y$, then $f(x \pm w) \le \|x \pm w\| = 1 = f(x)$ for all $w \in P_\alpha^{-1}(0)$. Hence $f(w) = 0$ for all $w \in P_\alpha^{-1}(0)$ and thus $f(y) = f(P_\alpha y + (I - P_\alpha)(y)) = f(P_\alpha y) = P^*_\alpha f(y)$ for all $y \in C_0[0,\omega_1]$. Therefore $f \in P^*_\alpha C_0[0,\omega_1]^*$. If $f \in S_{C_0[0,\omega_1]^*}$, by the Bishop-Phelps theorem, there are $f_n \in S_{C_0[0,\omega_1]^*}$ that attain their norms and $\|f_n - f\| \to 0$. If $f_n \in P^*_{\alpha_n} C_0[0,\omega_1]^*$ for some $\alpha_n < \omega_1$, then $f \in P^*_{\sup\{\alpha_n\}} C_0[0,\omega_1]^*$.

For $\alpha \le \omega_1$ and $x \in C[0,\omega_1]$, put

$$P_\alpha x(\beta) = \begin{cases} x(\beta) & \text{if } \ 0 \le \beta \le \alpha\,, \\ x(\alpha) & \text{if } \ \alpha < \beta \le \omega_1\,, \end{cases}$$

for $\beta \in [0,\omega_1]$. Then $P_\alpha,\ 0 \le \alpha \le \omega_1$, form a PRI for $C[0,\omega_1]$ endowed with the supremum norm.

Speaking about the space $C[0,\omega_1]$, let us note in passing that Semadeni proved in [24] that *the codimension of* $C[0,\omega_1]$ *in the Banach space of all weak** *sequentially continuous linear funtionals on* $C[0,\omega_1]^*$ *is one* and thus $C[0,\omega_1] \times C[0,\omega_1]$ *is not isomorphic to* $C[0,\omega_1]$.

Although the standard norm of $C_0[0,\omega_1]$ is an LFS norm, by considering $x = \chi_{[0,\omega_0]}$, we can see that *the supremum norm of* $C_0[0,\omega_1]$ *does not locally depend on finitely many coordinates.* Indeed, otherwise, let $\{f_i\}$ be a finite number of elements of $C_0[0,\omega_1]^*$ that comes from the definition of the local dependence on finitely many coordinates for the supremum norm and x. Consider the infinite dimensional subspace Z of $C_0[0,\omega_1]$ formed by all functions in $C_0[0,\omega_1]$ with supports in $[0,\omega_0]$ and take a nonzero point h in the intersection of Z with $\{f_i\}_\perp$. It is not true that $\|x+th\| = 1$ for all small $|t|$, which should be so by Definition 1. As $C[0,\omega_1]^*$ is not weak* separable, it can similarly be shown that *the supremum norm of* $C[0,\omega_1]$ *does not locally depend on countably many coordinates* (consider $x = \chi_{[0,\omega_1]}$).

This is in contrast with the following fact that comes from the work of Talagrand [25] (cf. e.g. [16], [8, Ch. II]). Let the operator T from $C_0[0,\omega_1]$ into $c_0[0,\omega_1]$ be defined by

$$Tx(\alpha) = \begin{cases} x(\alpha+1) - x(\alpha) & \text{if } \ \alpha < \omega_1\,, \\ 0 & \text{if } \ \alpha = \omega_1\,. \end{cases}$$

Then *the norm* $\|\cdot\|$ *defined on* $C_0[0,\omega_1]$ *by*

$$\|x\| = \sup_{t\in[0,\omega_1]} \{|x(t)| + |Tx(t)|\}$$

is an equivalent norm on $C_0[0,\omega_1]$ *that locally depends on finitely many coordinates.*

In order to see this, assuming that $\|x\|_\infty = 1$, let $\alpha < \omega_1$ be the supremum of $\gamma < \omega_1$ such that $|x(\gamma)| = 1$. Observe that $\delta := \frac{1}{3}|Tx(\alpha)| > 0$ as $|x(\alpha+1)| < |x(\alpha)|$. Let $F \subset [0,\omega_1]$ be the (finite) set of all ordinals $\beta \in [0,\omega_1]$ such that $|Tx(\beta)| \ge \delta$. Then $\sup_{t\in F}\{|x(t)| + |Tx(t)|\} \ge (|x(\alpha)| + |Tx(\alpha)|) = 1 + 3\delta$ and $\sup_{t\in[0,\omega_1]\setminus F}\{|x(t)| + |Tx(t)|\} \le 1+\delta$. As the functions $y \mapsto \sup_{t\in F}\{|y(t)| + |Ty(t)|\}$ and $y \mapsto \sup_{t\in[0,\omega_1]\setminus F}\{|y(t)| + |Ty(t)|\}$ are both Lipschitz, there is a neighborhood U of x such that $\sup_{t\in F}\{|y(t)| + |Ty(t)|\} > 1 + 2\delta > \sup_{t\in[0,\omega_1]\setminus F}\{|y(t)| + |Ty(t)|\}$ for every $y \in U$. Hence the norm $\|\cdot\|$ depends on U only on coordinates in F. As $C[0,\omega_1]$ is isomorphic to $C_0[0,\omega_1]$, the space $C[0,\omega_1]$ *admits an equivalent norm that locally depends on finitely many coordinates.*

5. Assuming the Continuum Hypothesis, Kunen constructed a nonseparable Asplund $C(K)$ space such that $C(K)^*$ is hereditarily weak* separable ([18], [17]). *This $C(K)$ space does not admit any equivalent norm that would locally depend on finitely many coordinates.* Indeed, by Theorem 7(iii), on the dual ball B of such a norm, the weak* and norm topologies coincide at each point that attains its norm. Let all points of such sphere S that attain their norm be denoted by P and let C be a countable weak* dense set in P (which is thus norm dense in P). As P is norm dense in S by the Bishop-Phelps theorem, C is norm dense in S. Thus S, and hence $C(K)^*$, would be norm separable, a contradiction. We do not know if Kunen's $C(K)$ admits an LFS norm.

Results

If an infinite dimensional Banach space X admits a bump fuction that locally depends on finitely many coordinates of X^*, then X contains an isomorphic copy of c_0 and it is an Asplund space, i.e., each separable subspace of X has separable dual ([3, p. 198] and [10]). The following statement is related to this result.

Theorem 4. *Assume that a Banach space X admits a continuous bump function φ that locally depends on countably many coordinates and that X is not isomorphic to any subspace of ℓ_∞. Then X contains an isomorphic copy of c_0.*

Proof. Assume that X does not contain any isomorphic copy of c_0, admits a continuous bump function that locally depends on countably many coordinates and that X is not isomorphic to any subspace of ℓ_∞. We will derive a contradiction. Let f be a continuous function on X such that f locally depends on a countably many coordinates, $f(0)=0$ and $f(x)=1$ whenever $\|x\|\geq 1$. Assume, without loss of generality, that f is an even function (otherwise consider $x\mapsto \frac{1}{2}(f(x)+f(-x))$). Put $U=\{x\in X : f(x)\neq 1\}$.

Applying a compact variational principle [3, Theorem V.2.6], we find a compact symmetric set $K\subset U$ and $\delta_0>0$ such that for every $\delta\in(0,\delta_0)$ there is $\Delta_\delta>0$ such that

$$f_K(x)>\Delta_\delta \qquad \text{whenever} \;\; x\in X \;\; \text{and} \;\; \|x\|=\delta\,,$$

where $f_K = \max\{f(k+\cdot) : \ k \in K\}$.

Claim. The function f_K is continuous on X and locally depends on countably many coordinates.

In order to verify the claim, we first check that $x_n \to x \in X$ implies that $f(x_n+k) \to f(x+k)$ uniformly for $k \in K$. Thus f_K is continuous. Now fix any $x \in X$. We will show that f_K locally depends on countably many coordinates. Without loss of generality, we may assume that $x = 0$. (Actually, 0 is the only point that will be used later.) Given $y \in X$ and $\delta > 0$ we say that δ is good for y if there are a continuous function $\psi^y : \ell_\infty \to \mathbb{R}$ and $f_1^y, f_2^y, \ldots \in B_{X^*}$ such that

$$f(z) = \psi^y(f_1^y(z), f_2^y(z), \dots) \qquad \text{whenever} \ \ z \in X \ \ \text{and} \ \ \|z-y\| < \delta\,.$$

Now, for every $k \in K$ let $\delta_k > 0$ be good. From compactness we find $k_1, \ldots, k_m \in K$ so that $K \subset \bigcup_{i=1}^m \big(k_i + \frac{1}{2}\delta_{k_i} B_X\big)$. Put $\delta = \frac{1}{2}\min\{\delta_{k_1}, \ldots, \delta_{k_m}\}$. We will show that δ is good for every point of K. So fix any $k \in K$. Find $i \in \{1, \ldots, m\}$ so that $k \in k_i + \frac{1}{2}\delta_{k_i} B_X$. Then, if $z \in X$ and $\|z\| < \delta$, we have

$$k + z \in k_i + \frac{1}{2}\delta_{k_i} B_X + \delta B_X \subset k_i + \delta_{k_i} B_X\,,$$

and so,

$$f(k+z) = \psi^{k_i}(f_1^{k_i}(k+z), f_2^{k_i}(k+z), \dots)\,.$$

Put

$$\mathcal{F} = \{\psi^{k_1}, \ldots, \psi^{k_m}\}\,,$$
$$\mathcal{G} = \{(f_1^{k_1}, f_2^{k_1}, \ldots), \ldots, (f_1^{k_m}, f_2^{k_m}, \ldots)\}\,.$$

Let $\{h_j : \ j \in \mathbb{N}\}$ be a countable dense subset in K. From the above we can see that for every $j \in \mathbb{N}$ there are $\psi_j \in \mathcal{F}$ and $(g_1^j, g_2^j, \ldots) \in \mathcal{G}$ such that

$$f(h_j + z) = \psi_j(g_1^j(h_j+z), g_2^j(h_j+z), \ldots)$$

whenever $z \in X$ and $\|z\| < \delta$. Thus

$$f_K(z) = \sup_{j\in\mathbb{N}} \psi_j\big(g_1^j(h_j) + g_1^j(z), g_2^j(h_j) + g_2^j(z), \dots\big)$$

whenever $z \in X$ and $\|z\| < \delta$. Note that the family $\{g_i^j : \ j, i \in \mathbb{N}\}$ is countable. So it remains to prove that the function $\psi : \ell_\infty(\mathbb{N}\times\mathbb{N}) \to \mathbb{R}$ defined by

$$\psi(u_i^j : j, i \in \mathbb{N}) = \sup_{j\in\mathbb{N}} \psi_j\big(g_1^j(h_j) + u_1^j, g_2^j(h_j) + u_2^j, \dots\big)$$

is continuous. To check this, we observe that $\{\psi_j : j \in \mathbb{N}\} \subset \mathcal{F}$, and that, from compactness, for every $j \in \mathbb{N}$ the functions

$$(u_1^j, u_2^j, \dots) \mapsto \psi_j\big(g_1^j(k) + u_1^j, g_2^j(k) + u_2^j, \dots\big)$$

are equicontinuous with respect to $k \in K$. Therefore ψ is continuous, and the local dependence of f_K on a neighbourhood on countably many coordinates is proved.

By the claim find $\delta \in (0, \delta_0)$, a continuous function ψ on ℓ_∞, and $f_1, f_2, \dots \in B_{X^*}$ such that $f_K(z) = \psi(f_1(z), f_2(z) \dots)$ whenever $z \in X$ and $\|z\| \leq \delta$. Note that

$$0 = f_K(0) = \psi(f_1(0), f_2(0), \dots) = \psi(0, 0, \dots) .$$

Find $\gamma > 0$ so that $\psi(u_1, u_2, \dots) < \frac{1}{2}\Delta_\delta$ whenever $\sup\{|u_1|, |u_2|, \dots\} < \gamma$. As X is not isomorphic to any subspace of ℓ_∞, there is $x \in X$ such that $\|x\| = \delta$ and $\sup\{|f_1(x)|, |f_2(x)|, \dots\} < \gamma$. Then

$$\Delta_\delta \leq f_K(x) = \psi(f_1(x), f_2(x), \dots) < \frac{1}{2}\Delta_\delta ,$$

a contradiction. ∎

COROLLARY 5. *Let X be a Banach space. Assume that X^* admits an equivalent (not necessarily dual) LFS norm. Then X^* is isomorphic to a subspace of ℓ_∞.*

Proof. If not, then X^* would contain an isomorphic copy of c_0 by Theorem 4. Thus X^* would then contain an isomorphic copy of ℓ_∞ (cf. e.g. [21, p. 103]). The space ℓ_∞ does not admit any LFS norm (see Example 2). ∎

PROPOSITION 6. *Let X be a separable Banach space such that X^* admits an equivalent (not necessarily dual) LFS norm. Then X does not contain any isomorphic copy of ℓ_1.*

Proof. Assume that X contains an isomorphic copy of ℓ_1. Then X^* contains an isomorphic copy of $\ell_1(c)$, by the result of Pełczyński (cf. e.g. [12, p. 213]). As a subspace of X^*, $\ell_1(c)$ would then admit an LFS norm. This is not the case (see Example 2). ∎

A variant of the following result for Gâteaux differentiable norms was proved in [7].

Theorem 7. (i) *Assume that the norm of a Banach space X is an LFS norm. Let $M \subset B_{X^*}$ be such that $\overline{M}^{w^*} = B_{X^*}$. Then given $f \in B_{X^*}$, there is a countable set $C \subset M$ such that $f \in \overline{C}^{w^*}$.*

(ii) *Assume that the norm of a Banach space X is an LFS norm and that B_{X^*} in its weak* topology is separable. Then X^* is weak* sequentially separable.*

(iii) *Assume that the norm of X locally depends on finitely many coordinates. Let $f \in S_{X^*}$ attain its norm. Then the norm and weak* topologies on B_{X^*} coincide at f.*

Proof. (i) According to Josefson-Niezesweig theorem (cf. e.g. [4, Ch. XII, Exercise 2 (i)]), any element of B_{X^*} is the weak* limit of a sequence from S_{X^*}. Using this result and the Bishop-Phelps theorem, we can see that it suffices to show that if $M \subset B_{X^*}$ is such that $\overline{M}^{w^*} = B_{X^*}$ and $g \in S_{X^*}$ is such that $g(x) = 1$ for some $x \in S_X$, then there is a countable set $C \subset M$ such that $g \in \overline{C}^{w^*}$. Let B be a countable set in B_X containing x and such that the set $\hat{B}$ of corresponding cosets is dense in X/W, where $W = \{f_i\}_\perp$ is chosen for x as in Definition 1. Now, if $h \in W$ is such that $x \pm h \in U$, where U is as in Definition 1, then $\|x \pm h\| = \|x\|$. Let $\{g_n\} \subset M$ be so chosen that $g(x) = \lim g_n(x)$ and $g = \lim g_n$ in the topology of pointwise convergence on B. From the proof of Šmulyan's lemma on the differentiability of norms (cf. e.g. [3, p. 3] or [12, p. 92]), we get $\sup_{B_W}(g_n - g) \to 0$ as $n \to \infty$. In particular, $g_n(w) \to g(w)$ as $n \to \infty$ for all $w \in W$. Thus $g_n \to g$ in the weak* topology.

(ii) Let C be a countable set in B_{X^*} that is weak* dense in B_{X^*}. Let D be the set of all finite linear combinations of elements of C, with rational coefficients. We note that D will still be countable. Let A denote the set of all norm attaing elements of X^*. By the first part of the proof of (i) we know that for every $f \in A$ there is a sequence $d_n \in D$, $n \in \mathbb{N}$, such that $\|d_n\| \le \|f\|$ and $d_n \to f$ weak*.

Now fix any $f \in X^*$. We will show that this f can also be reached as the weak* limit of a sequence from D. Find $f_1 \in A$ such that $\|f - f_1\| < \frac{1}{2}$. Assume that for some $i \in \mathbb{N}$ we have already found elements $f_1, \dots, f_i \in A$ such that $\|f_2\| < \frac{1}{2}, \dots, \|f_i\| < \frac{1}{2^{i-1}}$ and $\|f - f_1 - \dots - f_i\| < \frac{1}{2^i}$. Find then $f_{i+1} \in A$ such that $\|f_{i+1}\| < \frac{1}{2^i}$ and $\|f - f_1 - \dots - f_i - f_{i+1}\| < \frac{1}{2^{i+1}}$. Then $\|f - f_1 - \dots - f_i\| \to 0$ as $i \to \infty$.

Further we will immitate a standard method from working with Baire one functions, cf. e.g. [22, Ch. XV.1, Theorem 4]. For every $i \in \mathbb{N}$ find

$d_{in} \in D$, $n \in \mathbb{N}$, such that $\|d_{in}\| \leq \|f_i\|$ and $d_{in} \to f_i$ weak*. Put then $g_n = d_{1n} + \cdots + d_{nn}$, $n \in \mathbb{N}$. We note that every g_n belongs to the countable set D. So it remains to prove that $g_n \to f$ weak*. Fix any $\epsilon > 0$ and any $0 \neq x \in X$. Find $k \in \mathbb{N}$ so that $2^{-k+1} < \frac{\epsilon}{3\|x\|}$ and $\|f - f_1 - \cdots - f_k\| < \frac{\epsilon}{3\|x\|}$. Find then $m > k$ so that

$$|\langle d_{in} - f_i, x\rangle| < \frac{\epsilon}{3k} \qquad \text{whenever} \quad i \in \{1, \ldots, k\} \quad \text{and} \quad n > m\,.$$

Then, if $n > m$, we have

$$\begin{aligned}
\big|\langle g_n - f, x\rangle\big| &\leq \sum_{i=1}^{k} \big|\langle d_{in} - f_i, x\rangle\big| + \sum_{i=k+1}^{\infty} \|d_{in}\|\|x\| + \Big|\Big\langle \sum_{i=1}^{k} f_i - f, x\Big\rangle\Big| \\
&< k \cdot \frac{\epsilon}{3k} + \sum_{i=k+1}^{\infty} \frac{1}{2^{i-1}}\|x\| + \Big\|\sum_{i=1}^{k} f_i - f\Big\|\|x\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3\|x\|} \cdot \|x\| + \frac{\epsilon}{3\|x\|} \cdot \|x\| = \epsilon\,,
\end{aligned}$$

and the proof of (ii) is finished.

(iii) Let $x \in S_X$ be such that $f(x) = 1$. Let U, $f_1, \ldots, f_n \in B_{X^*}$ and ψ be chosen as in Definition 1 for the point x. Let $W = \{f_i\}_\perp$. If $h \in W$ is such that $x \pm h \in U$, then $\|x \pm h\| = \|x\|$. Similarly as above in this proof, we find that whenever $f_\mu \to f$ in the weak* topology, then $\sup_{B_W}(f_\mu - f) \to 0$. Therefore $f_\mu \to f$ in the norm as X/W is finite dimensional. ∎

Theorem 7 (i) has the following consequence.

Corollary 8. *Assume that the dual ball of an LFS norm in its weak* topology is a Valdivia compact. Then X is a WLD space.*

Note that the space $\ell_1(c)$ shows that the LFS norm here cannot be replaced by a norm that locally depends on countably many coordinates. Indeed, the dual ball of the standard norm of $\ell_1(c)$ is a Valdivia compact in its weak* topology (use Goldstine's theorem) and the canonical norm of $\ell_1(c)$ depends on countably many coordinates (see Example 2 above). However, $\ell_1(c)$ is not WLD. Also, it is not true that in the statement of Theorem 7 (iii), the norm and weak* topology coincide at each point of the sphere, see Example 4. Indeed, it is known that $C[0, \omega_1]^*$ does not admit any dual norm with the latter property. (Otherwise, $C[0, \omega_1]^*$ would then admit a dual locally uniformly rotund norm ([23]) which is impossible by Talagrand's result (cf. e.g. [3, p. 313])).

For Gâteaux differentiable norms, a variant of the following result can be found in [3, p. 58].

THEOREM 9. *Assume that the norm of a Banach space is an LFS norm. Then* $\operatorname{dens} X^* \leq \operatorname{card} X$.

Proof. Given $x \in S_X$, the cardinality of the set of supporting functionals to B_X at x is less than or equal to c. Indeed, let $\{x_n\} \subset X$ be a sequence such that $\{\hat{x}_n\}$ is dense in X/W, where W is from Definition 1. As in the proof of Theorem 7 (i), we can show that if f and g are two supporting functionals to B_X at x, then $f(w) = g(w)$ for every $w \in W$. Thus for any two different supporting functionals f, g to B_X at x, there must be n available such that $f(x_n) \neq g(x_n)$. By the Bishop-Phelps theorem, the density of X^* is then less than or equal to $c \cdot \operatorname{card} X$ which is less than or equal to $\operatorname{card} X$. ∎

Assuming the Continuum Hypothesis, one can give a short proof of the following result, which for the case of Gâteaux differentiable norms is usually proved by the smooth variational principle (cf. eg. [8, Ch. II], for the smooth variational principle cf. e.g. [3, p. 9]). Note that this principle cannot hold true in $c_0(\Gamma)$ for LFS norms if Γ is uncountable (otherwise, one could apply it to a strictly convex norm on $c_0(\Gamma)$ and get a contradiction).

COROLLARY 10. *Assume the Continuum Hypothesis. Let the norm of a Banach space X be an LFS norm. Then B_{X^*} in its weak* topology is sequentially compact.*

Proof. We follow [13]. Assume that a sequence $\{f_n\} \subset B_{X^*}$ has no weak* convergent subsequences. For every subsequence $\{f_{n_k}\}$ of $\{f_n\}$, choose a point x in B_X such that $\{f_{n_k}(x)\}$ is not convergent. We thus get a subspace Y of X of density character c such that B_{Y^*} is not sequentially compact in its weak* topology. Moreover, the norm of Y is an LFS norm. Hence in order to finish the proof, we can assume that the density character of X is c and that B_{X^*} in its weak* topology is not sequentially compact. Čech and Pospíšil showed in [2] that if a compact space K is not sequentially compact, then $\operatorname{card} K \geq 2^{\aleph_1}$. By using Theorem 9 and this result, under the Continuum Hypothesis, we have

$$\begin{aligned}\operatorname{card} X^* &\leq (\operatorname{dens} X^*)^{\aleph_0} \leq (\operatorname{card} X)^{\aleph_0} \leq ((\operatorname{dens} X)^{\aleph_0})^{\aleph_0} \\ &= (c^{\aleph_0})^{\aleph_0} = c^{\aleph_0} = c = \aleph_1 < 2^{\aleph_1} \leq \operatorname{card} B_{X^*} = \operatorname{card} X^*,\end{aligned}$$

a contradiction. ∎

Note that the statement of Corollary 10 does not hold true in general for norms that locally depend on countably many coordinates. This is can be seen on the space $\ell_1(c)$ (Example 2), as $B_{\ell_\infty(c)}$ is not sequentially compact in its weak* topology (cf. e.g. [12, p. 65]).

Note that Corollary 10 implies that *under the Continuum Hypothesis, $\ell_\infty = C(\beta\mathbb{N})$ is not a quotient space of any Banach space with an LFS norm.* Indeed, otherwise $\beta\mathbb{N}$ would be sequentially compact. It is well known that this is not the case: $\beta\mathbb{N}$ contains no convergent sequence as otherwise, by Tietze extension theorem, c_0 would be isomorphic to a quotient of ℓ_∞, which is not the case, cf. e.g. [12, p. 254]. Similarly, *under the Continuum Hypothesis, $\ell_\infty/c_0 = C(\beta\mathbb{N} \setminus \mathbb{N})$ admits no LFS norm* for otherwise $\beta\mathbb{N} \setminus \mathbb{N}$ would be sequentially compact by Corollary 10.

The following theorem relates to some results in [27].

THEOREM 11. *Let X be a Banach space and let $M \subset B_X$ be such that $\overline{\operatorname{span}}\, M = X$. Let $G \subset X^*$ be the set of all points of X^* that are countably supported on M. Assume that X admits a continuous bump function that locally depends on countably many elements of G. Then X is WLD.*

Proof. Let φ be a continuous bump function on X that locally depends on countably many elements of G. We will prove that each element of X^* has a countable support on M. This will show that B_{X^*} in its weak* topology is a Corson compact. Thus it suffices to show the following

Claim: Given $f \in X^*$ and $\varepsilon > 0$, there is an $f_\varepsilon \in X^*$ such that its support on M is countable and $\|f_\varepsilon - f\| \leq \varepsilon$.

In order to prove the claim, let $f \in X^*$ and $\varepsilon > 0$ be given. Define the function Φ on X by

$$\Phi(x) = \begin{cases} \varphi^{-2}(x) & \text{if } \varphi(x) \neq 0\,, \\ +\infty & \text{if } \varphi(x) = 0\,. \end{cases}$$

It follows that Φ is a bounded below lower semicontinuous function on X such that $\{x \in X : \Phi(x) < +\infty\}$ is open and on its domain, Φ locally depends on countably many elements in G. By the Ekeland variational principle (cf. e.g. [3, Theorem I.2.4.]), there is $x_0 \in X$ such that

$$(\Phi - f)(x) \geq (\Phi - f)(x_0) - \varepsilon\|x - x_0\| \tag{1}$$

for all $x \in X$. Let U, $\{f_i\} \subset G$ and ψ be as in the definition of the local dependence for Φ at x_0. Let $W = \bigcap_i f_i^{-1}(0)$. Finally let $\delta > 0$ be such that

$x_0 + h \in U$ whenever $h \in X$ is such that $\|h\| < \delta$. If $h \in W$ and $\|h\| < \delta$, we have

$$\Phi(x_0+h) - \Phi(x_0) = \psi\big(f_1(x_0+h), \dots, f_n(x_0+h), \dots\big) - \psi\big(f_1(x_0), \dots, f_n(x_0), \dots\big) = 0\,.$$

Hence from (1), for $h \in W$ and $\|h\| < \delta$ we have

$$f(h) = f(x_0+h) - f(x_0) \le \Phi(x_0+h) - \Phi(x_0) + \varepsilon\|h\| = \varepsilon\|h\|\,. \tag{2}$$

Let $\tilde{f}$ be the restriction of f to W. By (2), for $\tilde{f}$ as an element of W^*, we have $\|\tilde{f}\| \le \varepsilon$. Let $\tilde{f}_0$ be a norm preserving Hahn-Banach extension of $\tilde{f}$ to X. Put $f_\varepsilon = f - \tilde{f}_0$. Then $f_\varepsilon \in W^\perp = \overline{\operatorname{span}}^{w^*}\{f_i\}$ and $\|f - f_\varepsilon\| = \|\tilde{f}_0\| \le \varepsilon$. As the total support of all $\{f_i\}$ on M is countable, so is the support of f_ε. This finishes the proof. ∎

Alster showed in [1] that a compact space K is homeomorphic to a weakly compact set in $c_0(\Gamma)$ for some Γ considered in its weak topology (i.e. K is an Eberlein compact) if K is a Corson compact and K is scattered, i.e. each subset of K has a relative isolated point. If K is a compact space such that $C(K)$ admits a continuous bump function that locally depends on finitely many coordinates, then $C(K)$ is an Asplund space ([10]). Thus K is then scattered (cf. e.g. [3, p. 258] or [12, p. 231]). Hence *K is an Eberlein compact if it is a Corson compact and $C(K)$ admits a continuous bump function that locally depends on finitely many coordinates.* The following statement is related to these results.

THEOREM 12. *Let K be a Corson compact such that $C(K)$ has density character $\aleph_1$. Then*

(i) *$C(K)$ is WLD if $C(K)$ admits a continuous bump function that locally depends on countably many elements of $\overline{\operatorname{span}}^{\|\cdot\|}K$.*

(ii) *$C(K)$ is WLD if $C(K)$ admits an equivalent LFS norm $\|\cdot\|$ that is pointwise lower semicontinuous.*

Proof. Since K is a Corson compact, $C(K)$ in its supremum norm has a projectional resolution of the identity P_α, $\alpha \le \omega_1$, such that $K \subset \cup_{\alpha<\omega_1} P^*_\alpha(C(K))$ (cf. e.g. [3, p. 254]).

(i) For $\alpha < \omega_1$, let M_α, be a countable dense set in the unit ball of $(P_{\alpha+1} - P_\alpha)(C(K))$. Then it is enough to put $M = \cup_{\alpha<\omega_1} M_\alpha$ and $G = \overline{\operatorname{span}}^{\|\cdot\|}K$ in Theorem 11.

(ii) As the norm $\|\cdot\|$ is pointwise lower semicontinuous, the space $\overline{\mathrm{span}}^{\|\cdot\|}K$ is 1-norming for the norm $\|\cdot\|$ (use the bipolar theorem, cf. e.g. [12, p. 163]). Since $K \subset \cup_{\alpha<\omega_1} P_\alpha^* C(K)^*$, by Proposition 13 below, $C(K)$ has a PRI in the norm $\|\cdot\|$, and, by [7, Lemma 2], the corresponding dual ball is Valdivia compact in the weak* topology. Thus the norm $\|\cdot\|$ satisfies the assumptions of Corollary 8 and so $C(K)$ is WLD. ∎

PROPOSITION 13. *Let $(X,\|\cdot\|)$ be a Banach space, having density $\aleph_1$, and admitting a projectional resolution of the identity $P_\alpha,\ \alpha \le \omega_1$ in the norm $\|\cdot\|$. Let $|\cdot|$ be an equivalent norm on X such that $\bigcup_{\alpha<\omega_1} P_\alpha^* X^*$ is a 1-norming subspace for $|\cdot|$. Then there exists an increasing "long sequence" $\beta_\alpha,\ \alpha \le \omega_1$ of ordinals from $[0,\omega_1]$ such that $P_{\beta_\alpha},\ \alpha \le \omega_1$ is a PRI on $(X,|\cdot|)$.*

Proof. We shall first show the claim: *For every $\alpha \in [0,\omega_1)$ there is $\beta \in (\alpha,\omega_1)$ such that $|P_\beta| = 1$, i.e.,*

$$|P_\beta x| \le |x| \qquad \textit{for every} \quad x \in X\,. \tag{3}$$

Fix one such α. By induction, we shall construct $\alpha_1 = \alpha < \alpha_2 < \alpha_3 < \cdots < \omega_1$ as follows. Let $n \in \mathbb{N}$ and assume that α_n was already chosen. Let $\{x_m^n :\ m \in \mathbb{N}\}$ be a (norm) dense set in $P_{\alpha_n}X$ (note that $\alpha_n < \omega_1$). For every $m \in \mathbb{N}$ we find a countable set $Y_m^n \subset \bigcup_{\alpha<\omega_1} P_\alpha^* X^* \cap B_{(X,|\cdot|)^*}$ such that

$$|x_m^n| = \sup \left\langle Y_m^n, x_m^n \right\rangle .$$

Since each Y_m^n is at most countable, there is $\alpha_{n+1} \in (\alpha_n,\omega_1)$ such that $Y_m^n \subset P_{\alpha_{n+1}}^* X^*$ for every $m \in \mathbb{N}$. This finishes the induction step. Put $\beta = \lim_{n\to\infty} \alpha_n$; then still $\beta < \omega_1$.

Let us show (3). Fix any $x \in X$. Let $\epsilon > 0$ be arbitrary. Since $|P_{\alpha_n}x| \to |P_\beta x|$, from the construction of β, there are $n,m \in \mathbb{N}$ and $\eta \in Y_m^n$ such that $|P_\beta x| - \epsilon < \langle \eta, P_\beta x\rangle$. But

$$\langle \eta, P_\beta x\rangle = \langle P_\beta^* \eta, x\rangle = \langle \eta, x\rangle\,.$$

Therefore

$$|P_\beta x| - \epsilon < \langle \eta, x\rangle \le |\eta|\,|x| \le |x|\,,$$

and finally, $|P_\beta x| \le |x|$. This proves our claim.

Next let β_0 be the β found in our claim for $\alpha := 0$. Let $0 < \alpha < \omega_1$ and assume that we found already β_γ for $0 \le \gamma < \alpha$. If α is a limit ordinal, put $\beta_\alpha = \lim_{\gamma\uparrow\alpha} \beta_\gamma$. If α is not a limit ordinal, put $\beta_\alpha = \beta$ where β was found by our claim for $\alpha := \beta_{\alpha-1}$. Now it is elementary to verify that the β_α satisfy the conclusion of our Proposition. ∎

We will now summarize some facts on $C[0,\omega_1]$. Some of them were discussed in Example 4.

THEOREM 14. (i) *The supremum norm of* $C_0[0,\omega_1]$ *is an LFS norm. However, it does not locally depend on finitely many coordinates.*

(ii) *The standard supremum norm of* $C[0,\omega_1]$ *does not locally depend on countably many coordinates.*

(iii) *There is a norm on* $C[0,\omega_1]$ *that locally depends on finitely many coordinates in* $[0,\omega_1]$.

(iv) *There is no continuous bump function on* $C[0,\omega_1]$ *that would depend locally on countably many elements of* $[0,\omega_1)$.

(v) *There is no LFS norm on* $C[0,\omega_1]$ *that would be lower semicontinuous in the topology of pointwise convergence on* $[0,\omega_1)$.

(vi) *The dual ball* $B_{C_0[0,\omega_1]^*}$ *in its weak** *topology is not a Valdivia compact.*

Proof. (i) − (iii) were proven in Example 4.

(iv) Put $M = \{\chi_{[\alpha+1,\omega_1]}\}_{\alpha\le\omega_1}$ and $G = [0,\omega_1)$ in Theorem 11.

(v) Assume that $C[0,\omega_1)$ admits an LFS norm $|\cdot|$ that is Y-lower semicontinuous where Y is the norm closed linear hull of $[0,\omega_1)$ in $C[0,\omega_1]^*$. From the bipolar theorem it then follows that Y is a 1-norming subspace for $|\cdot|$ (cf. e.g. [8, Ch. 13]). The closed linear hull Y of $[0,\omega_1)$ in $C[0,\omega_1]^*$ is 1-norming subspace for the LFS norm and each element of B_Y is countably supported on the set M defined above. Since B_Y is weak* dense in B_{X^*}, by Theorem 7 (i), there is a countable set $C \subset Y$ such that the Dirac measure δ_{ω_1} belongs to $\overline{C}^*$. Hence δ_{ω_1} would have a countable support on M, which is impossible.

(vi) This was discussed in Example 4. ∎

A result similar to that in Theorem 14 (v) for Gâteaux differentiable norms was shown in [9].

We will finish this paper with presenting a few open problems.

Open Problems

1. Assume that the norm of a Banach space X is an LFS norm. Does X admit an equivalent Gâteaux differentiable norm?

2. Assume that a Banach space X admits a norm that locally depends on finitely many coordinates. Does X admit a C^∞-smooth norm? This problem has a solution in the positive in the case of separable spaces ([15]).

3. Does $C[0, \omega_1]$ admit an equivalent locally uniformly rotund norm that is a limit, uniform on bounded sets, of norms depending locally on finitely many coordinates (or LFS1 norms)?

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