teorema
Vol. XXV/2, 2006, pp. 5-24

Paraconsistent Logic: A Proof-Theoretical Approach*

Hubert Marraud

RESUMEN

Una lógica es paraconsistente si admite teorías inconsistentes pero no triviales. Dada la definición usual de inconsistencia, la noción de paraconsistencia parece depender de la interpretación dada al signo '¬'. Como la lógica paraconsistente pone en cuestión propiedades de la negación consideradas básicas en otros contextos, cabe dudar que un operador sin esas propiedades sea realmente un negador. La conclusión es que no puede haber genuinas lógicas paraconsistentes. Esta objeción puede responderse desde una perspectiva (sub)estructural siempre que puedan construirse cálculos de secuentes paraconsistentes con reglas estructurales ligeramente distintas de las de la lógica clásica.

ABSTRACT

A logic is paraconsistent if it allows for non-trivial inconsistent theories. Given the usual definition of inconsistency, the notion of paraconsistent logic seems to rely upon the interpretation of the sign '¬'. As paraconsistent logic challenges properties of negation taken to be basic in other contexts, it is disputable that an operator lacking those properties will count as real negation. The conclusion is that there cannot be truly paraconsistent logics. This objection can be met from a substructural perspective since paraconsistent sequent calculi can be built with the same operational rules as classical logic but with slightly different structural rules.

I. INTRODUCTION

According to the most extended definition, a paraconsistent logic is a logic, which allows for non-trivial inconsistent theories. Let L be a logic whose consequence relation will be denoted by $\vdash_L$. A L-theory is, of course, a set of formulae closed under $\vdash_L$; a theory is trivial if it is just the set of all formulae of the corresponding language. The difficulties arise when, having defined triviality, one tries to define inconsistency. It is usually said that a set of formulae X is inconsistent if and only if there is at least one formula A such that both $X \vdash_L A$ and $X \vdash_L \neg A$.

Thus, the notion of inconsistency and hence the definition of paraconsistent logic seem to rely upon the interpretation given to the sign '¬'. At this point there are two different strategies. The first [Arruda (1980)] relativizes the notion of inconsistency as follows:

> Let O be some *unary operator*; a set of formulae X is O-inconsistent if and only if there is at least one formula A such that both $X \vdash_L A$ and $X \vdash_L OA$.

This characterisation is at once too wide and too narrow. On one hand, there would be $\diamond$-inconsistent theories, where $\diamond$ is the possibility operator, with the amazing consequence that any modal logic extending KT yields only $\diamond$-inconsistent theories[1]. On the other hand, the existence of such binary operators as Sheffer stroke or joint negation shows that there is no privileged link between inconsistency and unary operators. If a theory entails both A and A|A, it is as inconsistent as another entailing A and $\neg$A. Finally, no advance has been done, since the question "Can there be paraconsistent logics?" can be rephrased now as "Can there be $\neg$-paraconsistent logics?"

The alternative is to use the concept of negation operator:

> Let O be some *unary negation operator*; a set of formulae X is O-inconsistent if and only if there is at least one formula A such that both $X \vdash_L A$ and $X \vdash_L OA$.

The question becomes then what is a negation operator. Notice that this is a partial definition because it applies only to unary negation operators, leaving open the question whether there are negation operators of arity >1 (as Sheffer stroke). In any case, the immediate objection is that whatever negation symbol is used, it is more plausible to suppose that it is not a real negation than to accept the paraconsistent reading. This is just Slater's (1995) objection. What is at stake is whether one can identify a negation operator without appeal to principles like the pseudo-Scotus, i.e. A,$\neg$A $\vdash_L$ B, rejected by any paraconsistent logic. To argue for the need to distinguish a Boolean negation from a De Morgan negation, and so on, is not very convincing and what is worse, you are then threatened with Quine's thesis change of logic is change of subject matter.

> departure from the law of excluded middle would count as evidence of revised usage of 'or' or 'not' [...]. For the deviating logician the words 'or' and 'not' are unfamiliar or defamiliarised. [Quine (1960) p.396].
> Alternative logics are inseparable from mere change in usage of logical words. [Quine *op.cit.*, p.389].

To sum up, to define *paraconsistent logic* you have to define first *negation*. As paraconsistent logic challenges inferential properties of negation taken to be basic in other contexts, it is always disputable that an operator lacking those properties will count as real negation. The intended conclusion is that there cannot be any truly paraconsistent logics.[2]

II. Logic of Formulae vs. Logic of Inferences

The argument against paraconsistent logic (and in general arguments based on the thesis “change of logic is change of subject matter”) rests on the assumption that every logic is characterised by the set of its logical constants and hence differences between logics have to be explained as differences concerning their logical constants. This assumption fits into a conception of logic grounded on the concept of thesis (theorem, valid formula, logical truth, etc.) rather than on the concept of deduction. This conception of logic starts with Frege and it is opposed, not only with the pre-Fregean tradition (as W. and M. Kneale show) but also with the conception of logic starting with Gentzen and continued in present times by such authors as Belnap, Došen and Girard.

If Hilbert-style proof theory expresses the idea that logic is the science of logical truths, Gentzen-style proof theory expresses the rival thesis that logic is the science of formal deductions. In a Gentzen-style consecution calculus there are two different kinds of rules: structural rules and operational (or logical) rules. The first can be described independently of the constants of the object language and state structural properties of derivability (reflexivity, transitivity, etc.) Operational rules, on the other side, represent the different operations applicable to consecutions in the object language. Against the claim that the basic rules are operational rules, the approach favoured by Došen and linear logicians claims that the basic features of a logical system are determined by structural rules.

> In fact, contrary to popular belief, these rules are the most important of the whole calculus, for, without having written a single logical symbol, we have practically determined the future behaviour of the logical constants. [Girard, Lafont & Taylor (1989) p. 30]

My thesis is that paraconsistent logics can be built with the same operational rules as classical logic (and hence with the same logical constants) but with slightly different structural rules; in Došen’s terminology, that there are substructural paraconsistent logics.

III. A Gentzen-Style Paraconsistent Calculus

Let us consider the usual derivation of the pseudo-Scotus in a standard Gentzenization of classical logic:

$p \vdash p$	axiom
$p, \neg p \vdash$	$\neg \vdash$
$p, \neg p \vdash q$	$\vdash$ weakening

This derivation depends on the identity axiom, the interpretation of the sign '¬' as given by the rule $\neg\vdash$ and a principle about the syntactic nature of objects in a consecution, right-weakening. Therefore it seems possible to reject the pseudo-Scotus keeping the meaning of logical constants restricting either the identity axiom $A \vdash A$ or the right-weakening rule. So far as the identity axiom states an essential property of consequence (*viz.,* reflexivity), the key to achieve paraconsistency seems to lie in the weakening rules. It is often thought that weakening is related to monotonicity (an essential property of any consequence relation) in the same way as the identity axiom is related with reflexivity; but as we shall see, this is untrue.

As I have said, in a substructural approach structural rules and derivations (those, which can be described independently of logical constants,) are taken to be basic for the identity of a logic or a logical system. If a consecution is a triple $X \vdash Y$ where X and Y are finite sequences of formulae, the axioms and structural rules of paraconsistent propositional logics are the following.

Axioms:

$A \vdash A$, for any sentence letter A

Structural rules:

Permutation $P\vdash \quad \dfrac{X,A,B,Y \vdash Z}{X,B,A,Y \vdash Z}$ $\qquad \vdash P \quad \dfrac{X \vdash Y,A,B,Z}{X \vdash Y,B,A,Z}$

Contraction $W\vdash \quad \dfrac{X,A,A \vdash Y}{X,A \vdash Y}$ $\qquad \vdash W \quad \dfrac{X \vdash A,A,Y}{X \vdash A,Y}$

Mingle $M\vdash \quad \dfrac{X,A \vdash Y}{X,A,A \vdash Y}$ $\qquad \vdash M \quad \dfrac{X \vdash A,Y}{X \vdash A,A,Y}$

Combination $\dfrac{X \vdash Y}{X,A \vdash A,Y}$

These rules enable us to define two distinct paraconsistent logical systems. The basic system NWPL (Non Weakenable Paraconsistent Logic) is given by the identity axiom and the rules of permutation, contraction and mingle; the addition of the combination rule yields a second paraconsistent system NWPLC.

The main difference with respect to classical logic concern the weakening rules, which in classical logic allow the introduction of any formula on the right and the left of a consecution.

$$K\vdash \quad \frac{X \vdash Y}{X,A \vdash Y} \qquad\qquad \vdash K \quad \frac{X \vdash Y}{X \vdash A,Y}$$

In the present case, these rules have been replaced by the mingle rules, i.e., the reverse of the contraction rules, and, only for NWPLC, the combination rule. The last rule has some interest on its own for it specifies independently of any logical symbol how to combine two derivations into a single derivation.

We have to specify yet the operational rules. Depending on the accepted structural rules it is possible to state different sets of rules for the same logical constant. So, if weakening and contraction are available, you can use as disjunction introduction rules either the pair

$$\frac{X \vdash A, Y}{X \vdash A \vee B, Y} \qquad \frac{X \vdash B, Y}{X \vdash A \vee B, Y}$$

or the single rule

$$\frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}$$

However when weakening and contraction are dropped, as in linear logic, classical disjunction splits into tensor sum, with the rules

$$\frac{X \vdash A, B, Y}{X \vdash A + B, Y} \qquad \frac{W, A \vdash X \qquad Y, B \vdash Z}{W, Y, A + B \vdash X, Z}$$

and a direct sum, with the rules

$$\frac{X \vdash A, Y}{X \vdash A \oplus B, Y} \qquad \frac{X \vdash B, Y}{X \vdash A \oplus B, Y} \qquad \frac{X, A \vdash Y, \qquad X, B \vdash Y}{X, A \oplus B \vdash Y}$$

Dually, we shall have a tensor product (or cumulative conjunction),

$$\frac{X, A, B \vdash Y}{X, A \otimes B \vdash Y} \qquad \frac{W \vdash A, X \qquad Y \vdash B, Z}{W, Y \vdash A \otimes B, X, Z}$$

and a direct product (or alternative conjunction)

$$\frac{X, A \vdash Y}{X, A \& B \vdash Y} \qquad \frac{X, B \vdash Y}{X, A \& B \vdash Y} \qquad \frac{X \vdash A, Y, \qquad X \vdash B, Y}{X \vdash A \& B, Y}$$

What happens with negation and (material) conditional? Since negation is an unary operator, its rules of left-introduction and right-introduction have a single premise and so remain unaffected by the presence or absence of weakening and contraction. The standard rules for negation are:

$$\vdash\neg \quad \frac{X, A \vdash Y}{X \vdash \neg A, Y} \qquad \neg\vdash \quad \frac{X \vdash A, Y}{X, \neg A \vdash Y}$$

It could be argued then, and this is somehow surprising, that negation is less ambiguous than conjunction or disjunction. At first sight it seems that in the case of material conditional it is not possible either to distinguish between a direct conditional and a tensor conditional. The single-premise rules for conjunction and disjunction require the presence of only one of the immediate subformulae of the formula to be introduced and thus they allow two different versions (one for each subformula) while the single-premise rule for material conditional uses both immediate subformulae to built the inferred formula:

$$\frac{X, A \vdash B, Y}{X \vdash A \supset B, Y}$$

This creates an illusion that conditional, unlike conjunction or disjunction, expresses a deductive connection between its antecedent and its consequent. It is just an illusion of relevance for, given contraction and weakening, the pair can replace the usual rule for conditional right-introduction without loss

$$\frac{X, A \vdash Y}{X \vdash A \supset B, Y} \qquad \frac{X \vdash B, Y}{X \vdash A \supset B, Y}$$

Therefore, you are entitled to distinguish a tensor conditional $A \Rightarrow B =_{def.} \neg A + B$ and a direct conditional $A \rightarrow B =_{def.} \neg A \oplus B$.

Letting aside exponentials[3], we have still to characterise the zeroary operators **1**, ⊥, ⊤ and **0** (called 'units' in linear logic).

$$\frac{X \vdash Y}{X, \mathbf{1} \vdash Y} \qquad \vdash \mathbf{1}$$

$$\bot \vdash$$

$$\frac{X \vdash Y}{X \vdash \bot, Y}$$

$$\mathbf{0} \vdash X \qquad X \vdash \top$$

The operators **1**, ⊥, ⊤ y **0** are the neutral elements with respect to ⊗, +, & y ⊕, respectively.

IV. Weakening and Monotonicity

A consecution calculus, as any other deductive system, does not give a direct definition of a consequence relation. It does not make sense to say that

that Y follows from X if and only if the consecution X $\vdash$ Y is provable in the corresponding consecution calculus. Consecutions are pairs of finite sequences of formulae while consequence is a relation between sets of formulae, finite or not, and formulae. Usually, starting from the notion of provable consecution, the definition of consequence runs:

> A formula *A* follows from a set *X* of formulae if and only if there is some finite sequence of elements of *X* such that *X* $\vdash$ *Y* is a provable consecution.

Notice that this definition *stipulates* that consequence is a monotonic and finitary operation. This fact is notorious, since it is often thought that monotonicity is warranted by the weakening rules. Roughly the content of these rules is rather that whenever X is a *finite set*, A follows from X if and only the consecution X $\vdash$ A is provable. It should be noted by the way that the preceding definition lets aside questions concerning whether there may be many occurrences of the same formula in the sequence Y, questions obviously related with the content of weakening and contraction rules.

The fact that paraconsistent logics NWPL and NWPLC lack weakening in its general form does not entail that these are non-monotonic logics. The lost monotony can be recovered defining consequence as follows:

> X $\vdash_{\text{NWPL}}$ A if and only if there is some finite sequence Y of formulae from X such that the consecution Y $\vdash$ A is provable in NWPL.
>
> X $\vdash_{\text{NWPLC}}$ A if and only if there is some finite sequence Y of formulae from X such that the consecution Y $\vdash$ A is provable in NWPLC.

To keep away monotonicity from weakening has a relevantist flavour. On the described account of logical consequence, to say that A follows from X is to say that there is an valid argument with premises $B_1,\ldots,B_n$ in X and conclusion A, where *premise* means "premise effectively used" in the argument.

As it is well known, monotonicity and finitaryness are two of the three properties of Tarskian consequence operations. Given a language $\mathcal{L}$ and an operation C on the set $\wp$(F) of sets of formulae of $\mathcal{L}$, we say that C is a Tarskian consequence operation in $\mathcal{L}$ if and only if the following conditions are satisfied:

(T1) $X \subseteq C(X)$ (reflexivity).
(T2) If $X \subseteq Y$ then $C(X) \subseteq C(Y)$ (monotonicity).
(T3) $C(C(X)) \subseteq C(X)$ (transitivity).

If moreover for all X,

(F) $C(X) = \cup\{C(Y){:}Y \text{ is a finite subset of } X\}$, C is said to be finitary. Likewise, if

$$\text{(S) } e(C(X)) = C(e(X)),$$

for every homomorfism e from F to F, C will be called structural. A finitary and structural Tarskian consequence operation is called an standard consequence operation. These definitions are from Wójcicki (1980), although what I call a 'Tarskian consequence' is for him simply consequence. It is worth noting that Tarski (1930) includes among his axioms for consequence operations an axiom of trivialization: there is a formula A such that C(A) = F.

Let $A \in C(X)$ if and only if $X \vdash_{NWPL} A$; then the consequence operation C of NWPL is standard:

(1) By A1 and the definition of $\vdash_{NWPL}$, C is reflexive.
(2) By the definition of $\vdash_{NWPL}$, C is monotonic.
(3) The admissibility of the Cut rule,

$$\frac{X \vdash A, Y \qquad\qquad Z; A \vdash W}{X, Z \vdash Y, W}$$

proved in the usual manner shows that C is transitive (cfr. §.7 below).
(4) Again by the definition of $\vdash_{NWPL}$, C is finitary.
(5) Structurality follows from the formulation of the axioms and rules of NWPL using schemata of formulae instead of formulae.

Mutatis mutandis, the same holds for NWPLC.

V. THE SUFFICIENCY IDEA

In recent developments in Gentzen-style proof theory there is some opposition between the thesis of the primacy of structural rules and derivations and the principle called by Belnap "the sufficiency idea". The thesis of the primacy of structural rules emphasises the role of the context in the assessment of the validity of inferences. On the other hand, Gentzen's sufficiency principle, as explained by Prawitz, states that the meaning of a logical constant is given by rules dealing only with this constant.

Belnap [(1982) p.382] tries to reconcile both thesis invoking Wittgenstein notion of family resemblance. Letting aside some complications, the idea, in the case of negation, is that 'negation' is not the name of a logical constant, but a generic name for the members of a family of logical constants sharing the same rules of left and right-introduction. When these fixed structural rules are combined with different sets of structural rules, different negations arise.

So operational rules will be invariant while structural rules vary from a logical system to another.

What are the proper rules for negation? The rules $\neg\vdash$ and $\vdash\neg$.are inappropriate for they involve contextual elements, say, the position of the displayed formula in the antecedent/consequent sequence. This is clear when one compares those rules with these:

$$\neg\vdash \quad \frac{X,X' \vdash Y,A,Y'}{X,\neg A,X' \vdash Y,Y'} \qquad\qquad \vdash\neg \quad \frac{X,A,X' \vdash Y,Y'}{X,X' \vdash Y,\neg A,Y'}$$

This phenomenon can be interpreted in two ways. The first, in the line of the distinction between tensor and direct variants of a binary operator made in linear logic, says that when permutation is not presupposed, negation splits into a "positional negation" and a "permutable negation". The second, in the line of Belnap's display logic and Sambin, *et al.* basic logic, tries to purify those rules from any contextual element.

A (radical) move to avoid interference of the context in the definition of logical constants is to remove context and this can be done stating visible operational rules. According with Sambin, Batilotti and Faggian [(2000), p. 981],

> A rule satisfies visibility if it operates on a formula (or two formulae) only if it is (they are) the only formula(e), either in the antecedent of in the succedent of a sequent.

In fact they use a more restricted notion of visible rule, since they require also that the formula resulting from the application of the rule be isolated in the corresponding side of the inferred consecution. Thus, the visible rules for tensor product are:

$$\frac{A,B \vdash Y}{A\otimes B \vdash Y} \qquad\qquad \frac{X \vdash A \qquad\qquad Y \vdash B}{X,Y \vdash A\otimes B}$$

and those for direct product:

$$\frac{A \vdash Y}{A\&B \vdash Y} \qquad\qquad \frac{B \vdash Y}{A\&B \vdash Y} \qquad\qquad \frac{X \vdash A \qquad\qquad X \vdash B}{X \vdash A\&B}$$

How are then the visible rules for negation? If as Sambin *et al.* do (not as they say) visibility is required both in the premises and the conclusion of the rules, the only possible rules are:

$$\neg\vdash(\mathrm{v}) \quad \frac{\vdash A}{\neg A \vdash} \qquad\qquad \vdash\neg(\mathrm{v}) \quad \frac{A \vdash}{\vdash \neg A}$$

With A as the sole formula in the premise-consecution and ¬A as the sole formula in the conclusion-consecution. Consider next the following derivation in NWPL:

$$\dfrac{\dfrac{A \vdash A \quad \text{Axiom}}{\vdash \neg A, A} \vdash\neg}{\vdash \neg A + A} \vdash +$$

According with the Wittgenstenian account of logical constants espoused by Belnap, there should be some combination of basic negation rules with appropriate structural ones allowing us to put this derivation in a suitable form. The difficulty is that using the rule (v) $\vdash\neg$ to get

$$\dfrac{\vdash \neg A, A \quad \vdash\neg}{\vdash \neg A + A} \vdash +$$

You have to derive first $A \vdash$. But a logical system in which $A \vdash$ is provable for any formula A, would be, no doubt , quite bizarre.

VI. Display Logic

Belnap's display logic tries to isolate the properties of the logical constants (as expressed by operational rules) from contextual properties (as expressed by structural rules). This is done by means of consecution calculi having the display property. A consecution calculus has the display property if and only if for any consecution and any antecedent (consequent) constituent there is a tightly equivalent consecution in which that constituent is "displayed" as the antecedent (consequent) standing alone.

To build consecution calculi with the display property an enriched structural language is needed. The idea is that in a consecution calculus there are two kinds of connectives: logical connectives, which take formulae into formulae, and structure connectives, which take structures into structures (as the comma of the standard consecution calculi). The notion of a structure is defined recursively as follows:

(1) Any formula is a structure;
(2) If X is a structure then *(X) is a structure,
(3) If X and Y are structures, (X,Y) is a structure.

The key feature to realise the display property is the use of two structure connectives: the binary ','combining two structures, and the unary '*', permitting one to flip from one side of the turnstile to the other. Roughly, the idea is that you can move a structure from the antecedent to the consequent or vice versa marking it with ‘*’.

Structure connectives are for structures what logical connectives are for formulae; if operational rules have to do with the behaviour of logical constants, structural rules determine the behaviour of structure connectives. In other words, structural rules deal with structures, operational rules deal with formulae.

The structure connective is also used in negation rules, which become:

$$\neg\vdash(\mathrm{v})\ \frac{*A\vdash Y}{\neg A\vdash Y}\qquad\qquad \vdash\neg(\mathrm{v})\ \frac{X\vdash *A}{X\vdash \neg A}$$

Therefore, you get visible rules allowing for the occurrence in the consecutions of formulae other than the displayed formula. Now the demonstration of $\vdash\neg$ A+A goes:

$$\cfrac{\cfrac{\cfrac{\cfrac{A\vdash A\quad \text{Axiom}}{*A\vdash *A}\ \text{DE}}{*A\vdash \neg A}\ \vdash\neg}{\vdash \neg A,A}\ \text{DE}}{\vdash \neg A+A}\ \vdash+$$

In a consecution calculus in a metalanguage with two punctuation marks there will be two groups of structural rules: structural rules for ',' and display equivalences for ‘*’.

To different definitions of display equivalence have been proposed: the display schemes P and A[4]. Let us review first Belnap’s definition, the P scheme.

Display equivalence is the smallest equivalence relation that makes equivalent all consecutions listed on the same column below.

$X,Y\vdash Z$	$X\vdash Y,Z$	$X\vdash Y$
$X\vdash *Y,Z$	$X,*Y\vdash Z$	$*Y\vdash *X$
	$X\vdash Z,Y$	$**X\vdash Y$

The second column makes equivalents $X\vdash Y,Z$ and $X\vdash Z,Y$, thus introducing a property of ‘,’ independent from ‘*’ — right-permutation. This is a disadvantage. We need display equivalences to separate properties of the inferential context from properties of the logical constants; but we have see that Belnap’s postulates are suitable to that end only when permutation on the consequent is assumed. In fact the P and A schemes are equivalent given right and left permutation.

The A scheme, coming from Wansing, is free from this limitation. Wansing definition of display equivalence runs:

Display equivalence is the smallest equivalence relation that makes equivalent all consecutions listed on the same column below.

$$\begin{array}{lll} X,Y \vdash Z & X \vdash Y,Z & X \vdash Y \\ X \vdash Z,{*Y} & X,{*Z} \vdash Y & {*Y} \vdash {*X} \\ Y \vdash {*X},Z & {*Y},X \vdash Z & X \vdash {**Y} \end{array}$$

These remarks answer partially a question by Belnap [(1996) p.91]: “What are good questions to ask about the relations between the P and the A schemes? Surely the two schemes don’t just sit there. Are they good for different things, or can either do the work of the other, or…?.”

To put NWPLC in display clothes, besides using the A scheme, operational rules have to be adapted to the new consecutions made from structures instead of finite sequences. As for structural rules, to deal with structures requires additional rules of association and commutation.

NWPL^{+}

Axioms

A1 $\qquad p \vdash p,$ $\qquad$ For any sentence letter p

Structural rules

$$\mathrm{A}\vdash\ \frac{(X,Y),Z \vdash W}{X,(Y,Z) \vdash W} \qquad \vdash\mathrm{A}\ \frac{X \vdash (Y,Z),W}{X \vdash Y,(Z,W)} \qquad \mathrm{C}\vdash\ \frac{X,(Y,Z) \vdash W}{(X,Y),Z \vdash W} \qquad \vdash\mathrm{C}\ \frac{X \vdash Y,(Z,W)}{X \vdash (Y,Z),W}$$

$$\mathrm{P}\vdash\ \frac{X,Y \vdash Z}{Y,X \vdash Z} \qquad \vdash\mathrm{P}\ \frac{X \vdash Y,Z}{X \vdash Z,Y} \qquad \mathrm{W}\vdash\ \frac{X,X \vdash Y}{X \vdash Y} \qquad \vdash\mathrm{W}\ \frac{X \vdash Y,Y}{X \vdash Y}$$

$$\mathrm{D}\vdash\ \frac{X \vdash Y}{X,X \vdash Y} \qquad \vdash\mathrm{D}\ \frac{X \vdash Y}{X \vdash Y,Y} \qquad \mathrm{Comb}\ \frac{X \vdash Y}{X,Z \vdash Z,Y}$$

Display equivalences

$$\begin{array}{lll} X,Y \vdash Z & X \vdash Y,Z & X \vdash Y \\ X \vdash Z,{*Y} & X,{*Z} \vdash Y & {*Y} \vdash {*X} \\ Y \vdash {*X},Z & {*Y},X \vdash Z & X \vdash {**Y} \end{array}$$

Operational rules

$$\otimes\vdash\ \frac{A,B \vdash Y}{A\otimes B \vdash Y} \qquad \vdash\otimes\ \frac{X \vdash A \quad Y \vdash B}{X,Y \vdash A\otimes B}$$

$$\&\vdash\ \frac{A \vdash Y}{A\&B \vdash Y} \qquad \frac{B \vdash Y}{A\&B \vdash Y} \qquad \vdash\&\ \frac{X \vdash A \quad X \vdash B}{X \vdash A\&B}$$

$$+\vdash\ \frac{A \vdash X \quad B \vdash Y}{A+B \vdash X,Y} \qquad \vdash+\ \frac{X \vdash A,B}{X \vdash A+B}$$

$$\oplus\vdash\ \frac{A \vdash X \quad B \vdash X}{A\oplus B \vdash X} \qquad \vdash\oplus\ \frac{X \vdash A}{X \vdash A\oplus B} \qquad \frac{X \vdash B}{X \vdash A\oplus B}$$

$$\neg\vdash\ \frac{X \vdash {*A}}{X \vdash \neg A} \qquad \vdash\neg\ \frac{{*A} \vdash Y}{\neg A \vdash Y}$$

VII. CUT ADMISSIBILITY

An advantage of the display property is that it enables a very general proof of cut admissibility relying only in fundamental properties of structural and operational rules. The multiplicity of logical systems forces us to be careful on the several versions of the cut rule, equivalent for the case of classical logic but not for other systems. Here we consider the elimination rule in Belnap (1982),

$$\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}$$

called *unitary cut* in Gabbay (1994).

We define first the notions of parameter and congruence. A rule can be understood as a family of inferences, in the same way as a formula schema can be seen as a set of formulae. Thus the inference

$$\frac{p,q \vdash p,q}{p \otimes q \vdash p,q}$$

belongs to the family of inferences of the rule $\otimes\vdash$. The parameters of an inference are those constituents (formulae or structures) occurring as parts of occurrences of structures assigned to structural variables in the statement of the corresponding rule. Hence in the previous inference the parameters are the occurrences of p and q in the consequents of the premise and the conclusion. Likewise it is said that constituents occupying similar positions in occurrences of structures assigned to the same structure-variable are congruent in this inference. As congruence is obviously an equivalence relation, the congruence class of a constituent of an inference is the set of all constituents congruent to it.

A substructure of a structure is negative if it is inside an odd number of *'s; otherwise it is positive. The presence of '*' implies a revised definition of antecedent and consequent parts. As before, in a consecution $X \vdash Y$, X is the antecedent and Y is the consequent. An antecedent part of a consecution is a positive substructure of its antecedent or a negative substructure of its consequent. A consequent part of a consecution is a positive substructure of its consequent or a negative substructure of its antecedent.

It can be proved that cut is admissible in any logical system satisfying the following eight conditions. The proof can be found in Belnap (1982), §.4, and Restall (2000), §§. 6.2 and 6.3. It seems that the proof comes back to Curry (1963).

1. Every formula which is a constituent of some premise of an inference is a subformula of some formula in the conclusion of that inference.
2. Congruent parameters are occurrences of the same structure.
3. Each parameter is congruent to at most one constituent in the conclusion of the inference.
4. Congruent parameters are either all antecedent or all consequent parts of their respective consecutions.
5. If a formula is nonparametric in the conclusion of an inference, it is either the entire antecedent or the entire consequent of that conclusion.
6. Every rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are consequent parts.
7. Let Inf be an inference falling under a rule R and let A be a parametric antecedent part of a consecution of Inf. Let also be $X \vdash A$ be the conclusion of an inference such that A is not a parameter. If Inf' results from putting the structure X for all constituents of Inf in the congruence class of the antecedent part Y, Inf' also falls under R.
8. Let Inf and Inf' be two inferences.with conclusions $X \vdash A$ and $A \vdash Y$ respectively, with A nonparametric in both inferences. Then either $X \vdash Y$ is identical to one of $X \vdash A$ or $A \vdash Y$ or can be obtained from the premises of Inf and Inf' using the rules of the system together with the elimination rule with A as principal or cut formula.

Systems NWPL and NWPLC satisfy these conditions save, due to duplication rules, the third. In the case of NWPLC duplication rules are eliminable; given combination it can be shown:

> If there is a derivation π of $X \vdash Y$ in NWPLC and a derivation θ of $Z \vdash W$ in NWPLC then there is a derivation ξ of $X,Z \vdash Y,W$. Moreover, if π and θ are duplication free so is ξ.

Hence given a derivation in NWPLC of $X \vdash Y$ free from duplication, there is a duplication free derivation of $X,X \vdash Y,Y$. Using then associativity, commutativity, permutation and contraction one gets a derivation of $X \vdash Y$.

Combination is not a rule of NWPL, unlike associativity, commutativity, permutation and contraction. Using the later, the duplication rules can be replaced by the rule of restricted combination

$$\frac{X \vdash Y \qquad X \vdash Y}{X,X \vdash Y,Y}$$

which is safe with respect the third condition.

VIII. PARACONSISTENCY REVISITED

NWPLC is a system of paraconsistent logic so far as, for instance, it is not the case that $p,\neg p \vdash_{NWPLC} q$. However this is perhaps a necessary but not sufficient condition. Johansson's minimal logic, as points out Da Costa, does not permit the derivation of every formula from a contradiction and yet could not be considered as genuinely paraconsistent for it allows the derivation of the negation of every formula from a contradiction. The idea is that a truly paraconsistent logic should not justify the derivation of a formula from a contradiction appealing to the untenability of contradictions.

A proof-theoretical account paraconsistent logics independent from the notion of a negation is implicit in the preceding discussion. The essence of paraconsistent logic is that right weakening is not an admissible rule. As it stands this definition counts minimal logic as a paraconsistent logic. To refine this definition in order to exclude such cases, we have first to make precise the concept of "version of a rule". Once this is done, we can define a paraconsistent logic to be one admitting no version of the right weakening rule.

Let us extend the notion of a substitution from formulae to structures as defined on p. 14. By a substitution for a language it is meant a homomorfism s from the set of its formulae into the set of its formulae; this notion is then extended to apply to structures adding the following two clauses:

$s(*(X)) = *s(X);$
$s(X,Y) = (s(X),s(Y)).$

Notice that whenever A is a formula, s(A) is also a formula. *A version of the right-weakening rule* is a rule R having the form:

$$\frac{X \vdash Y}{X \vdash Z,Y}$$

such that for any (structure) substitution s,

$$\frac{X \vdash Y}{X \vdash s(Z),Y}$$

is a case of R.

Although this definition is enough for the present purpose, it lacks the desirable generality for it "fixes" the introduced structure Z in the left-most position of the consequent.

Right duplication is not a version of right weakening since it does not allow for the introduction in the consequent of any structure *with the same form* as structure Y. Nor is the NWPLC derivable rule

$$\frac{X \vdash Y}{X \vdash X,Y}$$

(starting from the hypothesis $X \vdash Y$ combination and left contraction yield $X \vdash X,Y$) for the same reason. Therefore NWPLC counts as a genuine paraconsistent logic according to our standard. On the contrary, Johansson's minimal logic will not qualify as paraconsistent; the rule

$$\frac{X \vdash}{X \vdash \neg A}$$

is admissible in minimal logic and as $s(\neg A) = \neg s(A)$, it is indeed a version of the right weakening rule.

IX. Properties of the Tensor Fragment

The tensor fragment $\{\otimes,+,\neg\}$ of NWPLC has a remarkable property, what I will call "downwards Scotianism". A logic or a logical system S is downwards Scotian if and only if for every formula A and every subformula B of A both $A,\neg A \vdash_S B$ and $A,\neg A \vdash_S \neg B$. In a paraconsistent downwards Scotian logic contradictions are located –a recurrent idea in the paraconsistent literature- in a precise sense. Although the theory T generated by $\{A,\neg A\}$, A being a formula of some language $\mathcal{L}$, is not trivial for $\mathcal{L}$, the restriction of T to the vocabulary of A is a trivial theory of this fragment of $\mathcal{L}$.

THEOREM. For any formula A of $\{\otimes,+,\neg\}$ and any subformula B of A, $A,\neg A \vdash_{NWPLC} B$ and $A,\neg A \vdash_{NWPLC} \neg B$.

Proof. First we have to distinguish different occurrences of the same subformula in a given formula. This can be achieved using some kind of indexing for occurrences of sentence letters in a formula. Thus, for instance, rewriting the formula $(p\otimes q) + \neg(p\otimes q)$ as $(p_1\otimes q_1) + \neg(p_2\otimes q_2)$. Once this has been done, the depth of an occurrence of a subformula B of A in this formula, in symbols $\mathcal{p}(B,A)$, is defined.

1. $\mathcal{p}(B,A) = 0$ if $B = A$.
2. $\mathcal{p}(A,A\otimes B) = \mathcal{p}(B,A\otimes B) = \mathcal{p}(A,A+B) = \mathcal{p}(B,A+B) = \mathcal{p}(A,\neg A) = 1$.
3. If $\mathcal{p}(A,B) = i$ and $\mathcal{p}(B,C) = j$, $\mathcal{p}(A,C) = i+j$, where A is an occurrence in B and B an occurrence in C.

EXAMPLES:

- $\wp(p_1, (p_1\otimes q_1) + \neg(p_2\otimes q_2)) = \wp(p_1, p_1\otimes q_1) + \wp(p_1\otimes q_1, (p_1\otimes q_1) + \neg(p_2\otimes q_2)) = 1 + 1 = 2.$
- $\wp(p_2, (p_1\otimes q_1) + \neg(p_2\otimes q_2)) = \wp(p_2, \neg(p_2\otimes q_2)) + \wp(\neg(p_2\otimes q_2), (p_1\otimes q_1) + \neg(p_2\otimes q_2)) = \wp(p_2, p_2\otimes q_2) + \wp(p_2\otimes q_2, \neg(p_2\otimes q_2)) + 1 = 1 + 1 + 1 = 3.$

The proof properly speaking begins establishing the theorem for $\wp(B,A) = 0$.

$A \vdash A$	
$A \vdash A,A$	$\vdash D$
$A,{*A} \vdash A$	DE
${*A} \vdash {*A},A$	DE
$\neg A \vdash {*A},A$	$\neg\vdash$
$A,\neg A \vdash A$	DE

$A \vdash A$	
$A,A \vdash A$	$D\vdash$
$A \vdash A,{*A}$	DE
${*A},A \vdash {*A}$	DE
${*A},A \vdash \neg A$	$\vdash\neg$
${*A} \vdash \neg A,{*A}$	DE
$\neg A \vdash \neg A,{*A}$	$\neg\vdash$
$\neg A,A \vdash \neg A$	DE

Then we do the same for $\wp(B,A) = 1$.

$A \vdash A$	$B \vdash B$
$A,B \vdash A\otimes B$	$\vdash\otimes$
$(A,B),A \vdash A,A\otimes B$	Combination
$A,(A,B) \vdash A,A\otimes B$	$P\vdash$
$(A,A),B \vdash A,A\otimes B$	$C\vdash$
$(A,A) \vdash (A,A\otimes B),{*B}$	DE
$A \vdash (A,A\otimes B),{*B}$	$W\vdash$
$A,B \vdash A,A\otimes B$	PE
$A\otimes B \vdash A,A\otimes B$	$\otimes\vdash$
$A\otimes B,{*A\otimes B} \vdash A$	DE
${*A\otimes B} \vdash {*A\otimes B},A$	DE
$\neg A\otimes B \vdash {*A\otimes B},A$	$\neg\vdash$
$A\otimes B,\neg A\otimes B \vdash A$	DE

$A \vdash A$	
$A,B \vdash A,B$	Combination
$A,B \vdash A\otimes B$	$\vdash\otimes$
$A \vdash A\otimes B,{*B}$	DE
$A,A \vdash A\otimes B,{*B}$	$D\vdash$
$(A,A),B \vdash A\otimes B$	DE
$A,(A,B) \vdash A\otimes B$	$A\vdash$
$A,B \vdash {*A},A\otimes B$	DE
$A\otimes B \vdash {*A},A\otimes B$	$\otimes\vdash$
$A\otimes B,{*(A\otimes B)} \vdash {*A}$	DE
$A\otimes B,{*(A\otimes B)} \vdash \neg A$	$\vdash\neg$
${*(A\otimes B)} \vdash {*(A\otimes B)},\neg A$	DE
$\neg(A\otimes B) \vdash {*(A\otimes B)},\neg A$	$\neg\vdash$
$A\otimes B,\neg(A\otimes B) \vdash \neg A$	

A├ A	B├B	A├ A	
A+B├ A,B	+├	A,A├ A	D├
A+B,*B├ A	DE	B├ B	
A+B,*B├ A,A	├D	A├ *A,A	DE
A,(A+B,*B)├ A	DE	A+B├ (*A,A),B	+├
(*A,A+B),*B├ A	C├	A+B├ *A,(A,B)	├A
*A,A+B├ A,B	DE	A,A+B├ A,B	DE
		A,A+B├ A+B	├+
*A,A+B├ A+B	├+	A+B├ *A,A+B	DE
A+B├ A,A+B	DE	A+B,*(A+B)├ *A	DE
A+B,*(A+B)├ A	DE	A+B,*(A+B)├ ¬A	├¬
*(A+B)├ *(A+B),A	DE	*(A+B)├ *(A+B),¬A	DE
¬(A+B)├ *(A+B),A	¬├	¬(A+B)├ *(A+B),¬A	¬├
A+B,¬(A+B)├ A	DE	A+B,¬(A+B)├ ¬A	DE

A├ A		A├ A	
*A├ *A	DE	*A├ *A	DE
¬A├ *A	¬├	¬A├ *A	¬├
¬A├ ¬A	├¬	¬A├ ¬A	├¬
¬A├ ¬A,¬A	├D	¬A,¬A├ ¬A	D├
*¬A,¬A├ ¬A	DE	¬A├ *¬A,¬A	DE
*¬A├ ¬A, *¬A	DE	¬A,*¬A├ *¬A	DE
¬¬A├ ¬A,*¬A	¬├	¬A,*¬A├ ¬¬A	├¬
¬¬A,¬A├ ¬A	DE	¬¬A├ *¬A,¬¬A	¬├
		¬A,¬¬A├ ¬¬A	DE

Finally let us consider the case p(B,A) > 1, i.e., when B is not an immediate subformula of A. The key is the admissibility of the cut rule for NWPLC.

Using induction, if p(B,A)=n+1, there is some C such that p(B,C)+ p(C,A) = n+1 and hence p(B,C),p(C,A) < n+1. In this case, by induction hypothesis, consecutions A ;¬A├ C, A ;¬A├ ¬C, C ;¬C├ B y C ;¬C├ ¬B are derivable.

	C,¬C├ B	hypothesis
A,¬A├ C	C├ B,*¬C	hypothesis & DE
A,¬A├ B,*¬C		Cut
(A,¬A),¬C├ B		DE
A ;¬A├ ¬C	¬C├ *(A,¬A),B	hypothesis & DE

$$\dfrac{\dfrac{A\,;\neg A \vdash *(A,\neg A),B}{(A,\neg A),(A,\neg A) \vdash B}}{(A,\neg A) \vdash B} \qquad \begin{matrix}\text{Cut}\\ \text{DE}\\ \text{C}\vdash\end{matrix}$$

The derivation of $A,\neg A \vdash \neg B$ is similar.

Departamento de Lingüística, Lenguas Modernas, Lógica y Filosofía de la Ciencia y Teoría de la Literatura y Literatura Comparada
Universidad Autónoma de Madrid
Cantoblanco, E-28042 Madrid
E-mail: hubert.marraud@uam.es

NOTES

* This work was supported by the Spanish Ministerio de Ciencia y Tecnología, research project BFF2003-08998-CD03-03 "La geometría de la demostración".

[1] KT is obtained adding to a suitable axiomatization of classical propositional logic the axiom schemes **K K** $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and **T** $A \supset \Diamond A$, and the necessitation rule $\vdash A \to \vdash \Box A$.

[2] Bryson Brown claims that preservationist paraconsistent logics are not subject to this objection.

[3] These are the unary connectives ! (of course) and ? (why not).

[4] See Belnap (1996). The P scheme comes from Belnap (1982) while the A scheme is due to Wansing (1994).

REFERENCES

ARRUDA, A. (1977), 'A Survey of Paraconsistent Logic' in .Arruda, A. I, Chuaqui, R., and Da Costa, N.C.A. (eds.) *Mathematical Logic in Latin America*, Amsterdam, North Holland, 1980, pp.1-41.

BELNAP, N.D., Jr. (1982), 'Display Logic', in *Journal of Philosophical Logic,* 11, pp.375-417.

— (1996), 'The Display Problem', in Wansing, H. (ed.), *Proof Theory of Modal Logic*, Dordrecht, Kluwer, 1996, pp.79-92.

BROWN, B. (1999), 'Yes, Virginia, There Really Are Paraconsistent Logics', in *Journal of Philosophical Logic,*vol. 28, pp.489-500.

CURRY, H.B. (1963), *Foundations of Mathematical Logic*, New York, McGraw-Hill.

DOŠEN, K. (1989), 'Logical constants as punctuation marks', in *Notre Dame Journal of Formal Logic*, vol. 30, pp.362-381.

GABBAY, D.M. (1994), 'What is a Logical System?' in .Gabbay, D. M. (ed.)., *What is a Logical System?*, Oxford, Clarendon Press, pp.179-216.

GIRARD, J.-Y. (1987), 'Linear Logic', in *Theoretical Computer Science*, vol. 50, pp. 1-102.

GIRARD, J.-Y., LAFONT, Y. & TAYLOR, P. (1989), *Proofs and Types*, Cambridge, Cambridge University Press.

GIRARD, J.-Y., LAFONT, Y. & REGNIER, L., (eds.) (1995), *Advances in Linear Logic*, Cambridge, Cambridge University Press.

PRIEST, G., ROUTLEY,R. & NORMAN, J., (eds.) (1989): *Paraconsistent Logic: Essays on the Inconsistent*.. München, Philosophia Verlag.

QUINE, W. V. O. (1970), *Philosophy of Logic*. Englewood Cliffs, N.J., Prentice-Hall.

RESTALL, G. (1999), *An Introduction to Substructural Logics*, London and New York, Routledge.

SAMBIN, G., BATTILOTTI, G. & FAGGIAN, C. (2000), 'Basic Logic: Reflection, Symmetry, Visibility', in *Journal of Symbolic Logic* vol. 65, pp. 979-1013.

SCHROEDER-HEISTER, P. & DOŠEN, K., (eds.) (1993), *Substructural Logics*, Oxford, Oxford University Press.

SLATER, B. H. (1995), 'Paraconsistent Logics?', in *Journal of Philosophical Logic* vol. 24, pp.451-454.

TARSKI, A. (1930), 'On some fundamental concepts of metamathematics' in *Logic, Semantics and Metamathematics, Papers from 1923 to 1938*, translated by J. H. Woodger, Oxford, Clarendon Press, 1956, pp. 30-37.

WANSING, H. (1994), 'Sequent calculi for normal modal propositional logics', in *Journal of Logic and Computation* vol. 4, pp. 124-142.

WÓJCICKI, R. (1988), *Theory of Logical Calculi*, Dordrecht, Kluwer.