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# The Inverse Problem for Flat Kinetic Minus Potential Lagrangians

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## 1. Introduction.

The inverse problem of the calculus of variations asks for necessary and sufficient conditions that a given system of second order ordinary differential equations should be the Euler-Lagrange equations of a regular Lagrangian function. In 1941 Douglas [3] gave an exhaustive and apparently complete treatment of the two degrees of freedom case. The subject was taken up again about twenty years ago by a number of authors: we cite [1,2,3,4,5,6] and the references therein as a representative sample of articles.

The purpose of the present note is to answer a restricted version of the inverse problem. We shall formulate a sufficient set of conditions which imply the existence of an essentially unique Lagrangian of classical mechanical type, that is, of the form flat-metric kinetic energy minus potential. In effect our result may be said to characterize ODE systems that are of generic classical mechanical type.

A general system of $n$ second order ODE will be written in the form

$$\ddot{x}^i = f^i(x^j, u^j), \tag{1.1}$$

where the velocity variable is denoted by $u^i$ rather than $\dot{x}^i$. A fundamental tensorial invariant associated to (1.1) known as the Jacobi endomorphism is given by

$$4\Phi^i_j = 2\frac{d}{dt}\left(\frac{\partial f^i}{\partial u^j}\right) - 4\frac{\partial f^i}{\partial x^j} - \frac{\partial f^i}{\partial u^k}\frac{\partial f^k}{\partial u^j}. \tag{1.2}$$

In (1.2) the summation convention on repeated indices applies as it does for the rest of the paper. Furthermore, $\frac{d}{dt}$ denotes the total time derivative operator, a notation which will be used interchangeably with a dot as is convenient.

In this paper, we shall be concerned with systems of the form (1.1) in which $f^i$ is independent of $u^j$. Although this condition is not invariant under general changes of the independent variables it is preserved by linear transformations. For such systems $\Phi$ reduces essentially to a "Jacobian" matrix.

For a system of type (1.1) Douglas [3] proved that the following conditions are necessary and sufficient for the existence of a Lagrangian: there should exist a non-singular symmetric $n \times n$ matrix $g_{ij}(x^k, u^k)$ that satisfies

$$g_{ij}\Phi^j_k = g_{kj}\Phi^j_i \tag{1.3}$$

$$2\dot{g}_{ij} + \frac{\partial f^k}{\partial u^i}g_{kj} + \frac{\partial f^k}{\partial u^j}g_{ki} = 0 \tag{1.4}$$

$$\frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j} = 0. \tag{1.5}$$

If such a matrix $g$ can be found, it is the Hessian with respect to the $u^i$ of the Lagrangian $L$ which can be found by quadrature. In the sequel we shall abbreviate eq. (1.3) as

$$g\Phi = (g\Phi)^t \tag{1.6}$$

to use a self-explanatory notation.

Equations (1.3)–(1.5) constitute a complicated algebro-differential system for the unknown $g$. Douglas [3] for $n = 2$ and Sarlet [6] for general $n$ proved that (1.3)–(1.5) entail a hierarchy of purely algebraic conditions:

$$g^k\Phi = (g^k\Phi)^t \, (k = 0, 1, 2, \ldots) \tag{1.7}$$

where the ${}^{(k)}\Phi$ are defined recursively by

$${}^{(0)}\Phi = \Phi \tag{1.8}$$

$${}^{(k+1)}\Phi =^{(k)} \dot{\Phi} - \frac{1}{2}\left[{}^{(k)}\Phi, \frac{\partial f}{\partial u}\right] \tag{1.9}$$

and the second term in (1.9) is the commutator of ${}^{(k)}\Phi$ and the matrix $\frac{1}{2}\frac{\partial f^i}{\partial u^j}$. For the type of systems we are considering, ${}^{(k)}\Phi$ is simply the total time derivative of $\Phi$ to order $k$. One further piece of notation is needed before we can proceed to our main result. We shall denote the Liouville or dilation field or differential operator $u^i \frac{\partial}{\partial u^i}$ by $\Delta$.

Now we can state our main theorem.

THEOREM 1.1. *Suppose that a second order ODE system* $(n > 1)$ *of type (1.1) in which the* $f^i$*'s are independent of* $u^j$ *and are smooth on some open domain* $U$ *in* $\mathbb{R}^n$ *is such that the first two algebraic conditions*

$$g\Phi = (g\Phi)^t$$
$$g\dot{\Phi} = (g\dot{\Phi})^t$$

*have a one-dimensional, non-degenerate solution space that is consistent with the third condition*

$$g\ddot{\Phi} = (g\ddot{\Phi})^t$$

*and that the entries of one such solution are smooth.*
*Suppose further that any one of the following conditions holds:*

1. $n = 2$ *and* $f^1, f^2$ *in* (1.1) *are not simultaneously null on an open subset of* $\mathbb{R}^2$ *(in particular if* $f^1, f^2$ *are real analytic).*
2. *Any first integral of the ODE which is homogeneous of degree zero in velocities is necessarily a constant number.*
3. *The distribution (in the sense of Frobenius' theorem) generated by the differential operators or vector fields* $\frac{d}{dt}$ *and* $\Delta$ *by taking successive Lie brackets, contains the entire vertical distribution spanned by the* $\frac{\partial}{\partial u^i}$ *on an open dense subset of the neighborhood* $U \times \mathbb{R}^n$.

*Then the ODE is of Euler-Lagrange type with essentially unique Lagrangian given by*

$$L = \frac{1}{2}\, g(u, u) - V$$

*where* $g$ *is a flat metric determined as described below. Furthermore once* $g$ *has been found, the potential* $V$ *can be obtained from the equation*

$$\frac{\partial V}{\partial x^i} = -g_{ij} f^j.$$

The reader will note that the Theorem is formulated in the time independent context though obvious modifications would make it applicable to non-autonomous systems as well.

## 2. Proof of the main result.

In this Section we shall prove the main Theorem. We begin with a Lemma.

Lemma 2.1.

1. *Let $L(x^i, u^i)$ be a regular Lagrangian that engenders an Euler-Lagrange vector field corresponding to forces that are independent of velocities. Then in the same system of coordinates the function $L$ satisfies the PDE*

$$\frac{\partial^2 L}{\partial x^i \partial u^j} - \frac{\partial^2 L}{\partial x^j \partial u^i} = 0.$$

2. *Given the hypotheses of (i) the forces are in fact determined by*

$$f^j = g^{ij}(1 - \Delta)\frac{\partial L}{\partial x^i},$$

*where $g^{ij}$ is the inverse of the Hessian of $L$.*

*Proof.* The forces $f^j$ are determined by the conditions

$$g_{ij} f^j = \frac{\partial L}{\partial x^i} - u^j \frac{\partial^2 L}{\partial x^j \partial u^i} \tag{2.1}$$

regardless of whether $f^j$ depend on $u^j$. Differentiating with respect to $u^k$ and splitting into symmetric and skew-symmetric parts gives

$$\begin{aligned} 2f^j \frac{\partial g_{ij}}{\partial u^k} + 2u^j \frac{\partial g_{ik}}{\partial x^j} + g_{ij}\frac{\partial f^j}{\partial u^k} + g_{kj}\frac{\partial f^j}{\partial u^i} + g_{ij}\frac{\partial f^j}{\partial u^k} - g_{kj}\frac{\partial f^j}{\partial u^i} \\ = 2\frac{\partial^2 L}{\partial u^k \partial x^i} - 2\frac{\partial^2 L}{\partial u^i \partial x^k}. \end{aligned} \tag{2.2}$$

Hence if the $f^j$ are independent of $u^j$ the above conditions collapse to $\dot{g} = 0$ and the condition stated in (i) of the Lemma.

(ii) If we contract the previous condition with $u^k$ we get

$$\Delta\left(\frac{\partial L}{\partial x^i}\right) = u^k \frac{\partial^2 L}{\partial u^k \partial x^i} = u^k \frac{\partial^2 L}{\partial x^k \partial u^i} \tag{2.3}$$

which enables the forces $f^j$ to be written in the stated form. ∎

*Proof.* From the first two algebraic conditions we obtain

$$\dot{g}\Phi + g\dot{\Phi} = (\dot{g}\Phi)^t + (g\dot{\Phi})^t \tag{2.4}$$

and

$$\dot{g}\dot{\Phi} + g\ddot{\Phi} = (\dot{g}\dot{\Phi})^t + (g\ddot{\Phi})^t. \tag{2.5}$$

Thus $\dot{g}$ also satisfies the first two algebraic conditions and hence

$$\dot{g} = \mu g \tag{2.6}$$

for some function $\mu$.

Now let $a$ and $b$ be any two entries of $g$. Then the previous condition implies that

$$\dot{a} = \mu a \tag{2.7}$$

$$\dot{b} = \mu b \tag{2.8}$$

and hence

$$\dot{a}b - a\dot{b} = 0. \tag{2.9}$$

It follows that if $b$ is non-zero then $\frac{a}{b}$ is a first integral.

Now we shall apply $\Delta$ to the first two algebraic conditions so as to obtain

$$\Delta g\Phi = (\Delta g\Phi)^t \tag{2.10}$$

$$\Delta g\dot{\Phi} = (\Delta g\dot{\Phi})^t. \tag{2.11}$$

Again it follows that

$$\Delta g = vg \tag{2.12}$$

for some function $v$. If $a$ and $b$ are as above then we must have that

$$\Delta\left(\frac{a}{b}\right) = 0, \tag{2.13}$$

that is, $\frac{a}{b}$ is homogeneous of degree zero.

Now we invoke conditions (i)–(iii) stated in the Theorem. If condition (ii) holds then we may conclude that $\frac{a}{b}$ is identically constant. If, however, condition (iii) holds then $\frac{a}{b}$ must be independent of $u^i$ on a dense open subset of $U \times \mathbb{R}^n$ and hence by continuity and denseness on $U \times \mathbb{R}^n$. Since $\frac{a}{b}$ is a first integral, it must be identically constant. Finally if condition (i) holds then $\frac{a}{b}$ is independent of $(u^1, u^2)$ except on the closed subset given by $f^1u^2 - f^2u^1 = 0$. By continuity $\frac{a}{b}$ must be independent of $(u^1, u^2)$ on $U \times \mathbb{R}^n$.

At this point we can assert that the putative Hessian $g_{ij}$ is of the form

$$g = \gamma g_0 \tag{2.14}$$

where $g_0$ is a constant solution of the algebraic conditions. To finish the proof we shall see what results from imposing the second and third Helmholtz conditions eq. (1.4) and (1.5). In fact (1.4) implies that $\gamma$ is also a first integral.

Finally eq. (1.5) implies that $\gamma$ must satisfy

$$\frac{\partial \gamma}{\partial u^j}\, \delta^m_k = \frac{\partial \gamma}{\partial u^k}\, \delta^m_j \tag{2.15}$$

where $\delta^m_k$ is the Kronecker symbol. On contracting $k$ with $m$ we find that

$$(n-1)\frac{\partial \gamma}{\partial u^j} = 0 \tag{2.16}$$

Since we are assuming that $n > 1$ it follows that $\gamma$ is independent of $u^j$ and hence is constant. Thus $g_{ij}$ is constant, all three Helmholtz conditions are satisfied and $L$ is a quadratic Lagrangian. Also part (i) of the Lemma implies that any term linear in $u^i$ that occurs in $L$ must actually be a total time derivative and thus is negligible according to the "essentially unique" clause of the Theorem. Finally part (ii) of the Lemma implies that the potential $V$ is determined from the condition

$$\frac{\partial V}{\partial x^i} = -g_{ij} f^j. \tag{2.17}$$

∎

We conclude the paper with a few remarks about the hypotheses of the Theorem 1.1. First of all the first algebraic condition imposes generically $\binom{n}{2}$ conditions on the $\binom{n+1}{2}$ components of $g$. The assumption that the first three algebraic conditions entail that $g$ is unique up to scaling corresponds to the generic situation in which there is a Lagrangian unique up to scaling by a constant.

Secondly, there certainly do exist systems with first integrals homogeneous of degree zero. For example in dimension $n$ any system with Euclidean kinetic energy and a central potential has $\binom{n}{2}$ linear integrals of motion corresponding to the components of angular momentum. By taking quotients we obtain integrals of degree zero. This class of systems does not contradict the Theorem, however, because the algebraic conditions are not satisfied. This point is discussed in detail by Henneaux and Shepley [4].

Thirdly, condition (iii) can be implemented in practice. For systems with forces independent of velocities the vector fields $\Delta$,

$$[\Delta, \Gamma] - \Gamma = -2f^i \frac{\partial}{\partial u^i} \tag{2.18}$$

$$[[\Delta, \Gamma] - \Gamma, [[\Delta, \Gamma], \Gamma]] = -8f^i \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial u^j} \tag{2.19}$$

are all vertical so if $n = 3$ their linear dependence can be checked. For example the Calogero and Toda systems in three dimensions may be shown to have unique Lagrangians whereas the Kepler system, of course, does not.

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