ABOUT MANIFOLDS AND DETERMINACY IN GENERAL EQUILIBRIUM THEORY

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Abstract

In this paper we prove that the social equilibrium set, of an exchange economy, with consumption space as a subset of a Banach space is a Banach manifold, and this characterization does not depend on the number of commodities. In the way to obtain this characterization we will show that the set of social weights of equilibrium, associated with a given distribution of the initial endowments, is finite.

Resumen

En este artículo se demuestra que el conjunto de equilibrio social de una economía de intercambio, con espacio de consumo contenido en un espacio de Banach, es una variedad en dicho Banach, cuya caracterización no depende del número de bienes. Con el fin de obtener esta caracterización, se demuestra que el conjunto de ponderaciones sociales de equilibrio asociadas a una distribución dada de dotaciones iniciales, es finito.

1. Introduction

In principle the structure of a set in a mathematical model is not mastered until it can be described locally, trough parameters or coordinates. This description is resumed by means of the concept of manifold. If a set is a manifold, then it is possible to use to understand its main characteristics, the differential analysis. Intuitively, manifolds in two or three dimensional spaces, are smooth curves or surfaces. In general. if a set M is a manifold, then locally, *i.e.* in a neighborhood of each point $p \in M$, but in general not globally, looks like a Banach space.

In this work we show that the social equilibrium set, *i.e.* the set of social weights and the set of initial endowments such that the excess utility function

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vanishes, is a Banach manifold. At the same time that we show this fact, we show that the equilibrium set is generically determined, this means that given the utility functions of the agents of the economy except for very particular combinations of parameter the number (the initial endowments) of distinct equilibria is finite *i.e.* the local uniqueness of the equilibrium social weights follow.

The determinacy of the equilibrium set is one of the main features of the classical models of General Equilibrium. For smooth economies with finitely many agents an commodities it is a well know fact, see for instance Debreu, G. (1970). The approach followed by G. Debreu, can't be follows in infinite dimensional models, in particular because the excess demand function is not necessarily a smooth function. Nevertheless it is possible to follow a similar approach considering the excess utility function in the place of the excess demand function. In this case we deal with the social equilibrium set more than with the walrasian equilibrium. This means that we work with the social weights more than with the prices, nevertheless, for a given economy there exists a biunivocal correspondence between the social equilibrium and the set of equilibrium prices.

2. The model

We consider an economy where each agent's consumption set is a subset of a Banach space. Agents will be indexed by $i \in I = \{1, 2, ...n\}$; and X_+ will denote the positive cone of the Banach space X. We do not assume separability in the utility functions $u_i : X_+ \to R$. Utility functions are in the $C^2(X, R)$ space, *i.e.* the set of the functions with continuous second F-derivatives, and we suppose that they are increasing functions it is to say that, each agent prefer more than less, formally, each first order F-derivative is positive. Where F-derivative define $f'(x)$ in the usual way of the linearization $f(x + h) = f(x) + f'(x)h + o(\|h\|)$. In order to assure the uniqueness of equilibrium allocation we will assume strictly quasi-concave utility functions. In addition, we suppose that for all $x \in X$ the inverse operator $(u_i'')^{-1}$ of u_i at x, exists. Here $u_i''(x)$ is identified with the quadratic form $(h, k) \to u''(x)hk$. In this work $C^k(X, Y)$ denote the space of $k - times$ continuously F-differentiable operators from X into Y, and $L(X, Y)$ denote the space of linear and continuous operators from X into Y. By $C^\infty(X, Y)$ we denote the set of functions belonging to $C^k(X, Y)$ for all integer k.

The consumption set of each agent is the same one, and it will be symbolized by X. The cartesian product of these n consumption sets is represented by Ω. So, a bundle set for the i-agent will be symbolized by $x_i \in X$ and an allocation will be denoted by $x = (x_1, x_2, ..., x_n) \in \Omega$. The i-agent endowments will be symbolized by w_i, and $w = (w_1, w_2, ..., w_n)$, symbolize the initial allocation. The total mounts of available goods will be denoted by $W = \sum_{i=1}^{n} w_i$. All of them contingent goods in time or state of the world.

With the purpose to obtain strictly positive equilibria, we will assume that utilities satisfy at least one of the following two, widely used assumptions in economics, conditions:

(i) (Inada condition) $\lim u'_j(x) = \infty$ if $x \to \partial(X_+)$, for each $j = 1, 2, ..., l$ and for each utility function, by $\partial(X_+)$, we denote the frontier of the positive cone. It assumes that marginal utility is infinite for consumption at zero.

(ii) All strictly positive bundle set is preferable to all bundle set with at least one zero component in one state of the world.

An economy will be represented by

$$\varepsilon = \{u_i, w_i, I\}.$$

As examples of economies with the properties above mentioned, consider those where the consumption set is $X_+ = C_{++}(M\ R^n)$ and utility functions are $u_i(x) = \int_M U_i(x(t), t)dt$, see [Chichilnisky, G. and Y. Zhou (1988)], and [Aliprantis, C.D; D.J. Brown and O. Burkinshaw (1990)].

It is well known that the demand function is a good tool to deal with the equilibrium manifold in economies in which consumption spaces are subset of Hilbert spaces, in particular R^l [Mas-Colell, A. (1985)]. But unfortunately if the consumption spaces are subsets of infinite dimensional spaces (not a Hilbert space), the demand function may not be a differentiable function [Araujo, A. (1987)], or it is not well definite because the price space is very large or the positive cone where prices are definite has empty interior. Despite in many of these cases it is possible, to characterize the equilibria set using the function of excess of utility , see for instance Accinelli, E. (1996), and it is possible using this function to introduce in infinite dimensional models differentiable techniques with wide generality. Then it is possible to solve problems defined in spaces of infinite dimension by means of techniques of differential calculus own of the finite case. And in this way to generalize the result obtained by Chichilnisky, G. and Zhou, Y. for smooth infinite dimensional economies to the case with no separable utilities.

In this work, following the Negishi approach, we will characterize the equilibrium set of the economy, as the set of zeroes of the excess utility function $e : \Delta \times \Omega \rightarrow R^{n-1}$. So, the equilibrium set will be denoted by

$$\varepsilon_q = \{(\lambda, w) \in \Delta \times \Omega : e((\lambda, w) = 0\}$$

Where Δ symbolize the social weight set,

$$\Delta = \left\{ \lambda \in R^n : \sum_{i=1}^n \lambda_i = 1,\ 0 \leq \lambda_i \leq 1;\ \forall i \right\},$$

and $w = (w_1, w_2, ..., w_n)$ are the initial endowments. In the considered hypothesis, the fact that each agent has non-null initial endowments, implies that the result of a process of maximization of the utility functions will be a strictly positive bundle set. Then each relative weight cannot be zero. So, without loss of generality, we can consider only cases where $\lambda \in \Delta_+ = int[\Delta]$.

In order to prove that ε_q restricted to $w \in \Omega_0$, where Ω_0 is an open and dense subset included in Ω is a Banach manifold, we will assume that the positive cone Ω^+ of the consumption space has non-empty interior. Typically examples of such spaces are $L^\infty(M, R^n)$ where M is any compact manifold, with the supremun norm, see Chichilnisky, G. and Zhou, Y. So, we show that in this cases, the set of regular economies is large, and its complement is a rare set. This is not a consequence of the Debreu theorems, here it follows from an alternative approach with particular interest in infinite dimensional cases. If the

only interest is to show that the set of the social equilibria, associated to each distribution of the initial endowments is finite, then we don't need to assume the non emptiness of the positive cone. In this case it is enough to allow for the possibility that *w* is not positive.

3. SOME OF NOTATION AND MATHEMATICAL FACTS

In this section we recalling some basic mathematical definitions that will be used later. Our main reference for considerations on Functional Analysis is Zeidler, E. (1993).

Definition 1 *Let f* : *Dom*(*f*) ⊆ *X* → *Y be a mapping between two Banach spaces,* (*B-spaces*) *X and Y over K, here Dom*(*f*) *is the domain of f, and let f* ′(*x*) *be the Fréchet derivative* (*F-derivative*) *at the point x for the map f*

1. $f' : D(f') \subseteq X \to L(X, Y)$ i.e. $f'(x)$ is a continuous linear map from X to Y.

2. f is called a **submersion** at the point x if and only if f is a C^1–mapping on a neighborhood of x, if $f'(x) : X \to Y$ is surjective and if the null space

$$Ker(f'(x)) = \{x \in X : f'(x) = 0\},$$

   splits X. The null space $Y_1 = Ker(f'(x))$ splits X means that $X = Y_1 \oplus Y_2$ (topological direct sum). f is called submersion on the subset $M \subseteq X$ iff f is a submersion at each $x \in M$.
   We will denote the image set of a linear operator $T : X \to Y$ by

$$R(T) = \{y \in Y : there\ exists\ x \in X : y = T(x)\},$$

   the dimension of $R(T)$ will be denoted by $rankT$, and the codimension of $(R(f))$ will be symbolized as $corankT = dim[X / ker(T)]$, where $X / ker(T)$ is the factor space.

3. The point $x \in X$ is called a **regular point** of f iff f is a submersion at x. Otherwise x is called **singular point**.

4. The point $y \in Y$ is called a **regular value** of f if and only if f^{-1} is empty or consists solely of regular points. Otherwise y is called **singular value**.

5. Let X be a Banach space, it follows that $f : U(x_0) \subset X \to$ R has a singular point at x_0 if an only if $f'(x_0) = 0$. Such point will be non-degenerate if and only if the bilinear form $(h, k) \to f''(x_0)hk$ is non-degenerate.

Recall that a linear map $T : X \to Y$ is called a **Fredholm operator** if and only if is continuous and both numbers the dimension of the $ker(T)$, $dim(Ker(T))$ and the codimension of the rank of f, $codim(R(T))$ are finite. The index of f is defined by: $ind(T) = dim(Ker(T)) - codim(R(T))$.

Definition 2 *A C^k, $0 \le k$ Banach manifold M, is a topological space with the following additional properties:*

1. Locally coordinate system. *For every point $p \in M$ there is an open neighborhood $U(p)$, a Banach space B_p and a mapping Φ_p which maps $U(p)$ homeomorphically onto an open set $V(p) \in B_p$.*

   If $q \in U(p)$ then $w = \Phi_p(q)$ is called de coordinate of q for Φ_p.

2. Coordinate transformation. *If $r \in U(p) \cap U(q)$ then r has a local coordinates $x = \Phi_p(r)$ and $y = \Phi_q(r)$. The change of these local coordinates is given by $y = \Phi_q[\Phi_p^{-1}](x)$.*

4. THE NEGISHI APPROACH

The Negishi approach start considering a social welfare function given by: $W\lambda : \Omega_+ \to R$ defined as:

$$(1) \qquad W_\lambda(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i).$$

where u_i is the utility function of the agent indexed by i, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in int[\Delta]$ (each λ_i represents the social weight of the agent in the market), and Ω_+ is the positive cone in the consumption space Ω.

As it is well know if $x^* \in \Omega$ solves the maximization problem of $W_{\lambda*}(x)$ for a given λ^*, subject to be a feasible allocation *i.e.*,

$$x^* \in \mathcal{F} = \left\{ x \in \Omega_+ : \sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} w_i \right\}$$

then x^* is a Pareto optimal allocation. Reciprocally it can be proved that if a feasible allocation x^*, is Pareto optimal, then there exists any $\lambda^* \in \Delta$ such that x^*, maximize $W_{\lambda*}$, see [Accinelli, E. (1996)]. There exists some Pareto optimal allocation where $x_i^* = 0$ for some $i \in \{1, 2, \ldots, n\}$ if each agent has positive no null endowments, these cases are possible if and only if the agents indexed in this subset be out of the market, *i.e.*, if and only if $\lambda_i = 0$. Then we can restrict ourselves, without loss of generality, to consider only cases where $\lambda \in \Delta_+$.

In this way characterized the set of Pareto optimal allocations, our next step is to choose the elements x^* in the Pareto optimal set such that can be supported by a price p and satisfying $px^* = pw_i$ for all $i = 1, 2, \ldots, n$ *i.e.*, an equilibrium allocation.

Suppose that the aggregate endowment of the economy is, $\sum_{i=1}^{n} w_i = W$. We will use the following notation:

For any $\lambda \in int [\Delta] = \{\lambda \in \Delta : \lambda_i > 0 \ \forall i \in I\}$,

$$(2) \qquad x(\lambda, W) = argmax \left\{ \sum_{i=1}^{n} \lambda_i u_i(x), \ s.t \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i \right\}$$

and let $e : \Delta \times \Omega \to R^n$ be the excess utility function, which coordinates are given by:

$$e_i(\lambda, W) = u'_i(x_i(\lambda, W))(x_i(\lambda, W)) - w_i).$$

Here $u'_i(x_i(\lambda, W)) : X \to R$ is the F-differential of the utility $u_i(x_i(\lambda, W))$.

Definition 3 *For fixed utility functions, for each $w \in \Omega_\in$, where*

$$\Omega_\in = \{w \in \Omega_+ : \in \leq w_i, i = 1, 2, \ldots,n\}, \text{for a given} \in$$

we define the set

$$\in q(w) = \{\lambda \in \Delta_+ : e_w(\lambda) = 0\},$$

it will be called the set of the **Equilibrium Social Weights**.
    In Accinelli, E. (1996) is show that the equilibrium social weights is a non-empty set.

Theorem 1 *Let $\lambda \in \in q(w)$, and let $x^*(\lambda)$ be a feasible allocation, solution of the maximization problem of W_λ and let $\gamma(\lambda)$ be the corresponding vector of Lagrange multipliers. Then, the pair $(x^*(\lambda), \gamma(\lambda))$ is a walrasian equilibrium and reciprocally, if (p, x) is a walrasian equilibrium then, there exists $\bar{\lambda} \in \in q$ such that x maximize $W_{\bar{\lambda}}$ restricted to the feasible allocations set, and p will be the corresponding vector of Lagrange multipliers i.e., $p = \gamma(\bar{\lambda})$.*
    The proof can be see in [Accinelli, E. (1996)].

5. THE EQUILIBRIUM SET AS A BANACH MANIFOLD

The first order conditions for (2) are:

(3)
$$\lambda_i u'_i(x_i(\lambda, W)) = \lambda_h u'_h(x_h(\lambda, W)), \ \forall h \neq i$$

$$\sum_{i=1}^n x_i(\lambda, w) = W,$$

where $W = \sum_{i=1}^n w_i$. It follows that for each *i,* the consumption of the i-agent, given by the function $x_i : \Delta \times \Omega \to X$ is, for all $\lambda \in int[\Delta]$ and $w \in \Omega$, a F-differentiable function.
    The following are well know properties of the excess utility function:

(1) $\lambda e(\lambda, w) = 0$.

(2) $e(\alpha\lambda, w) = e(\lambda, w), \ \forall \alpha > 0$.

See for instance Accinelli, E. (1996).
    From item (1) it follows that the rank of the jacobian matrix $J_\lambda e(\cdot, w)$ of the excess utility function $e(\cdot, w) : \Delta \to R^n$ is at most equal to $n - 1$. And as from

item (2) we know that if $e_i(\lambda, w) = 0 \ \forall \, i = 1, 2, \ldots, n-1$, then $e_n(\lambda, w) = 0$, we will consider the restricted function $\bar{e} : \Delta \times \Omega \rightarrow R^{n-1}$ obtained from the excess utility function removing one of its coordinates.

The following theorem holds:

Theorem 2 *If the positive cone of the consumption space, has a non-empty interior then, there exists an open and dense subset $\Omega_0 \subseteq \Omega \in$ such that*

$$\varepsilon q \, / \, \Omega_0 = \{(\lambda, w) \in \ int[\Delta] \times \Omega_0 : e(\lambda, w) = 0\}$$

is a Banach manifold.

Proof: To prove this theorem, we will prove the following assertions:

(i) There exist a residual set $\Omega_0 \subseteq \Omega$ such that, the mapping $\bar{e} : int[\Delta] \times \Omega_0 \rightarrow R^{n-1}$ is C^1, and zero is a regular value of e *i.e.* for all $(\lambda, w) \in \ int[\Delta] \times \Omega_0$, such that $e(\lambda, w) = 0$ the mapping $\bar{e}$ is a submersion.

(ii) For each parameter $w \in \Omega_0$, the mapping $\bar{e}(\cdot, w) : int[\Delta] \rightarrow R^{n-1}$ is Fredholm of index zero.

- Now, from Zeidler, E. (1993) section (4.19), the theorem follows.

   As a corollary of this theorem, it follows that: *For each $w \in \Omega_0$ the equation $e(\lambda, w) = 0$, $\lambda \in \ int[\Delta]$ has at most finitely many solutions λ of $e_w(\lambda) = 0$.*

(iii) *This corollary follows from the fact that: Convergence of $\bar{e}(\lambda_n, w_n) \rightarrow 0$ as $n \rightarrow \infty$ and convergence of $\{w_n\}$ implies the existence of a convergent subsequence of $\{\lambda_n\}$ in $int[\Delta]$.*

- *The oddness of this solutions follows using differential techniques is proved in* [Accinelli, E. (1996)].

Proof of the step (i): Consider the mapping from $int[\Delta] \times \Omega_0 \rightarrow R^{n-1}$ defined by the formula:

$$\lambda, w \rightarrow \bar{e}(\lambda, w),$$

where $\bar{e}(\lambda, w)$ is the vector (in R^{n-1}) defined by $n-1$ coordinates of the vector $e(\lambda, w)$.

We need to prove that 0 is a regular value of the restricted excess utility function $\bar{e}$. It is to say that the restricted excess utility function $\bar{e}$ is a submersion at each point $(\lambda, w) \in \lambda \times \Omega$, i.e., $\bar{e}'(\lambda, w) : int[\Delta] \times \Omega_0 \rightarrow R^{n-1}$ is surjective and the null space $Ker(e'(\lambda, w))$ splits X.

We begin showing that the linear tangent mapping is always onto, or equivalently that the rank of the linear map $\bar{e}'$ will be always equal to $n-1$. We will prove that the affirmation is true in a residual set Ω_0.

To see this consider a little change in the endowments given by $w(v) = w + va$, where $x \in \Omega_+$ with $a = (1, 1, \ldots, 1)$ and $v = (v_1, v_2, \ldots, v_n) \in R^n$ is a vector in a small open neighborhood U of zero, such that $v_n = \sum_{i=1}^{n-1} v_i$. The vector v will be thought as a state-independent parameter for redistributions of initial endowments. Observe that $\sum_{i=1}^{n} w_i(v) = \sum_{i=1}^{n} w_i = W$.

The excess utility function for the economy $\varepsilon(v) = \{u_i, w(v)_i\}$ will be:

(4) $$e(\lambda, v) = (e_1(\lambda, v_1), \ldots, e_n(\lambda, v_n)),$$

where

$$e_i(\lambda, v) = u_i'(x_i(\lambda, W))[x_i(\lambda, W) - w_i - v_i a].$$

Observe that the allocations that solve (2) for the economies $\varepsilon(v)$ and ε are the same.

It is easy to see that:

(5) $$\frac{\partial(e_1, e_2, \mathrm{K}, e_{n-1})}{\partial(v_1, v_2, \mathrm{K}, v_n)} = \begin{bmatrix} u_1' & 0 & \mathrm{K} & 0 \\ 0 & u_2' & \mathrm{K} & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ 0 & 0 & \mathrm{L} & u_{n-1}' \end{bmatrix}$$

The rank of this matrix is equal to $n - 1$, as the rank of a matrix is locally invariant, then for each $w + va$, there exists a neighborhood U_v such that the rank of $e'(\lambda, w(v))$ is equal to $n - 1$, for all $w \in U_v$. Since $e(\lambda, v)$ is arbitrarily close to $e(\lambda, w)$ this prove the denseness of Ω_0.

To prove that zero is a regular value for e we need to prove that $Ker(e')$ splits. In our case, as $R(e) = R^{n-1}$, the quotient space $(\Delta \times \Omega_0) / Ker(e')$ has finite dimension, then $codim[Ker(e')] < \infty$ and the splitting property is automatically satisfied, see Zeidler, E. (1993).

Proof of the step (ii): We will prove that, $\bar{e}(\cdot, w) : \Delta \to R^{n-1}$ is a Fredholm operator of index zero. This map will be a Fredholm operator if is a C^1–map and if $J_\lambda \bar{e}(\cdot, w) : \Delta \to L(\Delta, R^{n-1})$ is a linear Fredholm operator for each $\lambda \in \Delta$. Where $J_\lambda \bar{e}(\cdot, w) : \Delta \to R^{n-1}$ is the jacobian matrix of $e(\cdot, w)$. The index of $J_\lambda \bar{e}(\cdot, w)$ at λ is

$$ind(J_\lambda \bar{e}(\lambda, w)) = dim(Ker(J_\lambda \bar{e}(\lambda, w))) - codim(R(J_\lambda \bar{e}(\lambda, w))).$$

The operator, $(J_\lambda e(\lambda, w))$ is, for each $w \in \Omega_0$ a finite linear operator from $R^{n-1} \to R^{n-1}$ and then, for each $\lambda \in \Delta$ is a Fredholm map of index zero.

The economies $\varepsilon = \{w_i, u_i, I\}$ where $w \in \Omega_0$ will be called **Regular Economies**.

Proof of the step (iii): Note that under the assumptions of our model and as $w_i > 0$ $\forall i$, if $\lambda_n \to \lambda \in Fr(\Delta)$ then there exists some i such that $\bar{\lambda}_i = 0$ then $x_i(\lambda_n) \to 0$ and $u_i'(x(\lambda_n)) \to \infty$ when $\lambda_n \to \bar{\lambda}$. So $\|e_i(\lambda_n)\| \to \infty$.

To prove that the set of regular economies is an open and dense set in the space of the economies, it is not necessary to assume the non-emptiness of the interior of the positive cone of the consumption space. It is sufficient to allow for the possibility that *w* is not positive. In this work, we use this assumption to characterize the equilibria set as a Banach manifold with a positive consumption space.

6. Conclusions

The Negishi approach allows us consider the models of finite dimensional economies and those of infinite dimension of unified way. It shows to us that generically both types of models display a similar behavior.

These economies behave well, in the sense that in sufficiently small neighborhoods of them great changes do not take place. That is to say, small changes in the fundamentals do not imply great changes in the behavior of the economic system. These economies are called regular.

In contrast, a complementary set of these exists, where small changes of the foundations imply absolutely unpredictable changes, these are the calls singular economies. These economies also can be studied of unified way using the method of Negishi.

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