Newton's Methods from a Geometric Point of View

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(Research paper presented by A. Ibort)

1. Historical introduction

The main problem we are going to analyze is the computation of the zeroes of an application $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$. For many of the results in these pages we need only a small k ($k = 1$ or $k = 2$), but we are interested on the geometric behaviour of our applications, and so by convenience we always suppose $k = \infty$.

The study of the system $f(x) = 0$ has a long history. The case best understood is the affine situation in which the system reduces to:

$$A \cdot x = b, \qquad A \in \mathcal{M}_{n \times n}(\mathbb{R}), \qquad b \in \mathbb{R}^n.$$

The equation is called "linear system" and there exists a lot of work made in its comprehension [18]. However, for the general problem:

$$f(x) = 0 \quad \text{with} \quad f \in C^k(\mathbb{R}^n, \mathbb{R}^n), \tag{1.1}$$

unless we do suppose some restriction on the behaviour of f the problem becomes unsolvable. Being this the situation it is normal to simplify the problem by looking for points verifying weaker conditions as:

$$x \in Z_\epsilon(f), \tag{1.2}$$

with $Z_\epsilon(f) = \{x \in \mathbb{R}^n : \text{exists } x_0 \in \mathbb{R}^n \text{ such that } f(x_0) = 0 \text{ and } \|x - x_0\| < \epsilon\}$, or more frequently:

$$x \in Z_\epsilon^f(f), \tag{1.3}$$

with $Z_\epsilon^f(f) = \{x \in \mathbb{R}^n : \text{exists } x_0 \in \mathbb{R}^n \text{ such that } f(x_0) = 0 \text{ and } \|f(x)\| < \epsilon\}$.

Assuming some restrictions on f is possible to find points verifying (1.2) or (1.3). This is the standard analysis of the problem and there are quite works finding its points (see i.e. [1], [7], [19]). This will be one of our objectives in this paper. To proceed away we are going to transform problem (1.1) in a dynamical systems analysis.

1.1. NUMERICAL SCHEMES: NEWTON'S METHOD. The numerical schemes for solving equation (1.1) are based in the construction of a sequence whose limit is a zero of f. In its more general formulation, we build a sequence:

$$x_0 \in \mathbb{R}^n, \qquad x_{k+1} = G_k(x_k, x_{k-1}, \ldots, x_0), \tag{1.4}$$

where the application G_k normally only depends on the m last terms of the sequence, i.e. $G_k = G_k(x_k, \ldots, x_{k-m+1})$, such numerical schemes are called multistep of order m. The method is stationary if G_k is independent of k, i.e. $G_k = G$. Obviously the most usual case is $m = 1$, because of its simplicity.

The Newton's method for solving the equation (1.1) is a numerical scheme defined by the sequence:

$$x_{k+1} = x_k - (Jf(x_k))^{-1} \cdot f(x_k), \tag{1.5}$$

where $Jf(x)$ is the jacobian matrix of f evaluated at the point x. The computational cost of the method is very large, because in each step we need to compute n^2 derivatives and to invert a matrix of order n. It has caused that a lot of variants of the scheme have been developed to reduce its cost. One of the possible ways of doing it is to fix the jacobian at a point, so we only have to compute the inverse once. This gives the so called fixed Newton method:

$$x_{k+1} = x_k - (Jf(x_0))^{-1} \cdot f(x_k).$$

The problem with this scheme comes from the reduction in the order of convergence that it is produced. Other approximation was given by Broyden in his work [6], his computations are based in a Taylor approximation of $Jf(x_k)$ by considering the values in the precedent points. This scheme gets the same convergence speed that the classical method (1.5) but with less computations. There are a lot of results in these directions because Newton's method is the natural model for a large class of numerical analysis problems.

Other related problem is the study of the convergence behaviour of the scheme. There are not general results that assure the convergence of the sequence (1.5) for a given starting point $x_0 \in \mathbb{R}^n$. There are partial results

assuring this fact when are assumed some conditions on f (see for instance [5], [9], [12]). To get convergence the scheme can be changed a little. For instance it can be used:

$$x_{k+1} = x_k - \alpha_k (Jf(x_k))^{-1} \cdot f(x_k),$$

with α_k chosen to verify a modulus reduction condition as:

$$\|f(x_{k+1})\| < \|f(x_k)\|, \qquad \forall k \geq 0.$$

This is the approach of [12]. In this article we will show some other variations to get convergence in more general cases.

1.2. NEWTON'S VECTOR FIELD.   We introduce now the Newton's vector field which is going to be the basic analytic tool that we are going to study. We are going to arrive at the construction of this vector field from two different perspectives, which will give us the reasons of its importance.

1.2.1. NUMERICAL CONSTRUCTION.   Newton developed the first method based in derivatives to solve a non-linear equation $f(x) = 0$, with $f \in C^k(\mathbb{R}, \mathbb{R})$. The key idea is to find the zero by changing the function f by its Taylor first order approximation at a point x_0. Obviously, the zero of the Taylor approximation is not a zero of the original function, but at least a "good" approximation. The Taylor first order approximation of f is:

$$f(x_0) + f'(x_0)(x - x_0),$$

computing its zero we find:

$$x^* = x_0 - \frac{f(x_0)}{f'(x_0)},$$

iterating this process we get the Newton sequence. These ideas can be translated to dimension greater than 1, simply substituting $f'(x_k)$ by $Jf(x_k)$. The scheme obtained in this case is (1.5).

The following idea to keep in mind is the interpretation of (1.5) as a numerical approximation of a first order equation. In fact, it can be interpreted as an Euler method with step $h = 1$ of the differential equation:

$$\frac{dx}{dt} = -(Jf(x))^{-1} \cdot f(x). \qquad (1.6)$$

This equation is called the continuous Newton method. The vector field $N(f) = -(Jf(x))^{-1} \cdot f(x)$ is Newton's vector field. It is quite natural to

expect that the convergence properties of the Newton's scheme (1.5) are pre-served when we study the continuous Newton's method (1.6).

We will return again to this vector field. Now, we only want to remark that to make sense of the formula (1.6) we have to assure that $Jf(x)$ has an inverse. The points in $\mathbb{R}^n$ verifying this property are called regular points for f, and the vector field is only well defined on them.

1.2.2. ANALYTICAL CONSTRUCTION. Another way of expressing prob-lem (1.1) is to consider it as a vector field. Then to find zeroes of f is the same that to look for the equilibrium points of the dynamical system:

$$\frac{dx}{dt} = f(x). \tag{1.7}$$

Following the trajectories of (1.7) we can reach eventually an equilibrium point of the flux. The problem with this is that normally the computations to get the flux of (1.7) are more complex than the starting problem. To get a really effective method we have to assure that the equilibrium points of the dynamical system are asymptotically stable. This occurs if there exists a neighborhood of each equilibrium point such that every trajectory, cutting it, has as limit the equilibrium point. In this case integrating only one trajectory we obtain the searched zero. In section 2 we will formalize these ideas.

A good way of simplifying (1.7) would be to find a transformation of the field f preserving the equilibrium points and making all these points asymp-totically stable. We would want to get in the neighborhood of each equilibrium point x^* the field:

$$T(x) = -(x - x^*),$$

which is obviously asymptotically stable. To get it we multiply the vector field by a matrix A which modifies its jacobian. We impose that the later expression has the desired aspect in a first order approximation:

$$T(x) = -(x - x^*) + o(\|x - x^*\|).$$

We need the condition $JT(x^*) = -id$. The condition is so:

$$JT(x^*) = J(Af(x^*)) = A \cdot Jf(x^*) = -id,$$

and we obtain $A = -(Jf(x^*))^{-1}$ and the new vector field would be:

$$\frac{dx}{dt} = -(Jf(x^*))^{-1}f(x),$$

but this dynamical system uses the zero x^* as a data. A good way to obtain a similar result is to use the value of the jacobian matrix at x instead of x^*. Making this, the new vector field is the Newton field (1.6). Thus the continuous Newton's method can be interpreted as a transformation which obtains asymptotic stability in all the equilibrium points of a vector field, without changing them.

In the following section we are going to make an analysis of the properties of the Newton field, which we are going to define in a great generality in the context of fibre bundles with connection. Also we will present some classical results about the convergence of the trajectories [12], [5].

In the section 3 we will analyze the Newton field in the neighborhood of a critical point. In one way we will give conditions to get extensions of the vector field to points with non bijective jacobian. Also we will study the dependence of the method with respect to changes of coordinates. The other point of view will be to understand the set of singular points. We will give some conditions to guarantee convergence in the neighborhood of singular points. Finally, we analyze the general situation when we only can assure the existence of convergence cones.

2. Newton's method on manifolds

2.1. Notations.   Given a differentiable manifold M, a vector bundle E of rank r over M is a manifold and a surjective application $\pi \colon E \to M$ verifying:

1. $\pi^{-1}(x)$ is a r dimensional real vector space.

2. There exists a covering $\{U_\alpha\}$ of M by open sets and a family of applications ϕ_α such that $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$. Also we impose for $(\phi_\alpha)_{|\pi^{-1}(x)}$ to be linear.

The covering U_α is called a trivialization for E. We will define a field on E to be a section $s \colon M \to E$ of the projection π. The set of all differentiable fields on E is usually denoted by $\Gamma(E)$. This definition is natural, because a vector field on M is a section of the projection $\tau_M \colon TM \to M$, where TM is the tangent bundle to M. The space TM has a natural structure of vector bundle over M, the fields on TM are the classical vector fields on M. We will develop our theory in this context.

We need to give an additional structure to the vector bundle E to obtain a definition of the Newton field. We only need the concept of connection:

DEFINITION 1. A connection D on a vector bundle $\pi \colon E \to M$ is an application:

$$D \colon TM \times \Gamma(E) \longrightarrow E$$

$$(v, s) \longrightarrow D_v s$$

verifying:

1. $D_{v+w}s = D_v s + D_w s$, for all $v, w \in TM$, $s \in \Gamma(E)$.
2. $D_v(s + t) = D_v s + D_v t$, for all $v \in TM$, $s, t \in \Gamma(E)$.
3. $D_{\lambda v}s = \lambda D_v s$, for all $\lambda \in \mathbb{R}$, $v \in TM$, $s \in \Gamma(E)$.
4. $D_v(fs) = d_v(f)s + f D_v s$, for all $f \in C^\infty(M)$, $v \in TM$, $s \in \Gamma(E)$.
5. If $v \in \Gamma(TM)$, then $D_v s \in \Gamma(E)$.

Fixing a field in the second variable we obtain an application $Ds \colon TM \to E$ linear in the fibres. This is the natural generalization of the concept of jacobian of a vector field.

We can select a trivialization (U_α, ϕ_α) for E, imposing also that U_α must be a chart on M. Then the coordinates in the chart are $(x_1, \ldots, x_n)$ and $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ is a base of $T_x M$ at each point x of U_α. In E we can choose local sections $(e_1, \ldots, e_r)$ of $\pi \colon \pi^{-1}(U_\alpha) \to U_\alpha$ defining a base at each fibre. So we can describe the connection locally. If

$$v = \sum_i v_i \frac{\partial}{\partial x_i}, \qquad s = \sum_j s_j e_j$$

then:

$$D_v s = \sum_{ij} v_i D_{\frac{\partial}{\partial x_i}} s_j e_j = \sum_{ij} v_i((d_{\frac{\partial}{\partial x_i}} s_j)e_j + s_j D_{\frac{\partial}{\partial x_i}} e_j), \quad \text{and}$$

$$D_{\frac{\partial}{\partial x_i}} e_j = \sum_k \Gamma_{ij}^k e_k = \Gamma_{ij}.$$

The functions Γ_{ij}^k are called the Christoffel symbols of the connection and they determine it completely. In a simpler notation:

$$D_v s = \Sigma_j (d_v s_j + \Sigma_i A_i^j s_i) e_j, \tag{2.8}$$

where A_i^j is a matrix with a 1-form at each entry, constructed from the Christoffel symbols by $A_i^j = \sum_k \Gamma_{ik}^j dx^k$.

A field over a curve $c\colon [a,b] \to M$ is an application $s\colon [a,b] \to E$ verifying that $c = \pi \circ s$. Given a connection D we define the derivative of the field along the curve as $D_{\frac{dc}{dt}}s$. The definition is correct because any field over a curve can be extended to a field on E, and it is easy to show that the derivative of the field in the direction $\frac{dc}{dt}$ is independent of the extension. We can now define:

DEFINITION 2. A field v over a curve c in a vector bundle E with connection D is called parallel with respect to the curve if $D_{\frac{dc(t)}{dt}}v = 0$.

This notion generalizes the concept of parallel vector in $\mathbb{R}^n$. It can be proved that for every vector $v(p) \in \pi^{-1}(p) = E_{|p}$ there exists one and only one parallel field v along a given curve c such that $\pi(v) = c$. The notion depends on the choice of the curve and it happens that two curves c and c' such that $c(a) = c'(a)$ and $c(b) = c'(b)$ transport the vectors from $c(a)$ to $c(b)$ in a different way.

We are going to use in what follows a special class of connections. For this we need to fix a metric on E, i.e. to give a scalar product g at every fiber in E and to impose it a smooth variation. Then:

DEFINITION 3. A connection D on E is compatible with the metric g if it verifies:

$$d(g(s_1, s_2)) = g(Ds_1, s_2) + g(s_1, Ds_2), \qquad \forall s_1, s_2 \in \Gamma(E). \qquad (2.9)$$

Remark 1. If $E = TM$, given a metric g, there exists one and only one torsion free connection compatible with g, i.e. verifying the condition $D_v w - D_w v = [v, w]$, for all $v, w \in \Gamma(TM)$. For other bundles, it can be proved that there exists always connections compatible with a given metric.

2.2. DEFINITION AND BASIC PROPERTIES OF THE NEWTON'S VECTOR FIELD.   We are going to define a generalization of the Newton's vector field (1.6) to the case of vector bundles over manifolds. The key idea for our construction is:

DEFINITION 4. A connection $D\colon TM \times \Gamma(E) \to E$ is invertible for the field s at a point p if $Ds(p)\colon T_pM \to E_p$ is an isomorphism of vector spaces.

Obviously, this definition is useful only for the case $\operatorname{rank} E = \operatorname{rank} TM$, because otherwise it can not be satisfied. From now on we suppose that E is equipped with a connection D. For some proofs it will not be necessary

for D to be compatible with any metric; when it is necessary we will write it explicitly. The connection D will be called a connection for the method (resp. metric for the method)

DEFINITION 5. Given a field s, the regular set $\Omega(s)$ of s is the set of points of M such that the connection for the method is invertible for s.

$$\Omega(s) = \{x \in M : Ds(x) : T_x M \to E_x \text{ is bijective}\}.$$

DEFINITION 6. The singular set $\Sigma(s)$ of s is the complement of $\Omega(s)$, this is:

$$\Sigma(s) = \{\, x \in M : \ \mathrm{rank}\, Ds(x) < r \,\}.$$

It is easy to show that the regular set is open, because it is the inverse image of an open set by the continuous map $\det(Ds(x))$. Now we are going to define the Newton's vector field on the regular set. In the section 3 we will try to extend this definition to some points of the singular set.

LEMMA 1. *If $x \in M$ is an equilibrium point of s (i.e. $s(x) = 0$) we have that the connection for the method is invertible for s at x if and only if there exists a chart (U, ϕ) and a trivialization (U, Φ) of E in a neighborhood of x (by simplicity $\phi(x) = 0$) such that the vector map $\hat{f}$ defined as:*

$$\hat{f} : \mathbb{R}^n \xrightarrow{\ \phi^{-1}\ } M \xrightarrow{\ f\ } E \xrightarrow{\ \Phi\ } U \times \mathbb{R}^n \xrightarrow{\ pr_2\ } \mathbb{R}^n$$

has invertible jacobian matrix at $(0, \ldots, 0)$.

Proof. Using the local coordinates expresion of a connection (2.8) we know:

$$Ds(x) = ds(x) + As(x),$$

where ds is the exterior differential of the components of s (when we trivialize) and A is a matrix of 1-forms. If we are in an equilibrium point we have that $s(x) = 0$, and so we obtain $As(s) = 0$. Now, we only have to compute $ds(x)$, but this is the jacobian matrix of the application $\hat{f}$ at the point. From here the result follows. ∎

We have shown a little more, the result is true if it is true in any chart. Lemma 1 shows that at equilibrium points the invertibility of the connection is equivalent to the invertibility of the jacobian. Remember that at an equilibrium point the jacobian of the field is well defined [11], so the precedent lemma shows that this can be computed by any connection.

Now we can compute a vector field in the tangent bundle from a field on the bundle E:

LEMMA 2. *Given $s, t \in \Gamma(E_{\Omega(s)})$ then:*

$$v = (Ds(x))^{-1} \cdot t(x) \in \Gamma(T\Omega(s)).$$

Proof. Remember that $\Omega(s)$ is open, so for each $x \in \Omega(s)$ we can find a chart (U, ϕ) and trivialization (U, Φ) of E contained on it. In these coordinates we define a map:

$$\beta \colon \mathbb{R}^n \xrightarrow{\phi^{-1}} U \xrightarrow{Ds} \operatorname{End}(E_{|U}) \xrightarrow{\hat{\Phi}} U \times GL(\mathbb{R}^n) \xrightarrow{pr_2} GL(\mathbb{R}^n),$$

which is smooth, because it is the composition of smooth maps. $\operatorname{End}(E_{|U})$ is the vector bundle over U which sets up in each fibre the endomorphism of the vector space $E_{|p}$, $\hat{\Phi}$ is the natural extension of the trivialization mapping Φ of E to the bundle $\operatorname{End}(E)$.

It is well known that the map $\operatorname{Inv}(A) = A^{-1}$ is a smooth morphism in the manifold $GL(\mathbb{R}^n)$; therefore the map $\beta' = \operatorname{Inv} \circ \beta$ is smooth. Calling $\hat{t}$ to the field t written in coordinates, we have that the application $l = \beta' \cdot \hat{t}$ is smooth as product of smooth applications. So we have that $\hat{v} = l$ is smooth and then v is a smooth vector field. ∎

We have obtained all necessary tools to define the Newton's vector field using the connection for the method.

DEFINITION 7. Given $s \in \Gamma(E)$, we define the Newton's vector field associated to s as:

$$N(s)(x) = -(Ds(x))^{-1} \cdot s(x), \qquad \forall x \in \Omega(s) \tag{2.10}$$

The definition is an extension of that used for real and complex vector spaces. We will prove now the most important properties of the vector field $N(s)$.

LEMMA 3. *In the regular $\Omega(s)$, the equilibrium points of s and the equilibrium points of its associated Newton's vector field $N(s)$ are the same.*

Proof. In $\Omega(f)$, $Ds(x)^{-1} \colon E \to TM$ is an isomorphism of vector bundles. Therefore we have: $s(x) = 0 \Leftrightarrow N(s)(x) = 0$. ∎

If we have a metric for the method g compatible with D we can prove:

PROPOSITION 1. *The function* $V = g(s,s)$, *where g is a metric for the method, is a Liapunov function for all the equilibrium points of the Newton's vector field associated to s.*

Proof. We need to prove that for any trajectory $x(t)$ of the Newton's vector field (2.10) the following inequality is verified:

$$\frac{d(V \circ x)}{dt} \leq 0,$$

and the equality is obtained if and only if $x(t) = 0$. Applying the chain rule and remembering the condition of compatibility of the metric and the connection (2.9). In our case we have the following:

$$\frac{d(V \circ x)}{dt}(t) = dV(x(t)) \cdot \frac{dx(t)}{dt}$$

$$= dg(s(x(t)), s(x(t))) \cdot \frac{dx(t)}{dt} = 2g(Ds(x(t)), s(x(t))) \cdot \frac{dx(t)}{dt}$$

$$= -2g(Ds(x(t)), s(x(t)) \cdot (Ds(x(t)))^{-1} \cdot s(x(t)) = -2g(s(x(t)), s(x(t))).$$

Being g a metric the required condition is verified. So, we conclude that V is a Liapunov function for all the equilibrium points of the Newton's vector field. ∎

With this proposition we can conclude the following important corollary:

COROLLARY 1. *Given a field s on E, the equilibrium points of its Newton's vector field $N(s)$, computed with a connection compatible with a metric on E, are asymptotically stable.*

Proof. It is a direct consequence of the existence of Liapunov functions. For a proof of this result see [11]. ∎

Now, we have proved that all the equilibrium points of s which are in the regular set, are equilibrium points for $N(s)$ and that the converse is true. Also these equilibrium points are asymptotically stable, We can drop the assumption made in the compatibility of the connection and we obtain:

PROPOSITION 2. *Given a field s on E, equilibrium points of its Newton's associated field are asymptotically stable.*

Proof. We only have to compute the eigenvalues associated to the jacobian of the Newton's vector field at their equilibrium points. It is a standard fact in dynamical systems theory that if these eigenvalues have strictly negative real part, then the equilibrium point is asymptotically stable. Remember that the jacobian is well defined at an equilibrium point. We have to compute:

$$d(N(s))(x^*) = d(-(Ds)^{-1} \cdot s)(x^*).$$

So we will write it in coordinates. Denoting by $\mathcal{M}(A)$ the matrix associated to a linear application A, we obtain:

$$\mathcal{M}(d)(N(s))(x^*) = -J((ds + As)^{-1}s)(x^*)$$
$$= -J((ds + As)^{-1})s(x^*) - ((ds + As)^{-1}ds)(x^*).$$

At an equilibrium point $s(x^*) = 0$ and $As(x^*) = 0$, therefore we conclude:

$$dN(s)(x^*) = -id.$$

∎

This property makes the field very useful to compute equilibrium points of fields.

Remark 2. When we are in the linear case: $M = \mathbb{R}^n$ and $E = \mathbb{R}^n \times \mathbb{R}^n$ with the trivial connection, s is actually an application $f \colon \mathbb{R}^n \to \mathbb{R}^n$ and equilibrium points are zeroes of the application. In that case the Newton's vector field becomes the classical Newton field (1.6). All the precedent results apply to this simple case and any numerical approximation of the field gives us a numerical scheme to compute the zeroes of s.

When we suppose $E = TM$, s is a vector field on M and equilibrium points are important objects often arising in problems from physics.

The following result give us a clear geometric idea of the trajectories of the Newton's vector field.

PROPOSITION 3. *Given $x(t)$ a trajectory of the Newton's vector field $N(s)$ associated to a field s, the field over the curve $x(t)$ defined by $s(x(t)) \cdot e^t$ is parallel along the curve.*

Proof. Notice that:

$$t \to x(t) \Rightarrow \frac{d}{dt} \to N(s)(x(t)),$$

where $x(t)$ is the flux associated to the Newton's vector field $N(s)$. Remember the definition 2 for a field to be parallel, we test the condition in our concrete case:

$$D_{N(s)(x(t))}(s(x(t)) \cdot e^t) = (D_{N(s)(x(t))}s(x(t)))e^t + s(x(t)) \cdot d_{N(s)(x(t))}e^t$$
$$= -s(x(t)) \cdot e^t + s(x(t)) \cdot \frac{d(e^t)}{dt} = 0.$$

Remark 3. An important corollary is that a Newton's vector field compatible with any metric g has not periodic trajectories. If there is a periodic trajectory $x(t)$ then there exists $T > 0$ such that $x(t + T) = x(t)$. But it is easy to verify that:

$$\frac{dg(x(t), x(t))}{dt} = -2g(x(t), x(t)).$$

Solving this differential equation we obtain:

$$g(x(t), x(t)) = g(x(0), x(0)) \cdot e^{-2t}, \tag{2.11}$$

and so $\|x(t + T)\| = \|x(t)\|e^{-T}$, and this gives a contradiction.

If we are in the case $M = \mathbb{R}^n$ and $E = \mathbb{R}^n \times \mathbb{R}^n$ we can write the precedent proposition saying that the semi-rays:

$$L(f(v)) = \{\lambda \cdot f(v) \colon \lambda > 0\}$$

are invariants for the image $f(x(t))$ of the Newton's vector field. These semi-rays will be called 1 dimensional Newton leaves.

Observe that the definition of the method depends on the choice of connection. In fact, convergence domains associated to each zero depend greatly on this connection. Even more, the shape of the singular set is also variable. The only invariant for the connection is the character of the zeroes: singular or regular.

2.3. GLOBAL BEHAVIOUR OF NEWTON'S VECTOR FIELD. There is a simple way of redefining the Newton's vector field to obtain an extension of the field to all the manifold M provided that TM and E are orientable vector bundles. This condition is necessary for a global definition of this desingularised field. In a local setup it is very easy, it follows from the observation:

$$A^{-1} = \frac{1}{\det A} \operatorname{Adj}^T(A),$$

where $\text{Adj}^T(A)$ is the transpose of the adjoint matrix of A. Therefore we can define the following field:

$$N_d(s) = -\text{Adj}^T(Js(x)) \cdot s(x) = \det(Js(x)) \cdot N(s)(x). \qquad (2.12)$$

The field is defined at every point because the adjoint matrix is always defined. To adapt this argument for a manifold we only need to define the determinant in an intrinsic way. To do that notice that a morphism $\phi \colon E \to F$ of vector bundles can be extended to a morphism between the order k exterior vector bundles as follows:

$$\begin{aligned}
\Lambda^k \phi \colon \Lambda^k E &\to \Lambda^k F \\
e_1 \wedge \cdots \wedge e_k &\mapsto \phi(e_1) \wedge \cdots \wedge \phi(e_k)
\end{aligned}$$

Then in the case of Ds which is a morphism between TM and E we have that $\Lambda^n Ds$ is the equivalent to the determinant. If we fix a metric g on E we can set up a canonical isomorphism called the Hogde star:

$$* \colon \Lambda^k(E) \to \Lambda^{n-k}(E),$$

with the *unique condition* of the orientability of E. In the same way we can define a star operator for the bundle TM. Through these isomorphisms Ds becomes an application from the trivial bundle to the trivial bundle, i.e $\Lambda^0(E) = \Lambda^0(TM) = M \times \mathbb{R}$, we can so define the $\det(Ds)(x)$ as the number obtained through the canonical identification $\Lambda^0(E^*) \otimes \Lambda^0(TM) \simeq M \times \mathbb{R}$. Then the desingularised Newton's vector field makes sense as:

$$N_d(s) = \det(Ds) \cdot N(s), \qquad (2.13)$$

where the determinant is defined through the identifications given by the Hodge star operator of the metrics g and h. The desingularised Newton's vector field is defined over all the manifold M, because remembering the local expression (2.12) we can extend the definition to the singular set $\Sigma(s)$. We can relate Newton's vector field and its desingularised counterpart:

LEMMA 4. *The trajectories of the fields $N(s)$ and $N_d(s)$ coincide in the regular set, $\Omega(f)$.*

Proof. Taking the trajectories of the desingularised field $x(s)$, if we apply the change of coordinates on $\Omega(s)$:

$$\frac{ds}{dt} = \frac{1}{\det(Ds(x))},$$

then they become trajectories for the Newton field. ∎

Observe that the re-parametrization can transform unbounded time intervals in bounded time intervals. It is caused by the unbounded modulus of the Newton's vector field near the singular points. The desingularised field does not hold the orientation of the trajectories of the Newton's vector field. To preserve this orientation we need a positive determinant for the connection, otherwise the orientation is reversed.

Equilibrium points of the field (2.13) are the equilibrium points of s and the so called *extraneous singularities*, which are the points of the singular set $\Sigma(f)$ verifying:

$$s(x) \neq 0, \qquad N_d(s)(x) = 0.$$

They are in the singular set because to verify the first condition is necessary for the endomorphism Ds not to be invertible. It is useful to define the *essential singularities* as the points in the singular set which are not equilibrium points for s or extraneous singularities. From now on, we suppose that the field s is generic enough, i.e. it is a good application in the sense of [3]. So the singular set is a stratified manifold. The strata are the sets defined by:

$$\Sigma^k(s) = \{x \in M : \dim \operatorname{Ker} Ds(x) = k\}. \tag{2.14}$$

It is easy to verify that $\operatorname{codim}_M \Sigma^k = k^2$. Then $\Sigma = \bigcup_{k \geq 1} \Sigma^k$ is a null measure set. We can study properties Σ^k as the following:

DEFINITION 8. A trajectory $x \colon (0, b) \to M$ of the Newton's vector field is prolongable through the singular set if there exists another trajectory $y \colon (c, d) \to M$ such that:

$$\lim_{t \to b} x(t) = \lim_{t \to c} y(t) = x^* \in \Sigma(s).$$
$$\lim_{t \to b} \frac{dx(t)}{dt} = \lim_{t \to c} \frac{dy(t)}{dt}.$$

It implies that the trajectory admits an extension crossing the singular set. The important observation is the following:

PROPOSITION 4. *The Newton's vector field trajectories are never prolongable through the singular set when they cross the set through an essential singularity which lies on* $\Sigma^1(s)$.

Proof. To show this we are going to analize the desingularised field. Given a trajectory which crosses the singular set at x^*, if it is an essential singularity

then it is not an equilibrium point for the desingularised field. Then by the existence and uniqueness theorem there exists a unique trajectory $x_d(t)$ of the desingularised field which passes through this point. Also, it is not an equilibrium point, so we can prolong the trajectory in the point. The Newton's vector field has the same trajectories that the desingularised field. So the unique possible prolongation of $x(t)$ is the second part of the trajectory of $x_d(t)$, which we will call $y(t)$.

Now remember that for a point in $x^* \in \Sigma^1(s)$ we have (see [3]) that $d(\det(Ds)) \neq 0$, so the determinant changes its signum when a trajectory crosses the singular set. Therefore the two trajectories $x(t)$ and $y(t)$ converge to x^*. ∎

It is possible to cross the singular set through extraneous singularities or through high order essential singularities. For generic good applications it has been proved that extraneous singularities are a sub-manifold of codimension 1 in $\Sigma(s)$ [13]. So most of the points of the singular set are "walls" for the Newton's field. It has motivated the name "wall manifold" used for the singular set in works as [25].

We now classify the points of the singular set, according to its behaviour with respect to the trajectories of Newton's vector field.

DEFINITION 9. A point $x^* \in \Sigma(s)$ is a sink for the Newton's vector field, if there exists a non-prolongable trajectory $x\colon (a,b) \to \Omega(s)$ verifying:

$$\lim_{t \to b} x(t) = x^*.$$

DEFINITION 10. A point $x^* \in \Sigma(s)$ is a source for the Newton's vector field, if there exists a trajectory $x\colon (a,b) \to \Omega(s)$ verifying:

$$\lim_{t \to a} x(t) = x^*.$$

Proposition 4 proves that essential singularities in $\Sigma^1(s)$ can not be sources and sinks at a time. The precedent definitions give a classification in the singular set. We will denote $So(s)$ the set of source points in the singular set, and $Si(s)$ the set of sink points. The set $So(s) \cap Si(s)$ is not always empty but it is contained in the extraneous singularities or in the high order singularities ($\Sigma^k(s)$, $k > 1$). We can study in an analytic way the behaviour of the essential singularities:

PROPOSITION 5. *If $x^* \in \Sigma(s)$ verifies:*

$$d(\det(Ds(x^*)))(N_d(s)(x^*)) > 0, \qquad (2.15)$$

then x^* is a source for the Newton's vector field, but if the left hand side of (2.15) is strictly less than zero, then x^* is a sink for the Newton's vector field.

Proof. To get any of the two inequalities we need $d(\det(Ds(x^*))) \neq 0$, therefore $x^* \in \Sigma^1(s)$ and the singular set is locally a manifold. We are going to prove the first part, the second follows the same argument. Being x^* an essential singularity $(N_d(x^*) \neq 0)$ there exists a unique trajectory $x_d(t)$ of the desingularised field through x^*. Thus there are two trajectories of the Newton's vector field starting or ending at x^*, by Proposition 4 it can not happen that one starts at the point and the other ends. Take the trajectory lying in the open set : $\det(ds(x)) > 0$. So this trajectory preserves the sense of the desingularised trajectory. We have only to impose that this trajectory does not go to the singular set but a sufficient condition is:

$$d(\det(Ds))(x^*)(N_d(s)(x^*)) > 0,$$

which is condition (2.15). ∎

Points of the singular set verifying inequality (2.15) will be called transversal sources. The set of all these points is denoted by $\mathrm{SoT}(s)$. Clearly, $\mathrm{SoT}(s) \subset \mathrm{So}(s) \cap \Sigma^1(s)$. Again using tools of transversality we can assure the smoothness and to compute the dimension of the complementary of the transversal sources and sinks. We can impose conditions to obtain global convergence for the Newton's vector field using the previous definitions and results. One simple statement is the following:

PROPOSITION 6. *Let M be a compact manifold with metric g and E a vector bundle with metric h and compatible connection D and let $s\colon M \to E$ be a field with isolated equilibrium points. If $\Sigma(s) = \mathrm{SoT}(s)$ then all trajectories of the Newton's vector field converge to an equilibrium point of s.*

Proof. Suppose that it is not true. Then there exists a trajectory $x(t)$ whose limit is not an equilibrium point. Remember that:

$$\|f(x(t))\| = \|f(x(0))\|e^{-t}. \qquad (2.16)$$

The trajectory is complete, because the only possibility to avoid the completeness is to end in the singular set. This is impossible by the transversal

source condition. Now, if the trajectory is complete, using formula (2.16) we conclude that it has as limit a set of zeroes of s. But obviously it must be one element set by the isolation hypothesis. ∎

Remark 4. We can formulate similar results in $\mathbb{R}^n$ imposing additional conditions. One way to proceed is suppose that an application $f\colon \mathbb{R}^n \to \mathbb{R}^n$ admits an extension to an application $\bar{f}\colon S^n \to \mathbb{R}^n$ and then we can apply the precedent result. Imposing also that the compactification verifies that $\bar{f}\colon S^n \to S^n$ and $f(\infty) = \infty$, then the result again applies because the trajectories of the Newton's vector field obviously do not go to infinity in this case.

The hypothesis of the proposition 6 are very restrictive and a lot of applications do not verify them. It simply shows that Newton's vector field has limitations imposed by the existence of the singular set. In the following section we give an overview of other possibilities to get convergence. In this case we will generalize the field and we will adopt a probabilistic point of view.

2.4. PROBABILISTIC CONVERGENCE. There are many works computing the probability of the Newton's scheme to achieve convergence to a zero. We are going to give an overview of the proof of the first important result in this way, obtained by Hirsch and Smale in their article [12]. In this subsection we resume this work, as always adapting it to the manifold setting.

We will need some hypothesis along the proof. To have a geometric definitions of the determinants involved in the proof we need to assure the orientability of M and E. Also we can suppose that s is good in the sense of [3]. We will define this concept in subsection 3.2, but it is enough for our purposes that this assures us that $\Sigma(s)$ is a stratified manifold and so is a measure zero set. We need a metric g on E and respectively h on TM. We impose the compatibility of the connection for the method D with g. We define the sphere bundle associated to E as:

$$S(E) = \{x \in E \colon g(x,x) = 1\}.$$

We are going to study a different field, which preserves the flow of the Newton's vector field, but sometimes changes the orientation. We define the generalized Newton's vector field as:

$$G(s)(x) = -\operatorname{sign}(\det(Ds(x))) \cdot N(s)(x), \qquad (2.17)$$

this field coincides with the Newton's vector field when the determinant of the jacobian is positive and reverses the orientation if it is not the case. $G(s)$ is defined on $\Omega(s)$ as the Newton's vector field. But we have that it crosses the singular set through essential singularities, obviously losing the property (2.11).

First we define the map:

$$
\begin{aligned}
s_1 \colon M - Z(s) & \rightarrow & S(E) \\
p & \mapsto & \frac{s(p)}{\|s(p)\|_g},
\end{aligned}
$$

where $Z(s)$ are the equilibrium points of s, obviously there s_1 is not defined. We will call $\Omega(s_1)$ and $\Sigma(s_1)$ the sets of regular and singular points of s_1. We can derive s_1 as a section of E using D.

Let $x \in \Omega(s_1)$. We denote by $B(x)$ the trajectory through $x = x(0)$ of the generalized Newton's vector field, we consider that a trajectory can continue crossing the singular set. Finally $A(x)$ will be the connected component of the set $\Omega(s_1) \cap B(x)$ verifying $x \in A(x)$. $A(x)$ is a smooth curve. This curve is oriented with the vector field (2.17). One important observation is $G(s)(x) \in \operatorname{Ker} Ds_1(x)$, because $s_1(x)$ is parallel along $A(x)$ as a single corollary of Proposition 3. We introduce the following useful notation:

$$
\begin{aligned}
s^{-1}[a, b] & = & \{x \in M : a \leq \|s(x)\|_g \leq b\} \\
s^{-1}(a, b) & = & \{x \in M : \ a < \|s(x)\|_g < b\} \\
s^{-1}(a, b] & = & \{x \in M : \ a < \|s(x)\|_g \leq b\} \\
s^{-1}(c) & = & \{x \in M : \|s(x)\|_g = c\} \\
s_+^{-1}(c) & = & \{x \in M : c \leq \|s(x)\|_g\} \\
s_-^{-1}(c) & = & \{x \in M : \|s(x)\|_g \leq c\}.
\end{aligned}
$$

We have supposed that s is good, now we make also the following assumptions:

$$
(G) \begin{cases} 1) \ \|s(x_0)\|_g = c > 0, \\ 2) \ \det(Ds(x_0)) > 0 \\ 3) \ \det(D(s)) \geq 0, \quad \text{on a neighborhood of } s^{-1}(c). \end{cases} \tag{2.18}
$$

Under these assumptions we can prove:

LEMMA 5. $A(x_0) \cap s^{-1}(c) = \{x_0\}$ and $A(x_0)$ is diffeomorphic to $\mathbb{R}$.

Proof. By the asumption (G1) $A(x_0)$ cuts transversally to $s^{-1}(c)$ at x_0. Now by the assumption (G3) is impossible to get $s^{-1}(c)$ again. Therefore x_0 disconnects $A(x_0)$ and so it cannot be diffeomorphic to a circle. ∎

A direct corollary is:

COROLLARY 2. $\|s(y)\| < c$ *for all y in the semi-positive trajectory $A^+(x_0)$ of $A(x_0)$.*

We will say that $A(x)$ has a positive limit point y, if there exists a sequence t_i of real numbers, such that:

$$\lim_{i \to \infty} \eta(t_i) = y,$$

where $\eta \colon \mathbb{R} \to A(x)$ is a diffeomorphism. It is easy to see that every positive limit point is either a critical point of s_1 or a zero of s. The first because $A(x) \subset s_1^{-1}(s_1(x))$ and $A(y)$ is smooth by the implicit function theorem if s_1 is regular in y, the second because s_1 is not defined on the zeroes of s. We are supposing that $S(E)$ is trivial, if it is not the case we can locally trivialize this bundle and then to check that the argument is purely local.

LEMMA 6. *Assuming condition (G) above, we suppose also $0 < q < c$ and:*

1. $s^{-1}[q, c]$ *is compact.*
2. $s_1(x_0)$ *is a regular value for $(s_1)_{|s^{-1}[q,c]}$.*

Then $A^+(x_0) \cap s^1(q) \neq \emptyset$.

We have to define what we mean by a regular value of s_1. For this we need to set up a trivialization of the sphere bundle over all the manifold. It will be usually impossible because, in general, $S(E)$ is not a trivial bundle. But we can make it locally using the Newton's trajectories to trivialize the bundle and the manifold. We can obtain a locally finite covering $\{M_i\}$ which transforms $E_{|M_i}$ in $M_i \times \mathbb{R}^n$ with a trivial derivation at least in the Newton's vector field directions. To define a regular value we need to fix a M_j. Then we take a value v_j of $E_{|M_j} = M_j \times \mathbb{R}^n$. We obtain other values on each M_j by gluing with the transition functions. So we have a collection $\{v_i\}$ of vectors. It depends on the order followed to make the identifications. We will say that v_i is a regular value if it is a regular value for each M_i. It is clear that s_1 is constant in $A(x_0)$ with this trivialization. So $s_1(x_0)$ is a regular value for s_1 if all their anti-images in M_i obtained by the precedent identifications are regular. Now we start the proof.

Proof. If $A^+(x_0)$ is disjoint from $s^{-1}(x_0)$, it must be contained in $s^{-1}(q, c]$. By a) $A^+(x_0)$ has a limit point $y \in s^{-1}[q, s]$. So $f(y) = 0$ or $y \in \Sigma(g)$. The first is impossible, the second too. To see it we have only to notice that $g(x_0) \simeq g(y)$, using the trivialization defined before. So by hypothesis y is regular. We obtain a contradiction. ∎

A subset W_1 of W_2 has full measure if $W_2 - W_1$ has measure zero. The following result extends Theorem 1.5 in [12]:

THEOREM 1. *Let* $s: M \to E$ *and* $\alpha > 0$ *satisfying the following hypothesis:*

1. s *is proper.*
2. $\det(Ds)^{-1}\{0\}$ *has measure zero.*
3. $\det(Ds)(x) \geq 0$ *if* $\|s(x)\|_g \geq \alpha$.

Let $0 < \epsilon < \alpha$. Then there exists a subset $W(\epsilon) \subset s_+^{-1}(\alpha)$ which is open and of full measure, such that if $x_0 \in W(\epsilon)$, then $A(x_0) \simeq \mathbb{R}$ and $A^+(x_0)$ contains a point y with $\|s(y)\| = \epsilon$.

Proof. We will say that s is proper if $\|s\|_g$ is proper, i.e. the anti-image of a compact is compact. For each $\beta > \alpha$ we define:

$$\text{Bad}(\epsilon, \beta) = s_1(\Sigma(s_1) \cap s^{-1}[\epsilon, \beta]).$$

In each M_i we can see $\text{Bad}(\epsilon, \beta)$ as a compact subset os S^{n-1} through the precedent trivializations. By Morse-Sard lemma, $\text{Bad}(\epsilon, \beta)$ has measure zero in S^{n-1} because s_1 is differentiable, also in the union of all $M_i's$ because the index set is countable. Clearly each point in $S^{n-1} - \text{Bad}[\epsilon, \beta]$ is a regular value for $(s_1)_{s^{-1}[\epsilon,\beta]}$, and this set is of full measure in S^{n-1}.

Since $\Omega(s)$ has full measure in M, so has $\Omega(s_1)$ in $M - Z(s)$. We can easily prove then that $s_1^{-1}(S^{n-1} - \text{Bad}[\epsilon, \beta])$ has full measure in $\mathbb{R}^n - Z(s)$. Now define:

$$W(\epsilon, \beta) = s^{-1}(\alpha, \beta) \cap s_1^{-1}(S^n - \text{Bad}(\epsilon, \beta)) \cap (\det(Ds))^{-1}[0, \infty).$$

$W(\epsilon, \beta)$ is open and has full measure in $s^{-1}[\epsilon, \beta]$. Finally define:

$$W(\epsilon) = \bigcup_{\beta > \alpha} W(\epsilon, \beta).$$

Then $W(\epsilon)$ is open and has full measure in $s_+^{-1}(\alpha)$. For every $x_0 \in W(\epsilon)$ the hypothesis of Lemma 6 are fulfilled, so we obtain the desired result. ∎

Remarking that the proof implies the non existence of limit points in any $s^{-1}[\epsilon, \alpha]$ we obtain easily the following result which proves that the trajectories of the desingularised Newton's vector field generically converge to a simple point as we suppose before; this generalizes Theorem 1.6 in [12].

THEOREM 2. *Under the hypotheses of Theorem 1 there exists a Baire subset W of full measure in $E^+(\alpha)$ such that if $x_0 \in W$, then $A^+(x_0) \simeq \mathbb{R}$ and $A^+(x_0)$ leads to $f^{-1}(0)$ for all $x_0 \in W$. In particular, $f^{-1}(0)$ is nonempty.*

We have proved that, for proper fields verifying a positivity condition in the determinant of the jacobian, we have a generalized Newton's vector field which generically converges to zero. This result is important and it has been the base for a lot of computational works implementing these techniques in concrete schemes, as Hirsch and Smale make directly in the article [12].

Other works have exploited in the last years the geometrical consequences of Proposition 3. The most interesting results are contained in [9]. There Immo Dienner defines the concept of Newton leaf for an application $f \colon \mathbb{R}^n \to \mathbb{R}^n$. A Newton k-leaf is the anti-image of a k dimensional subspace $V \in \mathbb{R}^n$, i.e. $L(V) = f^{-1}(V)$. By Proposition 3 these leaves are preserved by the Newton's vector field, making possible to develop a topological-geometric theory. These ideas pass to the manifold setting when the bundle E is trivial, i.e. $E = M \times \mathbb{R}^k$ and the connection used is the standard. In other cases the theory has to take into account the infinitesimal holonomy induced by the curvature associated to the connection and the global holonomy induced by the topology of the manifold and of the vector bundle.

3. THE NEWTON'S VECTOR FIELD AT SINGULARITIES

Through this section we will study the behaviour of the Newton's vector field at neighborhoods of singular points of a field. We are going to make a local study. So we are going to analyze the vector case, i.e. our field will be identified with an application $f \colon \mathbb{R}^n \to \mathbb{R}^n$, the presence of curvature makes the analysis more technical, but we could repeat all the proofs in this context. To make easy the exposition we restrict ourselves to the zero curvature case. The main question which we will try to answer is: what can we expect about the convergence of the Newton's vector field when its zeroes lie in the singular set?

In subsection 3.1 we will take the road through the continuous extension of the field to the singular set. Tools of singularity theory will help us to obtain

some results.

In subsection 3.2 we will define new tools to analyze the behaviour of the field. The key idea will be the changes of coordinates which allow to check what properties of the field are geometric. Thanks to these, we introduce some classification results using Boardman's classification of singularities [4]. The main results show that some types of singularities never admit a continuous extension.

In subsection 3.3 we will work with the case obtained when there does not exist any continuous extension of the field. We present in a geometrical way some standard results [15, 21], precising their meaning and adding some generalizations.

3.1. CONTINUOUS EXTENSIONS OF THE FIELD. Our objective will be to analyze the conditions for the continuous extension of the Newton's field to a singular equilibrium point. We will suppose that our application $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is good, thus Σ is a stratified manifold. In this set the Newton's vector field is not defined, because the endomorphism Df is not bijective and it does not admit an inverse. Remember that in the particular case we are studying $Df = Jf$, it is the jacobian matrix associated to the application f. We will denote now the Newton's vector field associated to f as $F = N(f)$. Then we can write:

$$f\colon \mathbb{R}^n \to \mathbb{R}^n \qquad F\colon \Omega(f) \to \mathbb{R}^n.$$

So we can introduce the following:

DEFINITION 11. F is defined at x_0 along the regular set if there exists the limit along the set $\Omega(f)$ of the application F.

Remember that an application has limit at x^* along a set Γ if x^* is an accumulation point of Γ and if the following limit is well defined:

$$\lim_{\substack{x \to x^* \\ x \in \Gamma}} F(x) = L,$$

(i.e., for all U, neighborhood of L, exists V, neighborhood of x^* such that $x \in \Gamma \cap V \Rightarrow F(x) \in U$.) The set of points where F is defined along $\Omega(f)$ is denoted by $E_r(F)$. The following result precise the possibility of getting a continuous extension:

THEOREM 3. *The function $\overline{F}$ defined as:*

$$\overline{F}(x) = \begin{cases} F(x), & x \in \Omega(f) \\ \lim_{\substack{x \to x^* \\ x \in \Omega(f)}} F(x), & x \in E_r(F) - \Omega(f) \end{cases}$$

is continuous in $E_r(F)$. Also, it is the biggest set where it is possible to obtain a continuous extension of the field.

For a proof of this result see [25], [23].

From now on we will study applications $c = g/h$ defined as:

$$g, h \colon \mathbb{R}^n \to \mathbb{R}, \qquad g, h \in C^r(\mathbb{R}^n, \mathbb{R})$$
$$c = g/h \colon \mathbb{R}^n \setminus Z(h) \to \mathbb{R}, \qquad c \in C^r(\mathbb{R}^n \setminus Z(h), \mathbb{R}),$$

where $Z(h)$ is the zero set of h. There is a natural geometric condition to obtain a continuous extension of the quotient:

DEFINITION 12. c is almost-continuous at $x \in \mathbb{R}^n$ if c is continuous at x or $g(x) = 0$.

The set of points where c is almost continuous is called the almost continuity domain $E(c)$. To admit a continuous extension of c at a point x, it is necessary to be in the interior of $E(c)$. If we suppose that $g(x) \neq 0$ and $h(x) = 0$, then it is easy to show that c does not admit any continuous extension to x. The following result expresses that the condition is sufficient for the Newton's vector field in a lot of cases:

THEOREM 4. *Let $g(x) = - \mathrm{Adj}^T(Jf)(x) \cdot f(x)$ and $h(x) = det(Jf(x))$, being f good subanalytic application. Then $c = g/h$ can be defined continuously on $\overset{\circ}{E}(c)$.*

Remember that $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is subanalytic if there exist diffeomorphisms ϕ and Φ such that $\phi \circ f \circ \Phi$ is analytic. First remark that $Z(h) = \Sigma(f)$. If $x \in \Sigma^1(f)$ then f is always locally subanalytic, then we can prove Theorem 4 in this particular case:

PROPOSITION 7. *Let $x^* \in \Sigma^1(f) \cap E(f)$ then $c = g/h$ admits a continuous extension at x^*.*

Proof. To carry on the calculations will be convenient to write the condition required to be a member of $\Sigma^1(f)$ in a different way. An application is good if the 1-jet associated to f, defined as:

$$j^1(f)\colon \mathbb{R}^n \quad \to \quad \mathbb{R}^n \times \operatorname{End}(\mathbb{R}^n)$$
$$p \quad \mapsto \quad (f(p), Jf(p)),$$

is transverse to the stratified sub-manifolds Σ^k defined in $\operatorname{End}(\mathbb{R}^n)$ by means of the rank of the application. Precising a little more, in fact, we impose the transversality of $pr_2 \circ j^1(f)$ with $\Sigma^k = \{A \in \operatorname{End}(\mathbb{R}^n)\colon \operatorname{rank} A = n - k\}$. To get this transversality condition we need to avoid:

$$dj^1 f(x^*)(\mathbb{R}^n) \subset T_{pr_2 \circ j^1 f(x)}(\Sigma^1),$$

but it is equivalent to impose $d(\det(Jf))(x^*) = 0$. Obviously $\det(Jf(x^*)) = 0$, and therefore there exists $\alpha_1(x)$ eigenvalue of $Jf(x)$ such that $\alpha_1(x^*) = 0$. In fact, we write:

$$d(\det(Jf))(x^*) = d(\alpha_1 \cdots \alpha_n)(x^*) = d\alpha_1(x^*)\alpha_2(x^*) \cdots \alpha_n(x^*)$$
$$+ \alpha_1(x^*)d\alpha_2(x^*) \cdots \alpha_n(x^*) + \cdots + \alpha_1(x^*) \cdots d\alpha_n(x^*)$$
$$= d\alpha_1(x^*)\alpha_2(x^*) \cdots \alpha_n(x^*).$$

Therefore if we want to get $d(\det(Jf))(x^*) \neq 0$ we need to impose:

$$d\alpha_1(x^*) \neq 0, \qquad \alpha_j(x^*) \neq 0 \quad j = 2, \ldots, n.$$

These conditions are frequently used [15], [23]. Being $\Sigma^1(f)$ a codimension 1 sub-manifold we can select coordinates $(x_1, \cdots, x_n)$ in a neighborhood V of x^* verifying:

$$\Sigma^1(f) \cap V = \{(x_1, x_2, \ldots, x_n)\colon x_2 = \cdots = x_n = 0\}.$$

Applying Hadamard's lemma (see [3]) we obtain:

$$h(x_1, \ldots, x_n) = x_1 h_1(x_1, \ldots, x_n) + \cdots + x_n h_n(x_1, \ldots, x_n) h_n(x_1, \ldots, x_n).$$

The idea of the proof of this lemma is to make:

$$h(x_1, \ldots, x_n) = h(x_1, \ldots, x_n) - h(0, x_2, \ldots, x_n)$$
$$+ h(0, x_2, \ldots, x_n) - h(0, 0, x_3, \ldots, x_n)$$
$$+ \ldots$$
$$+ h(0, \ldots, 0, x_n) - h(0, \ldots, 0)$$

Now:

$$h(0, \ldots, 0, x_i, \ldots, x_n) - h(0, \ldots, 0, x_{i+1}, \ldots, x_n)$$
$$= \int_0^{x_i} \frac{\partial h}{\partial x_i}(0, \ldots, 0, t, x_{i+1}, \ldots, x_n) \, dt$$
$$= \int_0^1 \frac{\partial h}{\partial x_i}(0, \ldots, 0, sx_i, x_{i+1}, \ldots, x_n)x_i \, ds = x_i h_i(x_1, \ldots, x_n).$$

So in our case we can conclude:

$$h_i(x_1, \ldots, x_n) = 0, \qquad i = 2, \ldots, n.$$
$$h(x_1, \ldots, x_n) = x_1 g_1(x_1, \ldots, x_n).$$
$$g(x_1, \ldots, x_n) = x_1 f_1(x_1, \ldots, x_n).$$

The last equality is deduced from the application of Hadamard's lemma to g, which is zero in $\Sigma^1(f)$ by hypothesis.

Finally let us remember that $d(\det(Jf))(x^*) = dh(x^*) \neq 0$, then:

$$dg(x_1, \ldots, x_n) = dx_1 g_1(x_1, \ldots, x_n) + x_1 dg_1(x_1, \ldots, x_n).$$
$$dg(0, \ldots, 0) = g_1(0, \ldots, 0) \neq 0.$$

Thus:

$$\lim_{x \to x^*} \frac{g(x)}{h(x)} = \lim_{x \to x^*} \frac{x_1 f_1(x)}{x_1 g_1(x)} = \frac{f_1(x^*)}{g_1(x^*)},$$

so the proof is complete. ∎

Proof of Theorem 4. To generalize the proof to points of $\Sigma^k(f)$, with $k > 1$, we need to extend the key result that we have proved implicity in the precedent proposition:

LEMMA 7. *Let $C_x(\mathbb{R}^n)$ be the local ring of differentiable functions at x. The ideal constituted by the functions which are zero when restricted to a fixed codimension 1 sub-manifold is principal.*

In other words: all functions which are zero in the sub-manifold, for a "good" choice of coordinates, can be written as $f = x_1 f_1$. It is, (x_1) generates the ideal. So it is principal. Moreover:

LEMMA 8. *Let $C_x(\mathbb{R}^n)$ be the local ring of differentiable functions at x. The ideal constituted by the functions which are zero when restricted to a fixed codimension 1 stratified analytic sub-manifold is principal.*

For a proof see [16]. With this result Theorem 4 can be proved directly. First the sub-analycity of f implies that $\Sigma(f)$ is a stratified analytic submanifold. So we are in the conditions of Lemma 8. There will be a function s such that $g = g_1 \cdot s$ and $h = h_1 \cdot s$. Thus:

$$F(x^*) = \lim_{x \to x^*} F(x) = \lim_{x \to x^*} \frac{g(x)}{h(x)} = \lim_{x \to x^*} \frac{s(x)g_1(x)}{s(x)h_1(x)} = \lim_{x \to x^*} \frac{f_1(x)}{g_1(x)}.$$

Now we have only to prove that $g_1(x^*) \neq 0$. The argument is similar to that of the case $k = 1$. If it is not true, we will obtain:

$$(pr_2 \circ j^1 f)_*(T_x^* \mathbb{R}^n) \subset T_{pr_2 \circ j_1 f(x^*)}\Sigma,$$

where $T_{pr_2 \circ j_1 f(x^*)}\Sigma$ is the limit of all the tangents to Σ^j with $j < k$ in points going to x^*. Now if $h_1(x^*)$ is zero we obtain that it is impossible to get away from the singular set in a first derivative approximation, i.e.

$$\frac{\partial h(x^*)}{\partial v} = \frac{\partial s(x^*)h_1(x^*)}{\partial v} = s(x^*)\frac{\partial h_1(x^*)}{\partial v} + \frac{\partial s(x^*)}{\partial v}h_1(x^*) = 0, \qquad \forall v \in \mathbb{R}^n.$$

And it implies the non tranversality of $(pr_2 \circ j^1 f)_*(T_x^* \mathbb{R}^n)$. So if we want to guarantee this condition, we need to impose $h_1(x^*) \neq 0$. Then, the proof is ended. ∎

From the precedent theorem we can obtain a clear corollary:

COROLLARY 3. *If f is a good subanalytic application with has a singular zero x^*, its Newton's vector field admits a continuous extension to x^* if and only if there exists a neighborhood of the point x^* where all the singularities are straneous or zeroes of f.*

Proof. By Theorem 4 a necessary and sufficient condition to obtain a continuous extension is that F is contained in $E(F)$ but that is just the condition imposed in the neighborhood. Because

$$F(x) = \frac{-\operatorname{Adj}^T(Jf)(x) \cdot f(x)}{\det(Jf(x))},$$

and we need to impose that $\operatorname{Adj}^T(Jf)(x) \cdot f(x) = 0$ in $\Sigma(f)$, it is possible if and only if x a strenuous singularity or x is a zero of f. ∎

The possibility of a continuous extension is very important for Newton's vector field. First, the numerical methods used to integrate the field are well behaved in this case, because the derivatives are bounded. Second, we obtain asymptotic stability in the extended field. This is formalized in the following two results.

LEMMA 9. *If we extend $N(f)$ to its domain of continuous extension, then the isolated zeros of f and $N(f)$ are the same.*

Proof. In the regular set the result was proved before. Suppose now that x^* is point where there exists a continuous extension of $N(f)$. Adapting equation (1.6) we obtain:

$$-Jf(x) \cdot N(f)(x) = f(x).$$

Therefore if x^* is a zero for $N(f)$ we have that it is also a zero for f. Now we go the other implication. Suppose $f(x^*) = 0$ and $Nf(x^*) \neq 0$. Then by the existence and unicity theorem of ODE's there exists only one trajectory $x(t)$ of the Newton's vector field through x^*. Making $x(0) = x^*$:

$$f(x(t)) = f(x(0)) \cdot e^{-t}.$$

But for $t > 0$, because x^* is an isolated zero, we obtain $x(t) = x^*$. But this contradicts the affirmation $N(f)(x^*) \neq 0$. ∎

Now we study the asymptotic stability:

LEMMA 10. *Let x^* be a singular isolated zero of f which admits a continuous extension of the Newton's vector field, then the function $V = \|f\|^2$ is strictly Liapunov for the Newton's vector field in a neighborhood of x^*, so x^* is asymptotically stable.*

Proof. Being x^* isolated, there exists $\epsilon > 0$ such that $B(x^*, \epsilon)$ does not contain another zero. By Proposition 3 we have:

$$\frac{dV(x(t))}{dt} = -2V(x(t)) < 0,$$

for all the points of $B(x^*, \epsilon)$. Thus V is strictly Liapunov and the point is asymptotically stable. ∎

We notice that the precedent condition is very strict and so generically Newton's vector field does not admit continuous extension to singular zeroes. In the following section we are going to precise this affirmation.

3.2. NEWTON'S VECTOR FIELD AND CHANGES OF COORDINATES. The main limitation of the precedent section is that it does not provide a criterium in terms of the derivatives of the field to characterize the continuous extension of the field. In the following lines we study the behaviour of the vector field in front of changes of coordinates. The idea is to extract the geometrical information contained in the vector field.

3.2.1. BASIC RESULTS. Newton's vector field can be interpreted intrinsically as an operation transforming fields over a manifold in vector fields over a subset of the manifold. But, there exists other interpretation as a transformation of functions. We will see that this interpretation is not geometric (depends on the choice of the chart) but it will lighten some aspects of the vector field.

DEFINITION 13. Given an application $f \colon \mathbb{R}^n \to \mathbb{R}^n$, and $\phi_1, \phi_2 \colon \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphisms verifying that $\phi_2(x) = 0 \Leftrightarrow x = 0$, we define the 0-transformation $f_{\phi_1}^{\phi_2} = \phi_2 \circ f \circ \phi_1^{-1}$.

Remark 5. This kind of transformation is nothing but the obvious action of the group $\mathrm{Diff}(\mathbb{R}^n)_L \times \mathrm{Diff}(\mathbb{R}^n)_R$, in our case with the additional condition of conservation of the zero. So they verify:

$$(f_{\phi_1}^{\phi_2})_{\Phi_1}^{\Phi_2} = f_{\Phi_1 \circ \phi_1}^{\Phi_2 \circ \phi_2}$$

DEFINITION 14. Given $f \colon \mathbb{R}^n \to \mathbb{R}^n$, its Newton's vector field associated in coordinates (ϕ_1, ϕ_2) is defined as $N(f_{\phi_1}^{\phi_2})$.

Making an explicit computation of the field we obtain:

$$df_{\phi_1}^{\phi_2}(x) = d\phi_2(f \circ \phi_1^{-1}(x)) \cdot df(\phi_1^{-1}(x)) \cdot d\phi_1^{-1}(x) \tag{3.21}$$

$$(d\tilde{f})^{-1}(x) = (d\phi_1^{-1}(x))^{-1} \cdot (df(\phi_1^{-1}(x)))^{-1} \cdot (d\phi_2(f \circ \phi_1^{-1}(x)))^{-1}$$
$$= d\phi_1(\phi_1^{-1}(x)) \cdot df(\phi_1^{-1}(x))^{-1} \cdot d\phi_2^{-1}(\phi_2 \circ f \circ \phi_1^{-1}(x))$$

$$N(f_{\phi_1}^{\phi_2})(x) = d\phi_1(\phi_1^{-1}(x)) \cdot (df(\phi_1^{-1}(x)))^{-1} \cdot$$
$$\cdot d\phi_2^{-1}(\phi_2 \circ f \circ \phi_1^{-1}(x)) \cdot (\phi_2 \circ f \circ \phi_1^{-1})(x). \tag{3.22}$$

We could consider $N(f_{\phi_1}^{\phi_2})$ as a field and then we can analyze if the pull-back of $N(f)$ through the diffeomorphisms $\phi = \phi_1 = \phi_2$ is equal to $N(f_{\phi_1}^{\phi_2})$, i.e. :

$$N(f_\phi^\phi) \overset{?}{=} d\phi(N(f) \circ \phi^{-1})$$

$$N(f_\phi^\phi)(x) = -d\phi(\phi^{-1}(x)) \cdot (df(\phi^{-1}(x)))^{-1} \cdot$$
$$\cdot (d\phi(f \circ \phi^{-1}(x)))^{-1} \cdot (\phi \circ f \circ \phi^{-1})(x)$$

$$d\phi N(f) \circ \phi^{-1} = -d\phi(\phi^{-1}(x)) \cdot (df(\phi^{-1}(x)))^{-1} \cdot f(\phi^{-1}(x)).$$

Comparing the two expressions we conclude:

LEMMA 11. *The Newton's vector field associated to an application f_ϕ^ϕ coincides with the pull-back of $N(f)$ through ϕ if and only if ϕ is a linear application.*

Other possibility is to consider $N(f)$ and $N(f_{\phi_1}^{\phi_2})$ as applications. In this case we can ask if $N(f_{\phi_1}^{\phi_2})$ is equal to the 0-transformation of f through the diffeomorphisms (ϕ_1, ϕ_2), i.e.:

$$N(f_{\phi_1}^{\phi_2}) \overset{?}{=} (N(f))_{\phi_1}^{\phi_2}. \tag{3.23}$$

Again the answer is negative because:

$$(N(f))_{\phi_1}^{\phi_2}(x) = \phi_2(-(df(\phi_1^{-1}(x)))^{-1} \cdot f(\phi_1^{-1}(x))). \tag{3.24}$$

And this expression is not, usually, equal to (3.22). However if we study linear changes of coordinates $\phi_1(x) = Ax$ and $\phi_2(x) = Bx$ then we obtain:

$$N(f_{\phi_1}^{\phi_2}) = -A \cdot (Jf(A^{-1}x))^{-1} f(A^{-1}x)$$
$$(N(f))_{\phi_1}^{\phi_2} = -B(Jf(A^{-1}x))^{-1} f(A^{-1}x).$$

Thus we can write:

LEMMA 12. *For linear changes of coordinates $(N(f))_A^B = N(f_A^B)$ if and only if $A = B$.*

Finally an useful observation is:

LEMMA 13. *If $t_v: \mathbb{R}^n \to \mathbb{R}^n$ is a translation with vector v we have that $(N(f))_{t_v}^{id} = N(f_{t_v}^{id})$.*

Now we observe that changes of coordinates in the origin preserve the basic properties of the field:

LEMMA 14. *Given* $f\colon \mathbb{R}^n \to \mathbb{R}^n$, *its Newton's vector field in coordinates* (ϕ, id) *is equal to the pull-back of* $N(f)$ *through the diffeomorphism* ϕ^{-1}.

Proof. Particularizing expression (3.22) we obtain:

$$N(f_\phi^{id})(x) = -(d\phi(x))^{-1} \cdot df(\phi^{-1}(x)) \cdot (f \circ \phi^{-1})(x)$$
$$= (d\phi(x))^{-1} \cdot N(f)(\phi^{-1}(x)) = (\phi^{-1})^*(N(f))(x).$$
∎

A pull-back through a diffeomorphism preserves continuity or discontinuity of vector fields. The conclusion is that, thinking on the extension of the field, changes of coordinates in the origin preserve the caracteristcs of the vector field. However for changes of coordinates of type (id, Φ) we have:

$$N(f_\Phi^{id}) = f^*(N(\Phi)),$$

where Φ is interpreted as a field. In this case we can not conclude anything about the continuous extension of the field after making the change. Because in the singular points the pull-back is not well defined. For the study of generic 0-transformations we can give the following result:

PROPOSITION 8. *Given* $f\colon \mathbb{R}^n \to \mathbb{R}^n$, *its Newton's vector field in coordinates* (ϕ_1, ϕ_2) *is equal to the pull-back of* $N(f_{id}^{\phi_2})$, *interpreted as a vector field, through the diffeomorphism* ϕ_1^{-1}.

Proof. We only have to compare the two expressions. Writting formula (3.22):

$$N(f_{\phi_1}^{\phi_2}) = d\phi_1(\phi_1^{-1}(x)) \cdot df(\phi_1^{-1}(x)) \cdot d\phi_2^{-1}(\phi_2 \circ f \circ \phi_1 - 1(x))$$
$$\cdot (\phi_2 \circ f \circ \phi_1^{-1})(x). \tag{3.25}$$

Also we compute the pull-back of $N(f_{id}^{\phi_2})$:

$$N(f_{id}^{\phi_2}) = (df(x))^{-1} \cdot d\phi_2^{-1}(\phi_2 \circ f \circ \phi_1 - 1(x)) \cdot (\phi_2 \circ f \circ \phi_1^{-1})(x)$$
$$(\phi_1^{-1})^*(N(f_{id}^{\phi_2})(x)) = d\phi_1(\phi_1^{-1}(x)) \cdot df(\phi_1^{-1}(x)).$$
$$\cdot d\phi_2^{-1}(\phi_2 \circ f \circ \phi_1 - 1(x)) \cdot (\phi_2 \circ f \circ \phi_1^{-1})(x).$$

The two expressions coincide. ∎

3.2.2. CONTINUOUS EXTENSION AND CLASSES Σ^I.   We are going to give some results providing conditions for the existence of continuous extensions of the Newton's vector field based on the derivatives of the point. For this we need the use of the geometric classification of singularities. Now we define the classes Σ^I associated to an application. From now on we fix $f \colon \mathbb{R}^n \to \mathbb{R}^n$ a smooth application.

DEFINITION 15. A point $x \in \mathbb{R}^n$ will be of class Σ^k if the dimension of the kernel of $Jf(x)$ is equal to i. The set of points of class Σ^k is denoted by $\Sigma^k(f)$.

DEFINITION 16. Let $\Sigma^I(f)$ be a submanifold in $\mathbb{R}^n$, with $I = \{i_1, \ldots, i_p\}$. Then the set $\Sigma^{\hat{I}}(f)$, with $I = \{i_1, \ldots, i_p, i_{p+1}\}$ is:

$$\Sigma^{\hat{I}}(f) = \Sigma^{i_{p+1}}(f|\Sigma^I(f)).$$

These sets are not always well defined, we need that they are manifolds to be able to use the recursive construction. The main result we will use is:

THEOREM 5. *There is a Baire subset in the space of smooth applications from $\mathbb{R}^n$ to $\mathbb{R}^n$, with the Whitney topology, for which the classes $\Sigma^I(f)$ are well defined and are smooth submanifolds. An application in this subset is called good.*

The dimension of these submanifolds can be computed, see for example [3].

Remark 6. For 2-dimensional applications the allowed classes of points for good applications are:

1. $\Sigma^0(f) = \Omega(f)$ the set of regular points.
2. $\Sigma^{1,0}(f)$ the set of fold points.
3. $\Sigma^{1,1}(f)$ the set of pleat points.

In [24] it is shown that using a convenient change of coordinates any fold point can be written as the $(0,0)$ for the application:

$$f(x,y) = (x^2, y).$$

Equally a pleat point can be always written as the $(0,0)$ for:

$$f(x,y) = (x^3 + xy, y).$$

This result will help us to study the possibility of continuous extensions of the Newton's field in the bidimensional case. The definition of the classes Σ^I only uses the first k derivatives of the application at a given point, where k is a function depending on the multi-index I. So we can know if a point x^0 is in a given class $\Sigma^I(f)$ for an application f computing only the k-jet of f at the point x^0. This motivates the definition of the Σ^I classes in the spaces of k-jets $J^k(\mathbb{R}^n, \mathbb{R}^n)$, this definition is independent of any f for k large enough. Our objective in the following pages can be expressed in the next two definitions:

DEFINITION 17. A class Σ^I in the space $J^k(\mathbb{R}^n, \mathbb{R}^n)$, with k large enough to have a good definition, is homogeneously extensible if for every application f with a zero x^0 in the given class, the Newton's vector field can be defined continuously at x^0.

DEFINITION 18. A class Σ^I in the space $J^k(\mathbb{R}^n, \mathbb{R}^n)$, with k large enough to have a good definition, is homogeneously non-extensible if for every application f with a zero x^0 in the given class, the Newton's vector field can not be defined continuously at x^0.

DEFINITION 19. A class Σ^I in the space $J^k(\mathbb{R}^n, \mathbb{R}^n)$, with k large enough to have a good definition, is non-homogeneously extensible if it is not homogeneous extensible neither nonextensible.

The definitions are useful if there exist homogeneous classes of singularities. An obvious case is the class $\Sigma^0 = \Omega$ which is obviously homogeneously extensible. We will prove some results about the existence of homogeneously non-extensible classes. First at all we give a simple example of non-homogeneously extensible class. It is the class $\Sigma^{1,0}$ in 2 dimensions. We give two examples of applications, the first with a continuous extension, the second without such extension.

Continuous extension. We take:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2^2 \end{pmatrix}.$$

The function is good. The singular set is $\Sigma(f) = \{(x_1, x_2) \colon x_2 = 0\}$. The Newton's vector field is:

$$N(f)\begin{pmatrix} cx_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2/2 \end{pmatrix}.$$

The point $(0,0)$ is a zero of class $\Sigma^{1,0}$ and admits a continuous extension of the vector field.

Discontinuous extension. We take:

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2^2 + x_1 \end{pmatrix}.$$

The function is good. Again the singular set is $\Sigma(f) = \{(x_1, x_2) \colon x_2 = 0\}$. The Newton's vector field is:

$$N(f) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -\dfrac{x_2^2 + x_1}{2x_2} \end{pmatrix}.$$

The point $(0,0)$ is a zero of class $\Sigma^{1,0}$ but does not admit a continuous extension of the vector field.

Therefore the class $\Sigma^{1,0}$ is non-homogeneous. Now we will give an example of non-extensible class. To understand the problem we introduce the following result:

THEOREM 6. ([17]) *For a good application* $f \colon \mathbb{R}^n \to \mathbb{R}^n$ *the following affirmations are equivalent:*

1. $x^0 \in \Sigma^{1,\overset{(n)}{\ldots},1}(f)$.

2. *There exist local coordinates which allow to write* f *as:*

$$y_1 = x_1, \ldots, y_{n-1} = x_{n-1},$$
$$y_n = x_1 x_n + x_2 x_n^2 + \cdots + x_{n-1} x_n^{n-1} + x_n^{n+1}.$$

This kind of singularities are called generalized pleat points.

We are going to prove:

THEOREM 7. *Let* $x^0 \in \mathbb{R}^n$ *be a generalized pleat point for* $f \colon \mathbb{R}^n \to \mathbb{R}^n$ *then its associated Newton's vector field never admits a continuous extension at* x^0.

Proof. We give a proof for the bidimensional case (pleat points). The general case uses the same arguments, for a complete proof we refer to [20].

By Theorem 6 we know that in a neighborhood of x^0 we have $f = \bar{f}_{\phi_1}^{\phi_2}$ where ϕ_1, ϕ_2 are diffeomorphisms and:

$$\bar{f}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^3 + x_1 x_2 \\ x_2 \end{pmatrix}.$$

By Proposition 8 we know that f will admit a continuous extension if and only if it occurs for the application $\bar{f}_{id}^{\phi_2}$ at $\phi_1^{-1}(x^0) = (0,0)$. We are going to prove that $\bar{f}_{id}^{\phi_2}$ is never continuous at $(0,0)$, so the proof would be complete.

Particularizing formula (3.22) for this particular case we obtain:

$$N(\bar{f}_{id}^{\phi_2})(x) = (d\bar{f}(x))^{-1} \cdot (d\phi_2(\bar{f}(x)))^{-1} \cdot (\phi_2 \circ \bar{f})(x) \tag{3.26}$$

Expanding the precedent expression and taking into account:

$$\phi_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} v_1(y_1, y_2) \\ v_2(y_1, y_2) \end{pmatrix}$$

$$d\phi_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \partial v_1/\partial y_1 & \partial v_1/\partial y_2 \\ \partial v_2/\partial y_1 & \partial v_2/\partial y_2 \end{pmatrix}$$

$$\left(d\phi_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^{-1} = \frac{1}{\det(d\phi_2(y_1, y_2))} \cdot \begin{pmatrix} \partial v_2/\partial y_2 & -\partial v_1/\partial y_2 \\ -\partial v_2/\partial y_1 & \partial v_1/\partial y_1 \end{pmatrix}$$

$$\left(d\phi_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\det(d\phi_2(y_1, y_2))} \cdot \begin{pmatrix} (\partial v_2/\partial y_2)v_1 - (\partial v_1/\partial y_2)v_2 \\ -(\partial v_2/\partial y_1)v_1 + (\partial v_1/\partial y_1)v_2 \end{pmatrix}.$$

We will obtain that:

$$\begin{aligned} N(f_{id}^{\phi_2})(x) = &- \begin{pmatrix} 1/(3x_1^2 + x_2) & -x_1/(3x_1^2 + x_2) \\ 0 & 1 \end{pmatrix} \\ &\cdot \frac{1}{\det(d\phi_2(y_1, y_2))} \begin{pmatrix} (\partial v_2/\partial y_2)v_1 - (\partial v_1/\partial y_2)v_2 \\ -(\partial v_2/\partial y_1)v_1 + (\partial v_1/\partial y_1)v_2 \end{pmatrix} \\ = &\frac{1}{\det(d\phi_2(y_1, y_2))} \begin{pmatrix} \frac{\frac{\partial v_2}{\partial y_2}v_1 - \frac{\partial v_1}{\partial y_2}v_2 + x_1\left(\frac{\partial v_2}{\partial y_1}v_1 - \frac{\partial v_1}{\partial y_1}v_2\right)}{3x_1^2 + x_2} \\ -\frac{\partial v_2}{\partial y_1}v_1 + \frac{\partial v_1}{\partial y_1}v_2 \end{pmatrix}. \end{aligned} \tag{3.27}$$

If we want that the expression (3.27) will admit a continuous extension we will only have to impose this for the first component, because the term which

divides the second is no null (ϕ_2 is diffeomorphism). So we have only to check when the following function:

$$h(x_1, x_2) = \frac{g(x_1, x_2)}{3x_1^2 + x_2}, \tag{3.28}$$

with,

$$g(x_1, x_2) = \frac{\partial v_2}{\partial y_2} v_1 - \frac{\partial v_1}{\partial y_2} v_2 + x_1 \left(\frac{\partial v_2}{\partial y_1} v_1 - \frac{\partial v_1}{\partial y_1} v_2 \right), \tag{3.29}$$

would admit a continuous extension. If there would exist an application ϕ_2 getting it, we could change linearly the coordinates with a matrix A, such that $A \circ \phi_2$ would have the identity as jacobian at $(0,0)$, because $d(A \cdot \phi_2) = A \cdot d\phi_2$. By Lemma 12 in these coordinates the Newton's vector field is again continuous. Therefore, we can suppose that the functions v_1, v_2 are of the form:

$$v_1(y_1, y_2) = y_1 + o(\|y\|)$$
$$v_2(y_1, y_2) = y_2 + o(\|y\|).$$

Remember that differentiation transforms $o(\|y\|)$ expressions in $o(1)$ inequalities. So we can write the expression (3.29) as:

$$\begin{aligned}
g(x_1, x_2) =& (1 + o(1))(y_1 + o(\|y\|)) - o(1)(y_2 + o(\|y\|)) + x_1 o(1)(y_1 + o(\|y\|)) \\
& - x_1(1 + o(1))(y_2 + o(\|y\|)) = y_1 + o(1)y_1 - o(1)y_2 + x_1 o(1)y_1 \\
& - x_1 y_2 - x_1 o(1)y_2 + o(\|y\|) = y_1 - x_1 y_2 + o(\|y\|).
\end{aligned}$$

Finally, the expression of g, remembering that $\bar{f} = (y_1, y_2)$, can be written as:

$$\begin{aligned}
g(x_1, x_2) &= \frac{y_1 - x_1 y_2 + o(\|y\|)}{3x_1^2 + x_2} \\
&= \frac{x_1^3 + x_1 x_2 - x_1 x_2 + o(\|y\|)}{3x_1^2 + x_2} = \frac{x_1^3 + o(\|y\|)}{3x_1^2 + x_2}.
\end{aligned} \tag{3.30}$$

We have only to check that g is always discontinuous at origin. Concretely we know:

$$g(x_1, x_2) = \frac{x_1^3 + w(y_1, y_2)}{3x_1^2 + x_2}, \qquad dw(0,0) = (0,0), \qquad w(0,0) = 0. \tag{3.31}$$

Making the following change of coordinates:

$$z_1 = x_1, \qquad z_2 = 3x_1^2 + x_2.$$

The inverse change is:

$$x_1 = z_1, \qquad x_2 = z_2 - 3z_1^2.$$

It is a local diffeomorphism as a direct application of the inverse function theorem. Components of $\bar{f}$ ($\bar{f} = (y_1, y_2)$) as functions in (z_1, z_2) are:

$$y_1 = z_1 z_2 - 2z_1^3, \qquad y_2 = z_2 - 3z_1^2. \tag{3.32}$$

Again we can rewrite g in the following way:

$$g(z_1, z_2) = \frac{z_1^3 + w(y_1, y_2)}{z_2}.$$

Suppose g is continuous at $(0,0)$, then we can divide by z_2 the numerator, because Hadamard's Lemma assures that every function h with zero set containing the axis $e = \{(z_1, z_2) \in \mathbb{R}^2 \colon z_2 = 0\}$ must be $h = z_2 h^0$. For this we need that w adopts the form:

$$w(z_1, z_2) = \cdots - z_1^3 + \cdots \tag{3.33}$$

But notice that $w(y_1, y_2) = o(\|y\|)$, and so this is impossible, because the smallest degree terms in w will be $y_1^2, y_1 y_2, y_2^2$, and in the three cases by means of (3.32) we conclude that the smallest degree terms, in the single variable z_1, are of kind z_1^4. Therefore it cannot annul the term z_1^3 and it is always impossible to obtain a continuous extension. ∎

3.3. BEHAVIOUR NEAR SINGULAR ZEROES. In the precedent pages we have given some criteria to determine when the Newton's field admits a continuous extension to the singular set. However, in a lot of practical cases this extension is not possible. In the following paragraphs we are going to study what can be said in these cases. We will show that it is possible to give algebraic conditions that will guarantee that a singular zero is almost as good as a regular one. In general we obtain conical convergence results as Theorem 8. We start with an useful definition:

DEFINITION 20. A singular zero x^0 of $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is a generalized sink if there exists a neighborhood V of x^0, such that for every positive-maximal trajectory $x \colon (a, b) \to \mathbb{R}^n$ of the Newton's vector field verifying $x(a) \in \Omega(f) \cap V$ then:

$$\lim_{t \to b} x(t) = x^0.$$

A trajectory is positive-maximal if it does not admit an extension at b. This definition is the natural generalization of the concept of sink, obviously the regular zeroes of f are generalized sinks. Now, we give a condition for a singular zero to be a generalized sink.

PROPOSITION 9. *Given an isolated singular generalized sink x^0 of $f\colon \mathbb{R}^n \to \mathbb{R}^n$, if there exists a neighborhood V of x^0 verifying that all the points in $V \cap (\Sigma(f) - x^0)$ are transversal sources, then x^0 is a generalized sink.*

Proof. Take $g = \|f\|^2$. Then it verifies for every trajectory of the Newton's vector field:

$$g(x(t)) = g(x(0))e^{-2t}.$$

Being x^0 isolated we can find $\epsilon > 0$ such that $V_\epsilon = \{x \in \mathbb{R}^n \colon \|f(x)\| \le \epsilon\}$ is contained in V. This V_ϵ is a neighborhood of f which makes x^0 to verify the condition of generalized sink, because all the trajectories in V_ϵ can be extended to infinity, otherwise there would be trajectories going to the singular set. Also V_ϵ is invariant for the Newton's vector field, so the trajectories have as limit x^0. ∎

Now we define formally a concept that we have already used in Proposition 5.

DEFINITION 21. Given $f\colon \mathbb{R}^n \to \mathbb{R}^n$, we define the repulsion function associated to it as:

$$R\colon \Sigma(f) \to \mathbb{R}, \qquad R(x) = d(\det(df(x)))N_d(f)(x).$$

This function is zero for points in $\Sigma^k(f)$ with k greater than 1, because the differential of the determinant is zero in these points. For points in $\Sigma^1(f)$ the value of the repulsion function at a point x shows if the singular set attracts or repulses the Newton's vector field at this point. We can extract a simple corollary of Proposition 9:

COROLLARY 4. *Let x^* a singular zero of f. If there exists a neighborhood V of x^*, such that:*

$$R(x) > 0 \quad \text{for all } x \in V \cap (\Sigma(f) \setminus \{x^*\}),$$

then x^* is generalized sink.

As a conclusion we can give an analytical criterium to determine the character of a singular zero:

PROPOSITION 10. *Let x^* be a singular zero of $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Then x^* is a critical point for R the repulsion function associated to f. Under the hypothesis of the Hessian of R non-degenerate, if the Hessian of R restricted to $\Sigma(f)$ is definite positive then x^* is a generalized sink, otherwise x^* is not a generalized sink.*

Proof. First one we compute the jacobian of the desingularised Newton's vector field:

$$
\begin{aligned}
J(N_d(f))(x^*) &= J(-\operatorname{Adj}^T(Jf) \cdot f)(x^*) \\
&= -J(\operatorname{Adj}^T(Jf))(x^*) \cdot f(x^*) - -\operatorname{Adj}^T(Jf)(x^*) \cdot Jf(x^*) \\
&= -J(\operatorname{Adj}^T(Jf))(x^*) \cdot 0 - \det(Jf)(x^*) \cdot id = 0.
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
dR(x^*) &= d(d(\det(Jf))(N_d(f)))(x^*) \\
&= d(d(\det(Jf)))(x^*)(N_d(f)(x^*)) + +d(\det(Jf))(x^*)d(N_d(f))(x^*) \\
&= d(d(\det(Jf)))(x^*) \cdot 0 + d(\det(Jf))(x^*) \cdot 0 = 0.
\end{aligned}
$$

So x^* is a critical point for the repulsion function. A sufficient condition to obtain a local minimum of R is to impose that the Hessian of R must be positive definite. If it is the case we are in the hypothesis of Proposition 4, and x^* is a generalized sink. If the Hessian is not positive there exist points in $\Sigma(f)$ with negative value of R arbitrarily near x^*. The trajectories of Newton's vector field through these points finish on them, so x^* is never a generalized sink. ∎

We will end the discussion enunciating a result for more general situations. It can be found in [22], it is a natural extension of results as [21], [7], [8].

THEOREM 8. *Let x^* be a singular zero of a good application $f\colon \mathbb{R}^n \to \mathbb{R}^n$, and suppose that $x \in \Sigma^1(f)$. We shall denote $X = \Im df(x^*)$, $N = \operatorname{Ker} df(x^*)$. Suppose that $\mathbb{R}^n = X \oplus N$ and P_N, P_X the projections over these two subspaces. If we impose the following two conditions:*

1. $f''(x^*)NN \cup X = \{0\}$,
2. $\|f''(x^*)nv\| \geq c_1\|n\|\|v\|$, $n \in N$, $v \in \mathbb{R}^n$ and $c_1 > 0$,

where $f''(x^*)$ is the Hessian of the application f. Then there exist constants $\theta, \rho > 0$ such that the local cone:

$$\mathcal{C}_{x^*,\rho}^{N,X,\theta} = \{x \in \mathbb{R}^n \colon \|x - x^*\| < \rho \text{ and } \|P_X(x - x^*)\| \leq \|P_N(x - x^*)\|\theta\},$$

verifies:

1. $\mathcal{C}_{x^*,\rho}^{N,X,\theta} \setminus \{x^*\} \in \Omega(f)$.

2. x^* is the unique zero of f in $\mathcal{C}_{x^*,\rho}^{N,X,\theta}$.

3. $\mathcal{C}_{x^*,\rho}^{N,X,\theta}$ is invariant for the Newton's vector field. Also the limit of every trajectory in the cone is the zero x^*.

The theorem is quite geometric. In fact it is easy to assure conditions for a point to not converge to a zero using Proposition 3. In the precedent case an idea of the proof is based on the following fact. The values of f restricted to $\Sigma(f)$ in a neighborhood of the point are in a codimension 1 submanifold $f(\Sigma_f) \subset \mathbb{R}^n$. So for small neighborhoods the image is almost a plane. Therefore if we find a point x such that their image $f(x)$ is not in this plane then by the Proposition 3 cannot converge to $\Sigma(f)$. But such x can be find along the N subspace. Theorem 8 formalizes this idea.

4. Conclusions

Along this article we have generalized the classical Newton's vector field to the manifold setting. We have showed that most of the results obtained in $\mathbb{R}^n$ can be translated without any problem to a manifold equipped with a bundle E and a connection on it. In this case we define the Newton's method as a transformation with takes a section s of E and gives a vector field $N(s)$ on TM. The new vector field keeps the equilibrium points of s but it gets the asymptotic stability of all of them. The additional hypothesis of existence of a compatible metric have allowed us to assure global results of convergence. We have characterized the behaviour of the singular set $\Sigma(s)$ respect to the Newton's vector field studying when this set attracts the field. We give precise analytic conditions as Propositions 10 and 6.

All the precedent discussion give new ideas in two different directions. First one the vector field defined on manifolds gives a new tool in differential geometry. Second one the generalization has a feed-back effect allowing to define generalized methods in $\mathbb{R}^n$. Using in $\mathbb{R}^n$ non-standard connections we can change largely the convergence domain of the Newton's vector field.

Other result adapted to the manifold setting is the probabilistic convergence theorems. These theorems give the probability of arriving to an equilibrium point, following the Newton's vector field starting at a generic point. We have adapted the most classical result due to Hirsh and Smale [12]. This gives very strong convergence properties for a large class of proper sections. The implications of this kind of results in topology and geometry are unexplored but it could give an engine to prove existence results in differential topology.

Another of the questions studied in this article have been the possibility of obtaining continuous extensions of the vector field to the singular set. We have generalized the work [23]. In that article they give a criterium to check when an equilibrium point in $\Sigma^1(s)$ admits a continuous extension of the vector field. Now we have generalized this study for equilibrium points of good applications, but in $\Sigma^k(s)$ with $k \geq 1$.

We have proposed in this article a new way of studying the precedent problem by means of singularity theory. We have used the Boardman's classification of singularities and we have proved that for certain class of singularities the vector field can be never extended continuously (see Theorem 7). The conclusions are quite negative and in fact there are many cases where the vector field does not admit extension in. It brings the problem to the study of the general case of singular zeroes, we have given analytical conditions to assure the good behaviour of this kind of zeroes. In the general case we only enunciate the most important results, they are really developed for the discrete Newton-Raphson scheme but can be adapted to this setting. In the future it could be interesting to develop more the geometric sight of the method to give independent proofs of results as Theorem 8.

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