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Bargaining Multiple Issues with Leximin Preferences

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RESUMEN
Tratamos en este trabajo problemas de negociación simultánea sobre varios asuntos, que formalizamos como productos cartesianos de problemas de negociación clásicos. Tanto en el caso en que las preferencias de los agentes son de tipo maximin como de tipo leximin, caracterizamos la clase de soluciones eficientes que cumplen una condición de separabilidad consistente en que la negociación simultánea proporciona resultados equivalentes a la negociación separada de cada uno de los asuntos. Estas soluciones del problema de negociación global se construyen a partir de soluciones clásicas en la clase de soluciones de camino monótono.

Palabras clave: Negociación simultánea, preferencias maximin, preferencias leximin.

ABSTRACT
Global bargaining problems over a finite number of different issues, are formalized as cartesian products of classical bargaining problems. For maximin and leximin bargainers we characterize global bargaining solutions that are efficient and satisfy the requirement that bargaining separately or globally leads to equivalent outcomes. Global solutions in this class are constructed from the family of monotone path solutions for classical bargaining problems.

Keywords: Global bargaining, maximin preferences, leximin preferences.
JEL classification: C7, C78.

1 Introduction

We address bargaining problems where a finite number of different issues must be resolved to reach an agreement. The following examples illustrate the situations we wish to address:

1. The bargainers must share a basket of ingredients to produce paella; each agent follows a different recipe, but all require a mix all ingredients in exactly the right proportions, and they all wish to cook as much paella as possible.

2. Bargaining at the World Trade Organization, when each representative is an egalitarian government bargaining on behalf of her country. Within each country, the resolution of each issues benefits disjoint subsets of the population, and transfers between these groups are not possible.

3. Bargaining under uncertainty without expected utility. There are m states of the world and agents bargain a contingent agreement for each state of the world at the ex-ante stage.

The literature addressing multiple issue bargaining has focussed mostly on environments where the agents preferences over benefit bundles are given by a utility function; then a global bargaining problem can be reduced to a classical bargaining problem where feasible agreements are allocations of utilities for each player. In this case, addressing the global problem is tantamount to uncovering the structure of the feasible set of utilities from that of single issue bargaining problems, and to discover the links between the classical solutions applied separately to the issues and classical solutions applied to the set of global feasible utilities. Most of the literature takes this approach, usually under the additional assumption that utilities are additive across issues.[1]

In contrast, we are mostly interested in solutions that apply directly to the global bargaining situation, either because detailed information about preferences is limited or manipulable; or because the preferences of the bargainers do not admit a utility representation that allows to bargain over a set of feasible utility allocations.

Bossert et al (1996) and Bossert and Peters (2001), that stress the motivation of bargaining under uncertainty, precede us in addressing global bargaining problems as cartesian products of classical bargaining problems.[2] In Bossert et al (1996) a class of strictly monotone path solutions is characterized by imposing maximin ex-ante efficiency. Bossert and Peters (2001) consider efficient solutions when agents have minimax regret preferences; in bilateral problems the class of monotone utopia-path solutions is characterized, but for more agents only dictatorial solutions remain.

[1] See Kalai (1977), Myerson (1981), Binmore (1984), Gupta (1989) and Ponsatí and Watson (1997).

[2] See also Bossert and Peters (2000).

A related class of problems is addressed in Hinojosa et al.(2005) and Mármol et al.(2005), where different families of solutions exhibiting a maximin efficiency propriety for multi-criteria bargaining models are proposed and explored. The main difference with our approach is that the bargaining sets do not admit a representation as a cartesian product of separate problems.

Our analysis of multiple issue bargaining problems begins with focus on environments with minimax preferences. We point out that the global solutions that assure efficient outcomes have a straightforward characterization in terms of efficiency on the intersection of the issue bargaining sets. Furthermore, we show that requiring equivalence in "minimax result", regardless of whether the issues are addressed separately or globally, characterizes the family of monotonic solutions.

Our main results relate to environments where the bargainers preferences are leximin. We characterize the set of efficient outcomes for these situations, and we propose a family of global solutions that attains efficient outcomes. This family is constructed by applying a classical solution to a sequence of classical bargaining problems: First the intersection of all the issue problems is considered, and (at least) one issue is resolved; subsequently the disagreement point and the bargaining set to address next is revised, and so on until a global agreement is obtained. We show that to attain the same outcome whether the issues are addressed separately or globally via this step by step procedure it is necessary and sufficient to use a monotonic classical solution.

The rest of the paper is organized as follows. We lay out the set up and discuss preliminary observations in Section 2. Solutions for global bargaining under maximin preferences are addressed in Sections 3. Sections and 4 and 5 address environments with leximin bargainers.

2 Multiple issue bargaining

A group of n agents, $i = 1, 2, \ldots, n$ bargain over m different issues $j = 1, \ldots, m$. Denote by $N = \{1, \ldots, n\}$ the set of agents, and by $M = \{1, \ldots, m\}$ the set of issues. The *issue bargaining problems* are classical bargaining problems; they are represented by pairs (S_j, d_j), where $S_j \subset \Re^n$ is the set of feasible benefits that can be allocated to the agents by mutual agreement on issue j and $d_j \in \Re^n$ is the allocation of benefits in case of disagreement on that issue. [3] The sets S_j are compact, strictly comprehensive[4] and $s > d_j$ for some $s \in S_j$. Denote the set of efficient allocations by $e(S_j)$ weakly efficient allocations by $we(S_j)$,[5] and the class of classical bargaining problems by $\mathcal{B}$. A *classical solution* is a

[3]The following notation will be used: $x, y \in \Re^n$: $x > y$ means that $x_j > y_j$, for $j = 1, \ldots, n$; $x \geq y$ means that $x_j \geq y_j$, for $j = 1, \ldots, n$ and $x \neq y$; and $x \geqq y$ means that $x_j \geq y_j$, for $j = 1, \ldots, n$. For a matrix $X \in \Re^{m \times n}$ we will denote by X_j the jth row, and by X^i the ith column. We will also denote the dominance relations between matrices $X, Y \in \Re^{m \times n}$, as follows: $X \geqq Y$ if $x^i_j \geq y^i_j$, $\forall i, j$; $X \geq Y$ if $X \geqq Y$ and $X \neq Y$; $X > Y$ if $x^i_j > y^i_j$, $\forall i, j$.

[4]Comprehensiveness suffices for many results, but we keep the stronger assumption throughout the paper for expositional simplicity.

[5]$e(S_j) = \{s \in S_j, \nexists s' \in S_j, s' \geq s\}$, $we(S_j) = \{s \in S_j, \nexists s' \in S_j, s' > s\}$

function $\varphi : \mathcal{B} \to \Re^n$ such that $\varphi(S_j) \in e(S_j)$. We further assume that agents can measure the benefit they obtain in each issue, and compare it to the benefits in the other issues.

A *global bargaining problem* is a pair (S, d), where $S = S_1 \times S_2 \times ... \times S_m$ is the set of feasible outcomes, each outcome allocating feasible benefits on every issue, while $d \in \Re^{m \times n}$ is the *status quo*, the allocation of benefits in case of disagreement over all issues. Thus, a *global* agreement is represented by an element of S, an $m \times n$ matrix:

$$X = \begin{pmatrix} x_1^1 & x_1^2 & \dots & x_1^n \\ x_2^1 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \dots & x_m^n \end{pmatrix}$$

The ith column of matrix X, $X^i = (x_1^i, x_2^i, \dots, x_m^i)^t \in \Re^m$, represents the benefits for player i, in each of the m issues. The jth row of matrix X, $X_j = (x_j^1, x_j^2, \dots, x_j^n) \in S_j \subset \Re^n$ represents an allocation of issue j benefits to each of the players.

Let $\mathcal{GB}$ denote the class of global bargaining games. We wish to explore *solutions*, criteria to select outcomes for each $(S, d) \in \mathcal{GB}$, and discuss their properties and performance. Formally a *solution* is a correspondence, $F : \mathcal{GB} \to \Re^{m \times n}$ such that $F(S) \subseteq S$, that selects a non-empty subset of feasible outcomes (possibly a singleton) for each global bargaining problem. For simplicity and without loss of generality we fix the disagreement point at the origin, and let $S_j \subset \Re^n_+$, so that global bargaining problems are represented by $S = \prod_{j \in M} S_j$, and denote the class of such bargaining games by $\mathcal{GB}_0$.

If we think of a global problem as m separate bargaining problems then a classical solution applied to each one of the issues will generate a solution to the global problem. However, in global bargaining each player can give up something in one issue to get more in another while such trade-offs are precluded when the issues are discussed separately. The global problem may also be approached as a bargaining problem among $n \times m$ agents; but in this case global payoffs - attained by the combination benefits over the m-issues - are also ignored.[6] It is, therefore, worthwhile to explore global solutions that genuinely address the issues jointly taking into account the interrelations in the benefits of different issues over the total payoffs of the agents, considering properties that seem appropriate to resolve global problems and exploring the links to classical solutions and their properties.

2.1 Pareto Optimality

A precise notion of efficiency for global problems requires a specification of the agents' preferences over bundles of issue benefits.

Prior to examining more detailed specifications for the individual preferences, let us consider the sets of outcomes that are undominated in the natural

[6]See Bergstresser and Yu (1977).

dominance relations $X \geq Y$ and $X > Y$. That is, the dominance relations based only on the assumption that individual preferences are monotone. We will refer to such undominated outcomes as Pareto optimal.

Pareto Optimality(PO): $X \in S \subseteq \Re^{m \times n}$ is *Pareto optimal* in S, if $\nexists Y \in S$, such that $Y \geq X$.

Note that because $S = S_1 \times \ldots \times S_m$, a global solution is Pareto optimal if and only if it allocates the benefits of each issue efficiently. Thus the following obviously holds.

Proposition 1 *For $S = S_1 \times ... \times S_m$, and $X \in S$, the following assertions are equivalent*
a) X is Pareto optimal in S.
b) X_j is efficient in S_j for all $j \in M$.

We may also want to consider weak Pareto optimality (WPO) defined as follows: $X \in S \subseteq \Re^{m \times n}$ is *weakly Pareto optimal* in S, if $\nexists Y \in S$, such that $Y > X$. The analogous equivalence to Proposition 1 holds for weak Pareto Optimality by replacing condition b) by X_j is weakly efficient in S_j for some $j \in M$. Nevertheless, for the class of strictly comprehensive problems we are addressing, as $e(S_j) = we(S_j)$, a global outcome is weakly Pareto optimal if and only if it allocates efficiently the benefits of at least one issue.

2.2 Monotonicity

The property of monotonicity plays a crucial role in the characterization of an important family of classical solutions, and will turn out very useful in our approach to global solutions:

Monotonicity(MON): A solution φ is *monotonic* if and only if $\varphi(T', d') \leqq \varphi(T, d)$ for all $(T', d'), (T, d) \in \mathcal{B}$, with $d = d'$ and $T' \subseteq T$.

Thomson and Myerson(1980) propose a general family of solutions with monotonicity proprieties defined as follows.[7] A *monotone path* (in $\Re^n_+$), G, is defined as the image of a function $\psi : \Re_+ \rightarrow \Re^n_+$ where $\psi(0) = 0$, ψ_i is continuous and nondecreasing for all $i \in N$, and $\sum_{i=1}^n \psi_i$ is increasing. The *monotone path solution* relative to the monotone path G, φ^G, is defined as $\varphi^G(T, d) = G \cap e(T)$.

Note that with the present definition monotone path solutions are well defined on the class of convex, compact and strictly comprehensive problems provided that $T \subseteq \Re^n_+$ and $d \geqq 0$. Note also that the results the solution yield do not depend on the disagreement point, but when the disagreement point is on the path, the solution is always individually rational.

Monotonicity characterizes the family of monotone path solutions, see Thomson and Myerson(1980). For completeness we sketch a proof of this characterization.

[7] Thomson and Myerson(1980) use a version of the monotonicity property which is slightly stronger than ours to characterize strictly monotone path solutions.

Proposition 2 *A solution φ in $\mathcal{B}_0$ is monotonic if and only if φ is a monotone path solution.*

Proof. Necessity is straightforward. It is easy to see that if φ is a monotone path solution then it verifies the monotonicity axiom.

To prove sufficiency, let φ be efficient and monotonic. For each $t > 0$ consider $T_t \subset \Re^n$, $T_t = \{y \in \Re^n, y \geq 0, \sum_{i=1}^n y_i \leq t\}$. Define $\psi(0) = 0$ and $\psi(t) = \varphi(T_t)$. Clearly, the image of ψ is a monotone path and φ the corresponding monotone path solution. ■

3 Maximin Bargaining

To begin, we consider situations as that of the Paella example, where the preferences of the bargainers over benefit bundles are determined by the minimum benefit achieved across all the issues.[8]

For these situations the global bargaining problem is very easily addressed as a classical problem via the intersection of the bargaining sets over all the issues. Formally, agent i, has preferences such that $X \succcurlyeq^i_{min} Y$ if and only if $z^i(X) \geq z^i(Y)$, where $z^i(X) = \min_{1\leq j\leq m}\{x^i_j\}$. Presently, global bargaining problems are reduced to a classical problem whose bargaining set is the intersection of the bargaining sets for all the issues. Note that compactness and comprehensiveness are closed under intersection. The function $z : \Re^{m\times n} \to \Re^n$ maps the multiple issue bargaining set S into a set $z(S) \subset \Re^n$,

$$z(S) = \{(z^1, \ldots, z^n) \in \Re^n,\, z^i = min_j\{x^i_j\}\, for\, some\, X \in S\}.$$

Note that a minimum payoff vector $z \in z(S)$ can be obtained from different feasible outcomes in S. It is immediate that the comprehensiveness of S_j implies that $z(S) = \cap_{j=1}^m S_j$.

Figure 1 displays the set $z(S)$ for a two person and three issue bargaining problem.

Let $z(X) = (z^1(X), z^2(X), \ldots, z^n(X))$. Consider the order relation in $\Re^{m\times n}$, $X, Y \in \Re^{m\times n}$, $X \succcurlyeq_{min} Y$, if $z(X) \geq z(Y)$. The set of efficient outcomes under the maximin criterion is now straightforward:[9]

Maximin Efficiency (MEF): An outcome $X \in S \subseteq \Re^{m\times n}$ is *maximin efficient* if $\nexists Y \in S$, such that $Y \succ_{min} X$.

It is easy to see that maximin efficiency implies weak Pareto-optimality. In general, it does not imply Pareto optimality.

The following is now immediate:

Lemma 3 *$X \in S$ is MEF if and only if $z(X) \in e(\cap S_j)$.*

[8] This is also the set up of Bossert et al (1996).

[9] This notion of efficiency is a strong version of "efficiency under uncertainty" in Bossert et al.(1996).

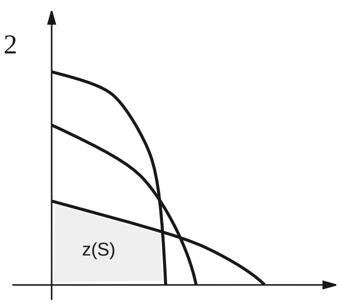

Figure 1: The set of minimal benefits

The following lemma observes that under strict comprehensiveness of the sets S_j, if $y \in \Re^n$ lies on the efficient frontier of $\cap_{j=1}^m S_j$, there exists at least one issue j for which y lies on the efficient frontier of the corresponding bargaining set S_j.

Lemma 4 *If $y \in \Re^n$, $y \in e(\cap_{j=1}^m S_j)$, then $y \in e(S_j)$ for at least one issue.*

Proof. Let $y \in e(\cap_{j=1}^m S_j)$ and suppose that for each j, there exists $\overline{y}_j \in S_j$ such that $\overline{y}_j \geq y$. It follows from the strict comprehensiveness of S_j that for all j, $\exists z_j \in S_j$ with $z_j > y$. Now, for each $i \in N$, denote $\varepsilon^i = \min_j \{z_j^i - y^i\}$ and consider $\overline{y}^i = y^i + \varepsilon^i$. $\overline{y} \leq z_j$ for all j and therefore, as a consequence of the comprehensiveness of S_j, $\overline{y}^i \in S_j$ for all j. It follows that $\overline{y}^i \in \cap_{j=1}^m S_j$ and $\overline{y}^i > y^i$, which is a contradiction to $y \in e(\cap_{j=1}^m S_j)$. ■

It is worth pointing out that the full strength of strict comprehensiveness is necessary for this result to hold as can be seen in Figure 2. Nevertheless, if the assumption is relaxed to comprehensiveness an analogous result can be established obtaining weak Pareto-optimality.

The following result characterizes maxmin efficient outcomes.

Proposition 5 *$X \in S$ is MEF if and only if $\exists k \in M$ such that $X_k \in e(S_k)$ and $X_j \geqq X_k$ for all $j \in M$.*

Proof. If $X \in S$ is MEF, it follows from lemmas 3 and 4 that $z(X) \in e(S_k)$ for some $k \in M$. Suppose that $X_k \neq z(X)$, then, as $X_k \geqq z(X)$, $X_k \geq z(X)$ holds, what contradicts the efficiency of $z(X)$ in S_k.

Conversely, when $X_j \geqq X_k$ for all $j \in M$, then $z(X) = X_k \in ef(\cap_{j=1}^m S_j)$ holds, and therefore $X \in S$ is maximin efficient. ■

To generate global solutions that satisfy MEF simply define the global solution F_φ^{min} as $F_\varphi^{min}(S)_k = \varphi(z(S))$ for all $k \in M$. Denote by F_φ^{sep} the global solution obtained from φ in each one of the issues. This is, for $S = S_1 \times \ldots \times S_m$,

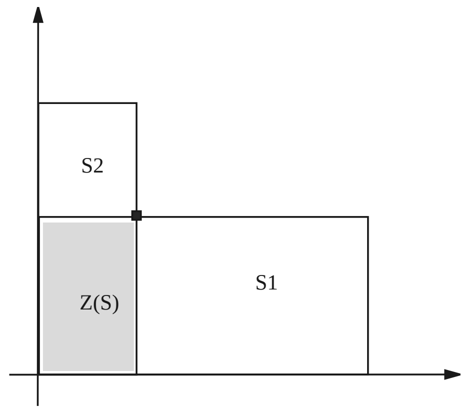

Figure 2: Efficient solution in z(S)

$F_{\varphi}^{sep}(S) = (\varphi(S_1), \ldots, \varphi(S_m))^t$. We want to identify what classical solutions induce the same maximin results under separate or global bargaining, i.e. what φ satisfy the following property:

Separate-global maximin equivalence (SGMEQ): A solution φ satisfies *Separate-Global maximin equivalence* when $\varphi(z(S)) = z(F_{\varphi}^{sep}(S))$ for all $S \in \mathcal{GB}_0$.

The following result establishes the equivalence between MON and SGMEQ.

Proposition 6 *Let φ be a classical solution in $\mathcal{B}_0$, φ verifies SGMEQ if and only if satisfies MON.*

Proof. Consider a classical solution φ that satisfies SGMEQ, and therefore, for $S_1, \ldots, S_m \in \mathcal{B}_0$, $\varphi(z(S)) = z((\varphi(S_1), \ldots, \varphi(S_m))^t)$ holds, and assume φ does not satisfy MON, i.e. there exist $T_1, T_2 \subseteq \Re_+^n$ such that $T_1 \subseteq T_2$, and $i \in N$ such that $\varphi(T_1)^i > \varphi(T_2)^i$. Consider the global bargaining problem $T = T_1 \times T_2 \in \mathcal{GB}_0$. In this case, $z(T) = T_1$ and it follows that $\varphi(z(T)) = \varphi(T_1)$. On the other hand, $z^i((\varphi(T_1), \varphi(T_2))^t) = min\{\varphi(T_1)^i, \varphi(T_2)^i\} = \varphi(T_2)^i < \varphi(T_1)^i = \varphi(z(T))^i$, contradicting SGMEQ.

Conversely, let φ satisfy MON and $S \in \mathcal{GB}_0$, as $z(S) \subseteq S_j$, $j \in M$, it follows that for all $i \in N$, $\varphi^i(z(S)) \leq \varphi^i(S_j)$,for all $j \in M$ and, therefore, $\forall i \in N, \varphi^i(z(S)) \leq min_j\{\varphi^i(S_j)\} = z^i(\varphi(S_1), \ldots, \varphi(S_m))$.

If SGMEQ fails $\exists i \in N$ such that $\varphi^i(z(S)) < min_j\{\varphi^i(S_j)\}$. From the efficiency of φ follows that $\varphi(z(S))$ is in the efficient frontier of at most a S_j, for this $j \in M$, $\varphi(S_j)$ is also efficient, and $\varphi^i(z(S)) < \varphi^i(S_j)$, hence $\exists k \in N$ such that $\varphi^k(z(S)) > \varphi^k(S_j)$, contradicting MON. ■

Now, using the equivalence established in Proposition 2, together with this last result the following characterization of the monotone path solutions can be established.

Theorem 7 *A solution φ in $\mathcal{B}_0$ verifies SGMEQ if and only if $\varphi = \varphi^G$, where G is a monotone path.*

4 The Leximin approach to Global Bargaining

We say that player i has leximin preferences over the global results, and denote $\succcurlyeq^i_{lex}$, if for $X, Y \in \Re^{m\times n}$

$$X \succcurlyeq^i_{lex} Y \Leftrightarrow X^i \geqslant_{lex} Y^i$$

where $\geqslant_{lex}$[10] denotes the leximin order in $\Re^m$.

In contrast to the case of maximin preferences, leximin preferences can not be represented by a utility function (see, for instance, Moulin (1988)) and therefore it is not possible to reduce the global problem to a classical bargaining problem on the utilities.

We now define a lexicographical ordering relation in $\Re^{m\times n}$ based on the successive minimum values attained by the agents.

Given $X \in \Re^{m\times n}$, consider the $m \times n$ matrix $Z(X)$, constructed as follows. For each column x^i - the vector of benefits of player i- reorder its components in increasing magnitude; this reordered vector is the ith column of matrix $Z(X)$. Thus, the first row of $Z(X)$, $z_1(X)$, contains the lowest element of each column of matrix X. The second row, $z_2(X)$, contains the second lowest element of each column of matrix X. In general, the elements of $z_k(X)$ are the k-th lowest element of each column of matrix X. We say that $X \succcurlyeq_{lex} Y$, if $z_k(X) \geq z_k(Y)$ for the first row, k, such that $z_k(X) \neq z_k(Y)$.

$\succcurlyeq_{lex}$ is a collective dominance relation that for $n = 1$ reduces to the leximin order in $\Re^m$, but it does not define a complete order in $\Re^{m\times n}$.

The following result states that, as in the case of m-dimensional vectors, the dominance relation $\geq$ among matrices is stronger than this lexicographic dominance relation.

Lemma 8 *Let $X, Y \in \Re^{m\times n}$. If $X \geq Y$, then $X \succcurlyeq_{lex} Y$.*

Proof. Since $X \geq Y$, it is clear that $Z(X) \geq Z(Y)$, and therefore, for the first row, k, such that $z_k(X) \neq z_k(Y)$, $z_k(X) \geq z_k(Y)$ holds. ■

The relationship between the dominance relation $\succcurlyeq_{min}$ induced by the first minimum values and $\succcurlyeq_{lex}$ induced by the successive minimums is established in the following result which is easy to prove.

Lemma 9 *Let $X, Y \in \Re^{m\times n}$. If $X \succcurlyeq_{min} Y$, then $X \succcurlyeq_{lex} Y$.*

Denote by $\succ^i_{lex}$ the asymmetric part of $\succcurlyeq^i_{lex}$.

Lemma 10 *Let $X, Y \in \Re^{m\times n}$. If $X \geq^i_{lex} Y$ for all $i \in N$, and $X \succ^i_{lex} Y$ for some $i \in N$, then $X \succcurlyeq_{lex} Y$.*

[10]For $a \in \Re^m$, let $r(a) \in \Re^m$ be the vector obtained reordering components in a in increasing order. For $a, b \in \Re^m$, $a >_{lex} b$ if there is $k = 0, \ldots, m-1$ such that $r_i(a) = r_i(b)$ for $i = 1, \ldots, k$, $r_{k+1}(a) > r_{k+1}(b)$; $a \geq_{lex} b$ if $a >_{lex} b$ or $r(a) = r(b)$.

Note that if $X \succcurlyeq^i_{lex} Y$ for all i, then $X \succcurlyeq_{lex} Y$ (but the converse is not true).

The partial ordering in $\Re^{m\times n}$, $\succcurlyeq_{lex}$, permits us to establish the following concept of efficiency, which is appropriate for global bargaining problems where the agents preferences are leximin.

LEXIMIN EFFICIENCY(LEF): A feasible outcome $X \in S \subseteq \Re^{m\times n}$ is *leximin efficient* if there is not another $Y \in S$ such that $Y \succcurlyeq_{lex} X$.

This concept of efficiency is stronger than MEF, which is based on a dominance relation where only the first minimum is considered. Whereas MEF implies weak Pareto optimality, LEF implies strong Pareto optimality as a consequence of lemma 8.

The following result establishes a characterization of the outcomes which are LEF.

Proposition 11 *$X \in S$ is LEF if and only if for all $k \in M$, $X_k \in e(S_k)$, and for all $j, k \in M$, either $X_j \geqq X_k$ or $X_j \leqq X_k$.*

Proof. Firstly, if for all $k \in M$, $X_k \in e(S_k)$ and for all $j, k \in M$, either $X_j \geqq X_k$ or $X_j \leqq X_k$, let π be the issue index permutation such that $X_{\pi(j)} \leqq X_{\pi(j+1)}, j = 1, \ldots, m-1$. It follows that $z_k(X) = X_{\pi(k)}$ for all $k \in M$. Suppose to the contrary, that X is not LEF, then there exists $Y \in S$, such that for some $k, k = 1, \ldots, m$, $z_j(Y) = z_j(X) = X_{\pi(j)}$ for all $j = 1, \ldots, k-1$, $z_k(Y) \geq z_k(X) = X_{\pi(k)}$. It follows from Lemma 4 and from the efficiency of X_k in S_k that, as $z_1(Y) \in \cap_{j\in M} S_j$ and $z_1(X) = X_{\pi(1)} \in ef(\cap_{j\in M} S_j)$, $Y_{\pi(1)} = X_{\pi(1)}$ holds. Analogously, $z_2(Y) = X_{\pi(2)} \in \cap_{j\neq 1} S_{\pi(j)}$ and necessarily, $X_{\pi(2)} = Y_{\pi(2)}$ and for $j = 1, \ldots, k-1$ $X_{\pi(j)} = Y_{\pi(j)}$. For $j = k$, $z_k(Y) \geq z_k(X) = X_{\pi(k)} \in ef(\cap_{j\neq 1,\ldots,k-1} S_{\pi(j)})$, $z_k(Y) \in \cap_{j\neq 1,\ldots,k-1} S_{\pi(j)}$, which is in contradiction with the efficiency in the intersection.

Conversely, if X is LEF, it is easy to see that $z_1(X) \in ef(\cap_{j\in M} S_j)$ and it follows that there exist $\pi(1)$, such that $z_1(X) = X_{\pi(1)} \in e(S_{\pi(1)})$ and hence $X_j \geqq X_{\pi(1)}$, for all $j \in M$. If $z_2(X) = z_1(X)$, then $X_{\pi(2)} = X\pi(1) \in e(S_{\pi(2)})$, in other case $z_2(X) \in \cap_{j\neq\pi(1)} S_j$, and applying recursively the reasoning the result follows. ■

The condition that characterizes global outcomes which are leximin efficient means that LEF are results solving all the issues efficiently and such that the outcomes corresponding to the different issues can be ordered with respect to the dominance relation $\geqq$ in $\Re^n$. That is to say, given two different issues, all the agents jointly obtain either more or less with respect to them.

The relationships between the different concepts of efficiency considered so far and their characterizations is summarized as:

Leximin efficient		Pareto optimal
$\forall j \in M,\ X_j \in e(S_j)$ $\exists \pi,\ X_{\pi(k)} \leqq X_{\pi(k+1)},\ \forall k = 1, \ldots, m-1$	$\Longrightarrow$	$\forall j \in M,\ X_j \in e(S_j)$
$\Downarrow$		$\Downarrow$
Maximin efficient		Weakly Pareto optimal
$\exists j \in M,\ X_j \in e(S_j)$ $X_j \leqq X_k,\ \forall k \in M$	$\Longrightarrow$	$\exists j \in M, X_j \in e(S_j)$

5 Global Bargaining that resolves issues step by step

In what follows we propose a family of global solutions that select leximin efficient outcomes.

Let $S = S_1 \times \ldots \times S_n$ where each S_j is strictly comprehensive and consider a classical solution, φ. We define a solution, F_φ, for global bargaining problems $(S, 0) \in \mathcal{GB}_0$, as follows:

$F_\varphi : \mathcal{GB}_0 \to \Re^{m \times n}$, with $F_\varphi(S) = X^* \in S$ where $X^* \in S$ is obtained by the following procedure.

Algorithm 4.1

Step 0: Let $d(0) = 0$, $I(0) = \{M\}$, $k = 1$.

Step k:

For $j \in \{j \in I(k-1),\ \varphi(\cap_{j \in I(k-1)} S_j, d(k-1)) \in e(S_j)\}$,
$X_j^* = \varphi(\cap_{j \in I(k-1)} S_j, d(k-1))$.
$I(k) = I(k-1) \setminus \{j \in M,\ \varphi(\cap_{j \in I(k-1)} S_j, d(k-1)) \in e(S_j)\}$.

- If $I(k) = \{\emptyset\}$, then $F_\varphi(S) = X^*$.
- If $I(k) \neq \{\emptyset\}$, then
  $d^i(k) = \varphi_i(\cap_{j \in I(k-1)} S_j, d(k-1))$ for all $i \in N$.
  $k := k + 1$.

The procedure uses a classical solution to select the agents' benefits for the issues sequentially. In each step, the classical solution is applied to the set of minimum benefits of the issues that are still unresolved, fixing the level of benefits for at least one issue. Then a new classical bargaining problem is considered, whose disagreement point consists of the levels of benefit assigned to the agents in the issue(s) fixed by the previous step.

It follows from Lemma 4 that at every step at least one issue is resolved and therefore, the values of X^* are selected at most in m steps. Note that if φ is defined uniquely, then F_φ is also defined uniquely. The procedure is illustrated in the following figure.

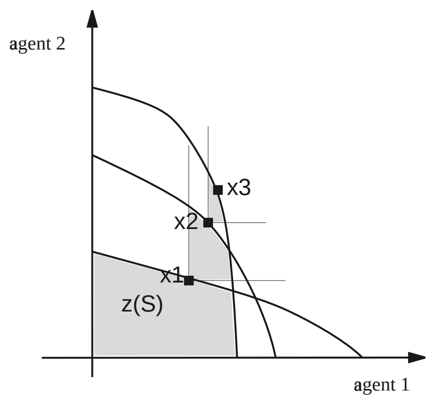

Figure 3: Step by step bargaining procedure.

Proposition 12 *A global bargaining solution F_φ selects a LEF outcome.*

Proof. The result follows from the construction of F_φ and the characterization of LEF outcomes established in Proposition 11. ■

In general, the solution to the global problem induced by a solution φ, F_φ does not coincide with the vector of solution obtained if bargaining takes place separately in each issue. If it does, we will say that φ verifies step by step equivalence.

Step by step equivalence (SSEQ): A classical solution φ satisfies *step by step equivalence* if and only if $F_\varphi(S) = F_\varphi^{sep}(S)$.

This axiom has the flavour of Kalai's step by step axiom (see Kalai, 1977). However, while Kalai addresses a additive utility bargaining problems where the bargaining set expands as new issues are added to the problem, our step by step equivalence axiom concerns global bargaining problems where the outcomes consist of the results obtained for each issue, which are valued in a leximin preference domain.

The following result states that SSEQ also characterizes the family of monotone path solutions. Figure 4 is an illustration.

Theorem 13 *A classical solution φ in $\mathcal{B}_0$ satisfies SSEQ, i.e. $F_\varphi(S) = F_\varphi^{sep}(S)$, if and only if $\varphi = \varphi^G$, where G is a monotone path.*

Proof. To proof that if $\varphi = \varphi^G$ with G a monotone path, then $F_\varphi(S)_k = \varphi(S_k)$, first note that $\varphi^G(T, 0) = \varphi^G(T, d)$ for all $d \in G \cap T$. As F_φ is leximin efficient, there exists π such that $F_\varphi(S)_{\pi(k)} \leqq F_\varphi(S)_{\pi(k+1)}$, for all $k = 1, \ldots, m-1$. $F_\varphi(S)_{\pi(1)} = \varphi(\cap_{j\in M} S_j) \in e(S_{\pi(1)})$. Besides, $\varphi(S_{\pi(1)}) \in e(S_{\pi(1)})$, and as a consequence of the monotonicity of φ, $\varphi(\cap_{j\in M} S_j) \leqq \varphi(S_{\pi(1)})$, and then necessarily $\varphi(\cap_{j\in M} S_j) = \varphi(S_{\pi(1)})$. In the second step of the construction of F_φ, the disagreement point is $\varphi(S_{\pi(1)})$ which is on the path. Now,

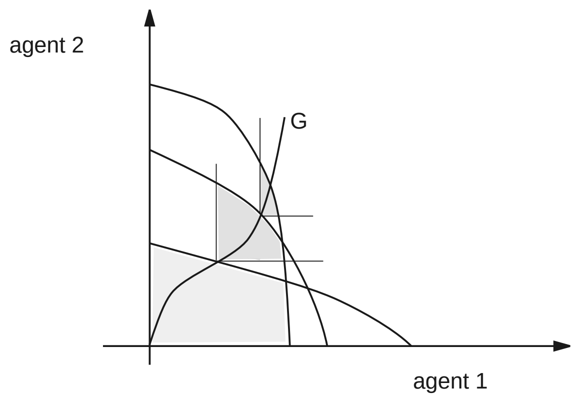

Figure 4: Monotone path solution

$F_{\varphi}(S)_{\pi(2)} = \varphi(\cap_{j\neq\pi(1)}S_j, \varphi(S_{\pi(1)})) = \varphi(\cap_{j\neq\pi(1)}S_j, 0) \in e(S_{\pi(2)})$, and also $\varphi(S_{\pi(2)}) \in e(S_{\pi(2)})$. It follows that $\varphi(S_{\pi(2)}) = F_{\varphi}(S)_{\pi(2)}$. Analogously the same result is obtained for each issue.

Conversely, from the leximin efficiency of the solution F_{φ} it follows that there exists π such that $\varphi(S_{\pi(k)}) \leqq \varphi(S_{\pi(k+1)})$ for $k = 1, \ldots, m-1$. For the first issue solved, $F_{\varphi}(S)_{\pi(1)} = \varphi(S_{\pi(1)}) \leqq \varphi(S_{\pi(k)})$ holds for all k, and therefore, $z(\varphi(S_1), \ldots, \varphi(S_m)) = F_{\varphi}(S)_{\pi(1)}$.

In addition $\varphi(z(S)) = F_{\varphi}(S)_{\pi(1)}$, and from Theorem 7 it follows that φ is a monotone path solution. ■

As a consequence of this last result, if classical solutions are restricted to those that satisfy homogeneity, i.e. $\varphi(\gamma(S)) = \gamma\varphi(S)\ \forall\gamma > 0$, then solutions verifying SSEQ are the proportional solutions. If symmetry is also required the unique solution is the egalitarian.

It is worth pointing out that SSEQ is a separability condition analogous to the separate/global equivalence condition in Ponsati and Watson(1997), where global bargaining problems with additive utilities are addressed. In our set up, symmetry, homogeneity and SSEQ characterize the egalitarian solution, whereas, for additive utilities, symmetry and homogeneity and Ponsati and Watson's separate/global equivalence characterizes the symmetric utilitarian solution. Binmore (1984) presents a similar result for the case of multiplicative utilities, there a related separability condition characterizes the Nash solution.

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