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The Best Algebraic Approximation in Hölder Norm

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1. Introduction

Let $C[-1,1]$ ($C_{2\pi}$) be Banach space of all real continuous (2π-periodic) functions f defined on $[-1,1]$ ($\mathbb{R}$), equipped with the sup norm. By Π_n (T_n), $n \in \mathbb{N}$, we denote the linear space of real algebraic (trigonometric) polynomials of degree at most n. The letter r will always denote a fixed positive integer and for $x \in [-1,1]$, we denote

$$\varphi(x) := \sqrt{1-x^2}. \tag{1}$$

For a function $f \in C[-1,1]$ ($f \in C_{2\pi}$), the best algebraic (trigonometric) approximation of f out of Π_n (T_n) is defined by

$$E_n(f) := \inf_{P\in\Pi_n} \|f-P\|_\infty \quad \big(T_n(f)) := \inf_{T\in T_n} \{\|f-T\|_\infty\big).$$

For an interval Ω of the real axis $\mathbb{R}$, $t>0$ and a function $f \in C(\Omega)$, the symmetric divided difference $\Delta_h^r f(x)$ is defined by

$$\Delta_h^r f(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+(\frac{r}{2}-j)h), \tag{2}$$

whenever $x \pm rh/2 \in \Omega$ and the usual modulus of smoothness of order r is defined by

$$\omega_r(f,t) := \sup_{h\in(0,t]} \sup_{x\in I(r,h,\Omega)} |\ \Delta_h^r f(x)\ |,$$

where $I(r,u,\Omega) := \{z \in \Omega : z \pm (ru)/2 \in \Omega\}$. When $\Omega = \mathbb{R}$, the restriction $x \in I(r,h,\Omega)$ is replaced by $x \in \mathbb{R}$. For functions $f \in C[-1,1]$ we will consider $\Omega = [-1,1]$ and for periodic functions f we consider $\Omega = \mathbb{R}$.

It is well known that, for a function $g \in C_{2\pi}$ and a real $\beta \in (0,1)$, the assertions $T_n(g) = O(n^{-\beta})$ and $\omega_1(g,\delta) = O(\delta^{\beta})$ are equivalent. Unfortunately, if we consider functions $f \in C[-1,1]$ and the modulus $\omega_1(f,t)$, such a result does no hold for the best algebraic approximation in $C[-1,1]$. This later phenomenon was first recognized by S. N. Nikolskii.

The remark above motivates the following problem: to characterize, in terms of a new modulus of smoothness, those functions $f \in C[-1,1]$ for which $E_n(f) = O(n^{-\beta})$. The problem has been studied by different authors. The main idea is to replace the classical translation h, which appear in Δ_h^r (see (2)), by a new one which take into account the end-point effect. We remit to [2] to avoid a long list of references.

Since the results of this paper are closed related with those of [2], we present the Ditzian-Totik modulus of smoothness in details. For function $f \in C[-1,1]$ and $t > 0$ define

$$\omega_r^{\varphi}(f,t) := \sup_{h\in(0,t]} \sup_{x\in J(r,h)} \mid \Delta^r_{\varphi(x)h} f(x) \mid, \tag{3}$$

where $J(r,h) := \{z \in [-1,1] : z \pm (rh\varphi(z))/2 \in [-1,1]\}$. It is known that ([2, p. 83]), for a function $f \in C[-1,1]$ and a real $\beta \in (0,r)$ we have

$$E_n(f) = O(n^{-\beta}) \quad \Longleftrightarrow \quad \omega_r^{\varphi}(f,t) = O(t^{\beta}). \tag{4}$$

The aim of this paper is to study approximation in Hölder (Lipschitz) norms by means of algebraic polynomials. Thus, for Hölder-type spaces of continuous functions we introduce a modulus of smoothness corresponding to (3).

Definition 1. For a fixed $\alpha \in (0,1)$ and φ given by (1), we define the Hölder class with respect to φ as the linear space $\mathrm{lip}_{\alpha}[-1,1] := \mathrm{lip}_{\alpha}^{\varphi}[-1,1]$ of all functions $f \in C[-1,1]$ such that

$$\lim_{t\to 0^+} \frac{\omega^{\varphi}(f,t)}{t^{\alpha}} = 0.$$

The space $\mathrm{lip}_{\alpha}[-1,1]$ is equipped with the norm

$$\|f\|_{\alpha} := \|f\|_{\infty} + \sup_{h>0} h^{-\alpha}\omega_{\varphi}(f,h)$$

and the modulus of smoothness

$$\theta^{\varphi}_{r,\alpha}(f,t) := \sup_{h\in(0,t]} \frac{\omega_r^{\varphi}(f,h)}{h^{\alpha}}. \tag{5}$$

Notice that, for each $n \in \mathbb{N}$, $\Pi_n \subset \mathrm{lip}_\alpha[-1,1]$. This allows us to define, for $f \in \mathrm{lip}_\alpha[-1,1]$, the best algebraic approximation in Hölder norm by

$$E_{n,\alpha}(f) := E^{\varphi}_{n,\alpha}(f) := \inf_{P\in\Pi_n} \|f-P\|_\alpha.$$

2. Main results

Our main result is the following theorem

Theorem 2. *Fix $\alpha \in (0,1)$, φ as in (1) and a positive integer r. There exist positive constants $C_{r,1}$ and $C_{r,2}$ such that, for every $f \in \mathrm{lip}_\alpha[-1,1]$ each $n > r$ and every $t \in (0,1)$,*

$$C_{r,1}\, E_{n,\alpha}(f) \leq \theta^{\varphi}_{r,\alpha}(f,\frac{1}{n}) \leq C_{r,2}\, t^{r-\alpha} \sum_{0\leq k\leq t} k^{r-\alpha-1} E_{k,\alpha}(f).$$

As a corollary we have (compare with (4)): For a function $f \in \mathrm{lip}_\alpha[-1,1]$ and $\beta \in (0, r-\alpha)$ it holds that,

$$E_{n,\alpha}(f) = O(n^{-\beta}) \quad \Longleftrightarrow \quad \theta^{\varphi}_{r,\alpha}(f) = O(t^{\beta}).$$

Our proof of Theorem 2 follows a general method presented by to P. L. Butzer and K. Scherer in [1] together with the ideas developed by Z. Ditzian and V. Totik in [2]. The arguments depend on a characterization of the modulus of smoothness (5) in terms of Petree K-functionals and some Jackson-type and Bernstein-type inequalities.

Let us denote by W^r the family of all functions $g \in C[-1,1]$, such that, on each closed interval $[a,b] \subset (-1,1)$, g is r-times continuously differentiable. We consider two seminorms in W^r defined by

$$|\, g\,|_{W^r_\varphi} := \sup_{x\in(-1,1)} |\varphi^r(x) g^{(r)}(x)| \quad \big(\,|\, g\,|_{W^r} := \sup_{x\in(-1,1)} |g^{(r)}(x)|\big).$$

Definition 3. Fix $\alpha \in (0,1]$ and φ as in (1). For $f \in \mathrm{lip}_\alpha[-1,1]$ and $t > 0$, the associated K-functionals are defined by

$$K_{r,\alpha}(f,t) := \inf_{g\in W^r} \left\{ \|f-g\|_\alpha + t\,|\, g\,|_{W^r_\varphi} \right\}$$
$$K^*_{r,\alpha}(f,t) := \inf_{g\in W^r} \left\{ \|f-g\|_\alpha + t\,|\, g\,|_{W^r_\varphi} + t^{(2r-\alpha)/(r-\alpha)}\,|\, g\,|_{W^r} \right\}.$$

The relations between the modulus of smoothness and the K-functional are stated in Theorem 4. We remark that the Jackson-type and Bernstein-Potapov-type inequalities given in Theorem 5 have independent interest.

THEOREM 4. *For each positive integer r and each $\alpha \in (0,1)$ there exist positive constants C_1 and C_2 (which depend on r) such that, for every $f \in \mathrm{lip}_\alpha C[-1,1]$ and $t \in (0,1/r]$*

$$C_1\theta_{r,\alpha}(f,t) \le K_{r,\alpha}(f,t^{r-\alpha}) \le K^*_{r,\alpha}(f,t^{r-\alpha}) \le C_2\theta_{r,\alpha}(f,t).$$

THEOREM 5. *Let r be a positive integer and $\alpha \in (0,1)$.*

(i) (Jackson-type inequality) *There exists a positive constant C_r such that, for each $n > r$ and every $g \in W^r$,*

$$E_{n,\alpha}(g) \le C_r n^{-r+\alpha}\left(| g |_{W^r_\varphi} + n^{-r} | g |_{W^r}\right).$$

(ii) (Bernstein-Potapov-type inequality) *There exists a positive constant $C := C(r,\alpha)$ such that, for each $n \in \mathbb{N}$ and $P \in \Pi_n$*

$$| P |_{W^r_\varphi} \le C n^{r-\alpha}\|P\|_\alpha.$$

In this paper we have studied spaces of Hölder continuous functions, but the arguments can be modified for obtaining analogous results in Hölder spaces of integrable functions (see, for example, Chapter 2 of [2]). Here we have worked with the weight function $\varphi(x) = \sqrt{1-x^2}$. But Theorem 4 can be extended to include more general weights as in [2]. Extensions to compact interval of the form $[a,b]$ $(a,b \in \mathbb{R})$ follow easily.

References

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