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Reflexive Spaces and Numerical Radius Attaining Operators*

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In this note we deal with a version of James' Theorem for numerical radius, which was already considered in [4]. First of all, let us recall that this well known classical result states that a Banach space satisfying that all the (bounded and linear) functionals attain the norm, has to be reflexive [16].

Before to state the results, let us recall the definition of numerical radius and introduce some basic notation. B_X and S_X will be the unit ball and unit sphere, respectively, of a Banach space X, X^* its topological dual and $L(X)$ the space of all bounded and linear operators on X (endowed with the usual operator norm). The *numerical range* of an operator $T \in L(X)$ is the set of scalars

$$V(T) := \{x^*(Tx) : (x, x^*) \in \Pi(X)\},$$

where $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. The *numerical radius* of such an operator T is the real number

$$v(T) := \sup\{|\lambda| : \lambda \in V(T)\}$$

and T *attains the numerical radius* if

$$\exists\ (x_0, x_0^*) \in \Pi(X) : |x_0^*(Tx_0)| = v(T).$$

The definition of the numerical range is due to Bauer [6] and one can find a good survey of the properties of numerical ranges in the monographs [10, 11]. As an example of these properties, let us say that the numerical radius is a

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continuous seminorm less than the norm. In the complex case, it is, in fact, a norm equivalent to the usual operator norm (Bohnenblust-Karlin Theorem [10, Theorem 4.1]).

It will be useful for us to use the set

$$\hat{\Pi}(X) := \{(x, x^*) \in S_X \times S_{X^*} : |x^*(x)| = 1\},$$

for which we have

$$v(T) = \sup\{\text{Re } x^*(Tx) : (x, x^*) \in \hat{\Pi}(X)\}.$$

It is clear that an operator T attains its numerical radius if, and only if, there exists an element $(x_0, x_0^*) \in \hat{\Pi}(X)$ such that $x_0^*(Tx_0) = \text{Re } x_0^*(Tx_0) = v(T)$.

James' Theorem can be stated in terms of operators as follows. A Banach space is reflexive if, and only if, each rank-one operator attains the norm. Then, a version of this result for numerical radius would characterize reflexive Banach spaces as those Banach spaces for which every rank-one operator attains its numerical radius. In [4] this version of James' Theorem was proved:

THEOREM 1. *A Banach space is reflexive provided that each rank–one operator on it attains its numerical radius.*

In fact, by refining the arguments used in [4], it can be showed that reflexivity is implied by the assumption that every rank-one operator whose range is contained in a fixed one-dimensional subspace attains its numerical radius. The proof of Theorem 1 given in [4] uses a "non sequential" version of Simons' inequality (see [23]). Here we will give an easier proof of the above result in the case that the Banach space is separable. However, the argument used in this special case contains the essential idea of the general one and it also has the advantage that Simons' inequality can be used instead of a quite general and more intricate inequality used in [4, Theorem 1].

In the following, co will denote the convex hull of a subset in a linear space and $\ell_\infty(B)$ will be the Banach space of real-valued bounded functions on B.

LEMMA 1. (Simons' inequality [22]) *Let B be a set, $A \subset B$ and $\{f_n\}$ a bounded sequence of functions in $\ell_\infty(B)$ such that for any sequence $\{t_n\}$ of non negative real numbers with $\sum_{n=1}^{\infty} t_n = 1$ there exists $a \in A$ satisfying*

$$\sum_{n=1}^{\infty} t_n f_n(a) = \sup_{b \in B} \sum_{n=1}^{\infty} t_n f_n(b).$$

Then

$$\sup_{a\in A}\limsup_{n} f_n(a) \geq \inf_{g\in \text{co}\{f_n\}} \sup\{g(b) : b \in B\}.$$

E. Oja got a slightly different proof of this inequality by Simons [19]. Recently, R. Deville and C. Finet got some improvement of Simons' inequality (see [15]). Also, a book by S. Simons [24] including some non sequential versions of Simons' inequality has appeared. These more general results, called minimax theorems, can be used to characterize weak compactness (see [23, 24]).

We now consider a stronger statement than Theorem 1:

THEOREM 2. *Let X be a Banach space such that for some $x_0 \in S_X$ the operator $x^* \otimes x_0$ attains its numerical radius, for any x^* in X^*, then X is reflexive.*

Proof of the separable case. We will argue by contradiction. So, let us suppose that X is separable and not reflexive but every operator $x^* \otimes x_0$ attains its numerical radius.

The Bishop-Phelps Theorem [8] (density of the set of norm attaining functionals for any Banach space) and the non reflexivity of X allow us to find $(x_0^*, x_0^{**}) \in \Pi(X^*)$ satisfying

$$\|x_0^{**} - x_0\| < 1 \qquad \text{and} \qquad x_0^{**} \notin X,$$

so,

$$|x_0^{**}(x_0^*) - x_0^*(x_0)| < 1$$

and then, taking into account that $x_0^{**}(x_0^*) = 1$, we have

$$x_0^*(x_0) \neq 0 \qquad \text{and} \qquad \alpha x_0^*(x_0) = 1, \tag{1}$$

for some scalar $\alpha \neq 0$. Now by virtue of the Hahn-Banach Theorem, there exist $\varphi \in S_{X^{***}}$ and $r > 0$ such that

$$\varphi(x) = 0, \quad \forall x \in X$$

and

$$\text{Re}\ \varphi(x_0^{**}) > r.$$

X is separable, so in X^{***} the topology of pointwise convergence on $X \cup \{x_0^{**}\}$ is metrizable on bounded sets. Since S_{X^*} is w^*-dense in $S_{X^{***}}$, there is a sequence $\{x_n^*\}$ in S_{X^*} converging to φ in $\sigma(X^{***}, X \cup \{x_0^{**}\})$. Then

$$\{x_n^*(x)\} \to 0, \quad \forall x \in X \tag{2}$$

and we can assume that

$$\text{Re } x_0^{**}(x_n^*) > r, \quad \forall n \in \mathbb{N}. \tag{3}$$

Now we make use of Simons' inequality, taking as sets

$$A := \hat{\Pi}(X) \qquad \text{and} \qquad B := \hat{\Pi}(X^*)$$

(A is considered as a subset of B by the natural inclusion of X into X^{**} and by considering the reverse order) and the sequence of bounded functions $f_n : \hat{\Pi}(X^*) \to \mathbb{R}$ given by

$$f_n(x^*, x^{**}) = \text{Re } x^{**}(x_n^*)x^*(x_0), \quad ((x^*, x^{**}) \in \hat{\Pi}(X^*)).$$

We check that the assumption in the previous lemma is satisfied. For each sequence $\{t_n\}$ with $0 \leq t_n \leq 1$ and $\sum_{n=1}^{\infty} t_n = 1$ we clearly have

$$\sum_{n=1}^{\infty} t_n f_n(x^*, x^{**}) = \text{Re } x^{**}\left(\sum_{n=1}^{\infty} t_n x_n^*\right) x^*(x_0), \quad \forall (x^*, x^{**}) \in \hat{\Pi}(X^*).$$

We are assuming that the rank-one operator $(\sum_{n=1}^{\infty} t_n x_n^*) \otimes x_0$ attains its numerical radius and it always happens for any operator T the coincidence $v(T) = v(T^*)$ (see [11, Corollary 17.3]). Then the convex series $\sum_{n=1}^{\infty} t_n f_n$ attains the supremum at a point of A. So we obtain

$$\begin{aligned} &\sup_{(x,x^*)\in\hat{\Pi}(X)} \limsup_n \text{ Re } x_n^*(x)x^*(x_0) \\ &\qquad \geq \inf_{y^*\in \text{co}\{x_n^*\}} \sup_{(x^*,x^{**})\in\hat{\Pi}(X^*)} \text{Re } x^{**}(y^*)x^*(x_0). \end{aligned} \tag{4}$$

Now then, by (2)

$$\sup_{(x,x^*)\in\hat{\Pi}(X)} \limsup_n \text{Re } x_n^*(x)x^*(x_0) = 0 \tag{5}$$

and by virtue of (3) and (1), if $y^* \in \text{co}\{x_n^*\}$ then

$$\text{Re } x_0^{**}(y^*)\frac{\alpha}{|\alpha|}x_0^*(x_0) > \frac{r}{|\alpha|},$$

so

$$\inf_{y^*\in \text{co}\{x_n^*\}} \sup_{(x^*,x^{**})\in\hat{\Pi}(X^*)} \text{Re } x^{**}(y^*)x^*(x_0) \geq \frac{r}{|\alpha|}. \tag{6}$$

To sum up, from (4), (5) and (6) we have

$$0 \geq \frac{r}{|\alpha|},$$

but $r > 0$ and we have a contradiction. ∎

Now, we will consider the reverse implication in James' Theorem for numerical radius. In a reflexive Banach space, it is obvious that any functional attains the norm, that is, any rank–one operator on it attains the norm. By considering the numerical radius (instead of the norm), we proved in [4] that a reflexive Banach space with basis can be renormed so that some rank–one operator does not attain its numerical radius. Here we will try to explain the idea of the proof.

THEOREM 3. *Any infinite–dimensional Banach space can be renormed so that there is a rank-one operator not attaining its numerical radius.*

Idea of the proof. If the space is not reflexive, then by Theorem 1, in fact, the previous statement holds for any equivalent norm.

So, we will assume that the Banach space, call it X, is reflexive. If one tries to construct an operator not attaining the numerical radius, things could be easier if one gets for this operator the coincidence of the numerical radius and the norm (otherwise it is difficult to get consequences by assuming that the operator attains its numerical radius). Let us observe that in any infinite-dimensional and reflexive Banach space that coincidence does not hold for every rank-one operator (see [18, Corollary 5]). But for normalized elements $z_0 \in S_X$, $x_0^* \in S_{X^*}$, equality

$$v(x_0^* \otimes z_0) = \|x_0^* \otimes z_0\| = 1$$

holds immediately if $|x_0^*(z_0)| = 1$, since $v(x_0^* \otimes z_0)$ is attained at $(z_0, x_0^*) \in \hat{\Pi}(X)$. The original idea consisted on constructing (after renorming) such an operator for which the numerical radius is attained just in the described situation. Following this idea, if $v(x_0^* \otimes z_0) = 1$ and the numerical radius is attained at $(x, x^*) \in \Pi(X)$, then, it happens

$$|x^*(z_0)| = 1 = |x_0^*(x)|$$

and if we rotate the elements x and x^* we get to a new couple still in $\hat{\Pi}(X)$ and satisfying

$$x^*(z_0) = 1 = x_0^*(x). \tag{7}$$

If we want that equalities (7) determine x^* (after rotation), it is enough to assume that z_0 is a smooth point. Let us call z_0^* the unique point of the unit sphere of X^* such that $z_0^*(z_0) = 1$. The smoothness of z_0 gives us $x^* = z_0^*$. Since $(x, z_0^*) = (x, x^*) \in \hat{\Pi}(X)$, x will be uniquely determined (up to rotation) if we assume that z_0^* is also smooth and so $x = \lambda z_0$ (for some $|\lambda| = 1$) and

$$(x, x^*) = (\lambda z_0, z_0^*).$$

By using again (7), $x_0^*(\lambda z_0) = x_0^*(x) = 1$, then the smoothness of z_0 gives us $\lambda x_0^* = z_0^* = x^*$. Finally, the couple (x, x^*) is, unless a rotation, (z_0, x_0^*).

So, it would be enough to construct the operator $x_0^* \otimes z_0$ satisfying

$$v(x_0^* \otimes z_0) = \|z_0\| = \|x_0^*\| = 1,$$

with z_0, z_0^* smooth and so that $x_0 \notin \mathbb{K} z_0$, for some $x_0 \in S_X$ such that $x_0^*(x_0) = 1$.

Of course, if the numerical radius of the operator is 1, then, there should be a sequence $\{(x_n, x_n^*)\} \subseteq \Pi(X)$ so that

$$\{x_n^*(z_0)\} \to 1, \qquad \{|x_0^*(x_n)|\} \to 1. \tag{8}$$

By using the inequality

$$2 \geq \|x_n + z_0\| \geq x_n^*(x_n + z_0)$$

and (8) it follows that $\{\|x_n + z_0\|\} \to 2$. Also, if x_0 is a w-cluster point of $\{x_n\}$, (8) will also give us $|x_0^*(x_0)| = 1$.

Conversely, if $\{\|x_n + z_0\|\} \to 2$ and $\{x_n\}$ converges in the w-topology to an element x_0 in the unit sphere, then there is a sequence of norm one functionals $\{x_n^*\}$ so that the sequences $\{x_n^*(x_n)\}$ and $\{x_n^*(z_0)\}$ converge to 1. The Bishop–Phelps–Bollobás Theorem [9] allows us to assume that, in fact, $x_n^*(x_n) = 1$ and so, if we fix an element x_0^* in the unit sphere of the dual so that $x_0^*(x_0) = 1$, we get

$$\lim_n x_0^*(x_n) = x_0^*(x_0) = 1, \qquad \lim x_n^*(z_0) = 1,$$

and so $v(x_0^* \otimes z_0) \geq \sup_n |x_0^*(x_n) x_n^*(z_0)| \geq 1$, that is, the numerical radius of the operator is 1. ∎

In the next statement proved in [4, Proposition 2] we collect the conditions we have just mentioned in order to construct a rank-one operator not attaining its numerical radius:

PROPOSITION 1. *Let X be a Banach space and assume that there exist $z_0, x_n \in B_X$ and $x_0 \in S_X$ satisfying*

(i) $\{\|x_n + z_0\|\} \to 2$,

(ii) z_0 *and* z_0^* *are smooth points, where* z_0^* *is the unique support functional of* B_X *at* z_0,

(iii) $\{x_n\} \to x_0$ *weakly and*

(iv) $x_0 \notin \mathbb{K} z_0$.

Then there exists $x_0^* \in S_{X^*}$ *such that the operator* $x_0^* \otimes z_0$ *does not attain its numerical radius.*

In a reflexive Banach space with a (normalized) Schauder basis $\{(e_n, e_n^*)\}$ it is easy to renorm in such a way that the conditions in the above proposition hold. For instance, we can take $z_0 = e_1$, $x_0 = e_2$ and $x_n = e_2 + e_n$. Clearly (iv) is satisfied and also does (iii), since the basis is shrinking in this case. In order to get conditions (i) and (ii) it suffices to construct a norm for which the unit ball contains ellipsoidal sections containing $z_0 = e_1$. This idea suggests to define a new norm by using the Minkowski functional of the set

$$B = \operatorname{co}\left\{\frac{1}{3K} B_X \cup \overline{\operatorname{aco}}\{e_2, e_2 + e_n : n \geq 3\} \cup A\right\},$$

where B_X is the unit ball of the original norm, K is the basic constant of $\{e_n\}$, $\overline{\operatorname{aco}}$ denotes closed absolutely convex hull and A is the ellipsoid given by

$$A = \left\{x \in X : \sum_{n=1}^{\infty} \frac{|e_n^*(x)|^2}{\varepsilon_n^2} \leq 1\right\},$$

for some sequence $\{\varepsilon_n\} \in \ell_1$ satisfying $\varepsilon_1 = 1$ and $\varepsilon_{n+1} < \varepsilon_n$, $\forall n \in \mathbb{N}$. A is compact and B is the closed unit ball of an equivalent norm satisfying the previous conditions (see [4, Example]).

The proof of the general renorming result can be found in [5].

Let us point out that there is a collection of results about denseness of numerical radius attaining operators (see for instance [7, 13, 21, 1, 2, 14]). It can be observed that any time a result is known for the more classical case, by considering norm attaining operators, there is also an appropriate version for the numerical radius. For instance, this happens with the original result by Lindenstrauss of denseness of operators whose second adjoints attain their norm (see [17, 25] and [3]). It also holds for the result by Bourgain [12] and the parallel version for numerical radius [3], the counterexample [17] and [21] and the renorming results [20] and [1]. James' Theorem seems to be the first known result that holds for the norm but not for the numerical radius.

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