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Valdivia Compact Spaces in Topology and Banach Space Theory *

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1. Introduction

Σ-products and their subspaces have been studied since 1950s. Probably the first important paper on this topic appeared in 1959. It was the paper of Corson [10] containing several results on normality of Σ-products of complete metric spaces. In the same year another paper, due to Glicksberg [19], appeared. It dealed with Čech-Stone compactifications of products. Among other things it contained a theorem saying that the Čech-Stone compactification of a Σ-product of compact spaces is the respective product. During the next twenty years Σ-products and their generalizations were studied by Russian mathematicians B. Efimov [15], A.P. Kombarov [38, 39, 40], V.I. Malyhin [41], A.V. Efimov and G.I. Certanov [14], S.P. Gul'ko [22]. We will not deal with Σ-product of general topological spaces but we pay attention to the Σ-product of real lines, which is the space

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}$$

equipped with the topology of pointwise convergence, inherited from the product space $\mathbb{R}^\Gamma$. The class of compact spaces lying in $\Sigma(\Gamma)$ for some Γ became interesting as, due to Amir and Lindenstrauss [2], it contains all *Eberlein compact* spaces. A compact space is called Eberlein if it is homeomorphic to a weakly compact subset of a Banach space. By the mentioned result of [2] a

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compact space is Eberlein if and only if it is homeomorphic to a subset of the space

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \forall \varepsilon > 0 \ \{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\}$$

for some set Γ. Due to a result of Grothendieck [20] (which is in this special case elementary) it does not matter whether we consider $c_0(\Gamma)$ with the weak topology or with the topology of pointwise convergence. In [6] Y. Benyamini, M.E. Rudin and M. Wage showed that Eberlein compacta are stable with respect to continuous images. In the same paper they gave a consistent example of non-Eberlein compact space lying in $\Sigma(\Gamma)$. In the same year E. Michael and M.E. Rudin [47] gave another somewhat simpler proof of the mentioned stability result. Further, they introduced the notion of *Corson compact* space for compact spaces lying in $\Sigma(\Gamma)$. They claimed that Corson compact spaces are stable with respect to continuous images, and that the proof is *virtually the same* as their proof for Eberlein compacta. Such a proof can be called *similar*, but some not completely trivial additional work is needed. In the same year S.P. Gul'ko [22] independently obtained a stronger result on stability of countably compact subsets of $\Sigma(\Gamma)$ to quotient mappings. Another two proofs of stability of Corson compacta to continuous images were given by G. Gruenhage [21]. It is done using some covering properties which characterize Corson compact spaces, and also with the aid of certain games.

In 1980s many functional-analytic properties of Corson compact spaces were established. This investigation had, roughly speaking, two directions. The first one was studying which properties of Eberlein compacta and weakly compactly generated spaces are still valid for Corson compacta and associated Banach spaces. The second one was studying how large is the gap between Eberlein compacta and Corson compacta. As for the latter, the first consistent example of a non-Eberlein Corson compactum was given already in [6]. Absolute examples were given by K. Alster and R. Pol [1] and M. Talagrand [60]. Further examples are in [61] and [5]. On the other hand, in [1] it is proved that the space of continuous functions on a Corson compact space is Lindelöf in the pointwise topology. Later R. Pol [59] gave a characterization of Corson compact spaces in terms of a Lindelöf type property of the space of continuous functions, see [3, Section IV.3]. For the weak topology, the situation is different. Under continuum hypothesis there exists a Corson compactum with the space of continuous functions not being weakly Lindelöf, see [5].

M. Valdivia in [63] started to study a class of Banach space defined by a Corson type property. These were those Banach spaces for which there is a linear, one-to-one weak* continuous mapping of the dual into $\Sigma(\Gamma)$ for a set Γ.

This class was given name *weakly Lindelöf determined* spaces by S. Argyros and S. Mercourakis in [4]. It follows from [51] (for an easy proof see [18]) that these are exactly the Banach spaces with Corson dual unit ball.

In [63] M. Valdivia constructed a projectional resolution of the identity operator on every weakly Lindelöf determined space. S. Argyros, S. Mercourakis and S. Negrepontis [5] observed that some properties of Corson compact spaces are shared by a larger class. In particular, they proved the existence of the retractional resolution of the identity on every compact subset of $\mathbb{R}^\Gamma$ which has a dense intersection with $\Sigma(\Gamma)$. This class was further studied by M. Valdivia [64], [65]. R. Deville and G. Godefroy [11] introduced the name *Valdivia compact* spaces for this class.

Valdivia compact spaces are closely related with Markushevich bases, see [65]. An independent investigation in this area was done by A. Plichko, see e.g. [54, 53, 55, 56].

Recently several results on structure of Valdivia compact spaces were obtained by the author. They contain strong non-stability properties of the class of Valdivia compacta [30, 32, 34]. Some positive stability results, together with a characterization of Valdivia compacta generalizing R. Pol's characterization of Corson compacta, are given in [31]. Interferences of Valdivia compacta with Asplund spaces are studied in [35].

The aim of this survey is to give an exposition on structure of Valdivia and Corson compact spaces. We would like to include elementary facts and mainly the recent results, as well as include basic open questions. The paper is organized as follows.

The rest of the first chapter contains definitions of basic notions and the development of several (more or less elementary, but very powerful) technical tools which will be often used in the sequel.

The second chapter is devoted to the characterization of Valdivia compact spaces and a related class of Banach spaces. This generalizes Pol's characterization of Corson compact spaces and has several applications in the following chapters.

The third chapter deals with topological properties of the class of Valdivia compacta. It contains many stability and non-stability results.

In the fourth chapter we study some classes of Banach spaces associated with Valdivia compacta. Some structural properties (resolutions of the identity, Markushevich bases) are included, as well as stability and non-stability results.

The fifth chapter is devoted to $C(K)$ spaces. It contains results on duality (i.e. on the relations of Valdivia properties of K and the space $C(K)$), as well as stability and non-stability results on $C(K)$ spaces.

In the last chapter we collect some illustrative examples of Valdivia compact spaces and Valdivia type Banach spaces.

We will use basic notions from topology and Banach space theory. All topological spaces are assumed to be Hausdorff. For a background on topology we refer to [16]. That on Banach spaces can be found in several classical books or in the nice recent lecture notes [24].

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1.1. Σ-subsets, Corson and Valdivia compacta – basic definitions In this section we give definitions of basic notions which will be used throughout all the paper. We begin by recalling the following notation.

Notations. Let Γ be an arbitrary set.
(i) For $x \in \mathbb{R}^\Gamma$ we put $\operatorname{supp} x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$.
(ii) We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \operatorname{supp} x \text{ is countable}\}$.

Next we recall the definition of Corson and Valdivia compact spaces. We do it using an auxiliary notion of Σ-subset. This turned out to be useful to make the expression easier.

Definition 1.1. Let K be a compact space.
(i) We say that $A \subset K$ is a *Σ-subset of K* if there is a homeomorphic injection h of K into some $\mathbb{R}^\Gamma$ such that $h(A) = h(K) \cap \Sigma(\Gamma)$.
(ii) K is called a *Corson compact space* if K is a Σ-subset of itself.
(iii) K is called a *Valdivia compact space* if K has a dense Σ-subset.
(iv) We say that K is a *super-Valdivia compact space* if each $x \in K$ is contained in some dense Σ-subset of K.

Let us remark that the ordinal segment $[0, \omega_1]$ is a Valdivia compact space which is not super-Valdivia, in fact its only dense Σ-subset is the open interval $[0, \omega_1)$ (see Example 1.10 below). Further, the cube $[0,1]^\Gamma$ is a super-Valdivia compact space which is not Corson for any uncountable Γ (see Theorem 3.29 below).

In study of Banach spaces naturally appear compact spaces which belong to more general classes. This is the reason why we introduce the following definition.

DEFINITION 1.2. Let κ be an infinite cardinal.

(i) If Γ is any set, we put $\Sigma_\kappa(\Gamma) = \{x \in \mathbb{R}^\Gamma : \operatorname{card} \operatorname{supp} x < \kappa\}$.

(ii) We say that $A \subset K$ is a *Σ_κ-subset of K* if there is a homeomorphic injection h of K into some $\mathbb{R}^\Gamma$ such that $h(A) = h(K) \cap \Sigma_\kappa(\Gamma)$.

(iii) K is called a *κ-Corson compact space* if K is a Σ_κ-subset of itself.

(iv) K is called a *κ-Valdivia compact space* if K has a dense Σ_κ-subset.

In this setting Corson means the same as $\aleph_1$-Corson, and Valdivia is simply $\aleph_1$-Valdivia. Similar classes with a different notation were studied for example in [15].

Corson and Valdivia compacta can be characterized using the notion of a separating family. Let us recall this notion.

DEFINITION 1.3. Let X be a set and $\mathcal{U}$ a family of subsets of X.

(i) For any $x \in X$ we put $\mathcal{U}(x) = \{U \in \mathcal{U} : x \in U\}$.

(ii) We say that $\mathcal{U}$ *separates points of* X if, whenever $x, y \in X$ are distinct, then there is $U \in \mathcal{U}$ containing exactly one of the points x, y; i.e. $\mathcal{U}(x) \neq \mathcal{U}(y)$ whenever $x \neq y$.

In the investigation of Valdivia compacta the following notions of tightness and closedness will be used very often.

DEFINITION 1.4. Let X be a topological space.

(i) We say that X is a *Fréchet-Urysohn* space if, whenever $x \in X$ and $A \subset X$ are such that $x \in \bar{A}$, then there is a sequence $x_n \in A$ with $x_n \to x$.

(ii) We say that the space X is *Fréchet-Urysohn with respect to G_δ subsets* if, whenever $x \in X$ and $A \subset X$ is a G_δ set such that $x \in \bar{A}$, then there is a sequence $x_n \in A$ with $x_n \to x$.

(iii) We say that the space X has *tightness* κ if, whenever $x \in X$ and $A \subset X$ are such that $x \in \bar{A}$, then there is a subset $B \subset A$ with $\operatorname{card} B \leq \kappa$ such that $x \in \bar{B}$.

(iv) We say that the space X has *strong tightness* κ if, whenever $x \in X$ and $A \subset X$ are such that $x \in \bar{A}$, then there is a subset $B \subset A$ with $\operatorname{card} B < \kappa$ such that $x \in \bar{B}$.

DEFINITION 1.5. Let X be a topological space and κ be an uncountable cardinal.

(i) We say that X is *κ-compact* if every open cover of X with cardinality strictly less than κ has a finite subcover. In particular, $\aleph_1$-compact spaces are called *countably compact*.

(ii) We say that $A \subset X$ is *κ-closed* if $\bar{C} \subset A$ for every $C \subset A$ with $\operatorname{card} C < \kappa$. In particular, $\aleph_1$-closed subsets are called *countably closed.*

Let us remark that a space X is κ-compact if and only if every centered family of closed subsets of X which has cardinality strictly less than κ has nonempty intersection. It is easy to check (cf. [33]) that a κ-closed subset of a κ-compact (in particular compact) space is κ-compact.

1.2. Elementary facts on Σ-subsets and Valdivia compacta In this section we collect basic facts on Σ-subsets, including several useful characterizations. These facts will be widely used in next chapters. We start by the following lemma.

Lemma 1.6. (i) *The set $\Sigma(\Gamma)$ is countably closed in $\mathbb{R}^\Gamma$ for every Γ. In particular, if A is any Σ-subset of a compactum K, then A is countably closed in K.*

(ii) *$\Sigma(\Gamma)$ is a Fréchet-Urysohn space for every Γ. In particular, any Σ-subset of a compact space is a Fréchet-Urysohn space.*

Proof. (i) This is obvious.

(ii) This follows immediately from [49, Theorem 2.1]. Let us give the proof. For every $x \in \Sigma(\Gamma)$ the set $\operatorname{supp} x$ is countable, so we can fix an enumeration $\operatorname{supp} x = \{\gamma_1(x), \gamma_2(x), \dots\}$. If $\operatorname{supp} x$ is finite, we fill up the sequence $(\gamma_k(x))$ with some element of Γ. Now let $A \subset \Sigma(\Gamma)$, $x \in \Sigma(\Gamma)$, $x \in \bar{A}$. We can construct by induction a sequence of $x_n \in A$ such that $|x_n(\gamma_k(x_l)) - x(\gamma_k(x_l))| < \frac{1}{n}$ for $0 \le l < n$ and $1 \le k \le n$, where $x_0 = x$. Then clearly $x_n \to x$ (since the convergence in the product topology is the coordinatewise one). ∎

This lemma enables us to prove the following uniqueness result [30, Proposition 2.4].

Lemma 1.7. *Let K be a compact space and A, B be two Σ-subsets of K. If $M \subset K$ is a set such that $A \cap B \cap M$ is dense in M, then $A \cap M = B \cap M$. In particular, $A = B$ whenever $A \cap B$ is dense in K.*

Proof. Let $x \in A \cap M$. By the assumption we have $x \in \overline{A \cap B \cap M}$. As A is Fréchet-Urysohn by Lemma 1.6(ii), there is a sequence $x_n \in A \cap B \cap M$ with $x_n \to x$. As B is countably closed by Lemma 1.6(i), we get $x \in B$. It follows that $A \cap M \subset B \cap M$. The inverse inclusion can be proved by interchanging roles of A and B. ∎

LEMMA 1.8. *Let A be a subset of $\Sigma(\Gamma)$. Then the following assertions are equivalent.*

(i) *A is countably compact.*

(ii) *A is closed and coordinatewise bounded in $\Sigma(\Gamma)$.*

(iii) *The closure K of A in $\mathbb{R}^\Gamma$ is compact and A is a Σ-subset of K.*

Proof. (i) $\Rightarrow$ (ii) Let A be countably compact. Each coordinate projection is a continuous function, hence it is bounded on A. Therefore A is coordinatewise bounded. If A were not closed in $\Sigma(\Gamma)$, there would exist $x \in \bar{A} \setminus A$. By Lemma 1.6(ii) there is a sequence $x_n \in A$ with $x_n \to x$. As, A is countably compact, the sequence x_n has a cluster point in A. But its only cluster point in $\Sigma(\Gamma)$ is x which does not belong to A. This is a contradiction.

(ii) $\Rightarrow$ (iii) Let A satisfy the assumptions of (ii). Then K, the closure of A in $\mathbb{R}^\Gamma$, is closed and coordinatewise bounded, hence it is compact. Further, $A = K \cap \Sigma(\Gamma)$ as A is closed in $\Sigma(\Gamma)$. This completes the proof.

(iii) $\Rightarrow$ (i) If A is a Σ-subset of K, then A is countably closed in K by Lemma 1.6(i), hence A is countably compact. ∎

Next we give some characterization of Σ-subsets. The point 2 generalizes the Rosenthal type characterization of Corson compact spaces (see e.g [47]).

PROPOSITION 1.9. *Let K be a compact space and A be a dense subset of K. Then the following assertions are equivalent.*

(i) *A is a Σ-subset of K.*

(ii) *There is a family $\mathcal{U}$ of open F_σ subset of K which separates the points of K, and $A = \{x \in K \mid \mathcal{U}(x)$ is countable$\}$.*

(iii) *A is homeomorphic to a countably compact subset of $\Sigma(\Gamma)$ for some Γ and $K = \beta A$.*

Proof. (ii) $\Rightarrow$ (i) Let $\mathcal{U}$ be such a family. As each $U \in \mathcal{U}$ is an open F_σ set, it is well-known that there is a continuous function $f_U : K \to [0,1]$ with $U = f_U^{-1}((0,1])$. Let us consider the mapping $h : K \to [0,1]^{\mathcal{U}}$ defined by $h(x)(U) = f_U(x)$ for $U \in \mathcal{U}$ and $x \in K$. Then h is continuous, as each f_U is continuous, and it is one-to-one, as $\mathcal{U}$ separates points of K. From the choice of f_U it is clear that $h(A) = h(K) \cap \Sigma(\Gamma)$.

(i) $\Rightarrow$ (ii) Suppose that $K \subset \mathbb{R}^\Gamma$ and $A = K \cap \Sigma(\Gamma)$. For $\gamma \in \Gamma$, $p \in \mathbb{Q}$, $p > 0$ and $j = 1,2$ put $U_{\gamma,p,j} = \{x \in K \mid (-1)^j x(\gamma) > p\}$ and $\mathcal{U} = \{U_{\gamma,p,j} : \gamma \in \Gamma, p \in \mathbb{Q}, p > 0, j = 1,2\}$. Then clearly $\mathcal{U}$ is a family of open F_σ sets which separates points of K. Let $B = \{x \in K : \mathcal{U}(x)$ is countable$\}$. It follows from the construction that $A \subset B$. By the already proved implication (ii)$\Rightarrow$(i) the set B is a Σ-subset of K, hence $A = B$ by Lemma 1.7.

(i) $\Rightarrow$ (iii) Suppose that $K \subset \mathbb{R}^\Gamma$ and $A = K \cap \Sigma(\Gamma)$. As K is compact, it is clearly coordinatewise bounded, so $K \subset P = \prod_{\gamma\in\Gamma}[-a_\gamma, a_\gamma]$ for suitable numbers $a_\gamma \in (0,\infty)$. By [19, Theorem 2] the space P is the Čech-Stone compactification of $P\cap\Sigma(\Gamma)$. Further, $\Sigma(\Gamma)$ is a normal space by [10, Theorem 1]. Now, as A is closed in $\Sigma(\Gamma)$, it follows by [16, Corollary 3.5.7] that $\bar{A} = K$ is the Čech-Stone compactification of A.

(iii) $\Rightarrow$ (i) Let $B \subset \Sigma(\Gamma)$ be countably compact and $g : A \to B$ be a homeomorphism. Let $h : K \to \bar{B}$ be the continuous extension of g. By Lemma 1.8 and the already proved implication (i)$\Rightarrow$(iii) we have that $\bar{B} = \beta B$, hence there is a continuous function $h' : \bar{B} \to K$ extending g^{-1}. Now it is obvious that $h' = h^{-1}$, so h is a homeomorphism and $h(A) = B = h(K)\cap\Sigma(\Gamma)$, which completes the proof. ∎

Now we are ready to give the following basic examples of Valdivia and non-Valdivia compacta.

Example 1.10. (i) The ordinal segment $[0,\omega_1]$ is a Valdivia compactum and the set $[0,\omega_1)$ is its unique dense Σ-subset.

(ii) The ordinal segment $[0,\alpha]$ is not Valdivia provided $\alpha \geq \omega_2$.

(iii) The quotient space made from $[0,\omega_1]$ by identifying points ω and ω_1 is not Valdivia. We will refer to this space as to *the interval* $[0,\omega_1]$ *with collated sequence.*

(iv) The quotient space made from $[0,\omega_1] \times \{0,1\}$ by identifying points $(\omega_1, 0)$ and $(\omega_1, 1)$ is not Valdivia. This space will be called *collated double-interval* ω_1.

Proof. (i) To see that $[0,\omega_1)$ is a Σ-subset it is enough to consider the embedding $h : [0,\omega_1] \to [0,1]^{[0,\omega_1)}$ defined by $h(\alpha) = \chi_{[0,\alpha)}$. To prove the uniqueness let us remark that $[0,\omega_1]$ has a dense set of isolated points, which are, of course, contained in every dense subset of $[0,\omega_1]$. Then it suffices to use Lemma 1.7.

(ii) This is a result of Yabouri, a proof is given in [11, Proposition II-2]. We give a simple proof based on Rosenthal type characterization of Valdivia compacta given in Proposition 1.9. Suppose that A is a dense Σ-subset of $[0,\alpha]$. Then A contains all isolated ordinals (by the density of A). Further, A does not contain the point ω_2, as A is Fréchet-Urysohn by Lemma 1.6 and ω_2 belongs to the closure of the set of all smaller isolated ordinals without being limit of any sequence. Let $\mathcal{U}$ be a family of open F_σ subsets of $[0,\alpha]$ determining A in the sense of Proposition 1.9. As $\omega_2 \notin A$, the set $\mathcal{U}(\omega_2)$ is uncountable. Let U_γ, $\gamma < \omega_1$ be distinct members of $\mathcal{U}(\omega_2)$. For every $\gamma < \omega_1$ there is $\beta_\gamma < \omega_2$ with $\omega_2 \in (\beta_\gamma, \omega_2] \subset U_\gamma$. Then $\beta = \sup_{\gamma<\omega_1} \beta_\gamma < \omega_2$. It follows that $U_\gamma \in \mathcal{U}(\beta + 1)$ for every $\gamma < \omega_1$, hence $\mathcal{U}(\beta + 1)$ is uncountable, which is a contradiction.

(iii) This is a result of M. Valdivia [66]. Let us denote this space by L. Suppose that A is a dense Σ-subset of L. Then A contains all isolated points of L, which are exactly the isolated ordinals. Further, A is countably closed, and each point of L (including the collated point $\{\omega, \omega_1\}$) is the limit of a sequence of isolated ordinals. It follows that $A = L$. Then it follows by Lemma 1.6 that L is Fréchet-Urysohn. But the collated point $\{\omega, \omega_1\}$ belongs to the closure of the interval (ω, ω_1) without being limit of any sequence from this set. This is a contradiction.

(iv) This is proved in [30, Example 3.4]. Let us denote this compactum by K and suppose that A is a dense Σ-subset of K. As A contains all isolated points and is countably closed (Lemma 1.6), it follows that $[0,\omega_1) \times \{0,1\} \subset A$. As the collated point $\{(\omega_1, 0), (\omega_1, 1)\}$ is limit of no sequence of points from $[0,\omega_1) \times \{0,1\}$, it follows from Lemma 1.6 that $A = [0,\omega_1) \times \{0,1\}$. Now, the characteristic function $\chi_{[0,\omega_1)\times\{0\}}$ has no continuous extension on K, hence A is not a Σ-subset of K by Proposition 1.9. Let us remark that $[0,\omega_1) \times \{0,1\}$ is homeomorphic to a countably compact subset of $\Sigma(\Gamma)$. This shows that the assumption $K = \beta A$ in condition 3 of Proposition 1.9 cannot be dropped. ∎

We continue by the following easy lemma, whose importance was observed already in [30, 31].

LEMMA 1.11. *Let K be a compact space and $A \subset K$ be a dense countably compact subset of K. Then $G \cap A$ is dense in G whenever $G \subset K$ is G_δ.*

Proof. Let $G \subset K$ be a G_δ set. Then there are open sets $U_n \subset K$ such that $G = \bigcap_{n\in\mathbb{N}} U_n$. Let $x \in G$ and W be an open neighborhood of x. We will show that $W \cap A \cap G \neq \emptyset$. To this end we construct by an easy induction open sets V_n, $n \in \mathbb{N}$ such that

$$V_1 \subset W, \quad x \in V_n,\, n \in \mathbb{N}, \qquad \bar{V}_{n+1} \subset V_n \cap U_1 \cap \dots \cap U_n,\, n \in \mathbb{N}.$$

As A is dense in K, we have $V_n \cap A \neq \emptyset$ for every n. Moreover, $\bar{V}_{n+1} \subset V_n$ and A is countably compact, hence $A \cap \bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$. But by the construction we have $\bigcap_{n\in\mathbb{N}} V_n \subset G \cap W$ which completes the argument. ∎

Let us remark that the previous lemma can be proved also by another argument, known in the theory of pseudocompact spaces. As A is countably compact, it is also pseudocompact (i.e. each continuous function on A is bounded). Suppose that G is a nonempty G_δ subset of K. By regularity of K we can without loss of generality suppose that G is closed. Then there is a non-negative continuous function g on K such that $G = g^{-1}(0)$. But g attains on A its minimum, hence $A \cap G \neq \emptyset$.

We preferred the direct argument, as it gives a better idea of how to prove a more general lemma which we formulate in the next section.

The above lemma has the following immediate consequences.

COROLLARY 1.12. *Let K be a compact space with a dense set of G_δ points. Then there is at most one dense Σ-subset of K.*

Proof. This follows immediately from Lemma 1.11 and Lemma 1.7. ∎

PROPOSITION 1.13. *Every continuous image of a super-Valdivia compact space is Fréchet-Urysohn with respect to G_δ sets.*

Proof. Let K be a super-Valdivia compact space, $G \subset K$ a G_δ set and $x \in \bar{G}$. As K is super-Valdivia, there is a dense Σ-subset A of K with $x \in A$. By Lemma 1.11 the intersection $A \cap G$ is dense in G, hence $x \in \overline{A \cap G}$. Now, A is Fréchet-Urysohn by Lemma 1.6 and thus there is a sequence $x_n \in G \cap A$ with $x_n \to x$. Hence K is Fréchet-Urysohn with respect to G_δ sets. Finally, it is enough to observe that spaces which are Fréchet-Urysohn with respect to G_δ sets are stable to closed continuous images. ∎

In the previous proposition super-Valdivia cannot be weakened to Valdivia, as witnesses the Valdivia compactum $[0, \omega_1]$. However we do not know the answer to the following question.

QUESTION 1.14. Let K be a Valdivia compact space which is Fréchet-Urysohn with respect to G_δ sets. Is then K super-Valdivia?

In Lemma 1.11 we only assumed that A is a dense countably compact subset of K. If we assume that A is even a dense Σ-subset of K we get a stronger statement, contained in the following lemma. It is proved in [34]. The proof is rather technical, we give only a sketch of it.

Lemma 1.15. *Let K be a compact space and $G = \bigcap_{n\in\mathbb{N}} \bar{U}_n$ where each U_n is an open subset of K. If A is a dense Σ-subset of K, then $G \cap A$ is dense in G.*

Proof. (Sketch) Let A be a fixed dense Σ-subset of K. If G is the closure of an open set, then $G \cap A$ is dense in G by density of A. Now suppose that $G = \bar{U} \cap \bar{V}$ where U and V are disjoint open sets. This special case was proved already in [30]. Without loss of generality suppose that $K = \bar{U} \cup \bar{V}$. Let $x \in \bar{U} \cap \bar{V}$ and assume $W \subset K$ is an open neighborhood of x such that $\bar{W} \cap \bar{U} \cap \bar{V} \cap A = \emptyset$. Then $A \cap \bar{W}$ is a dense Σ-subset of $\bar{W}$. The sets $A \cap \bar{W} \cap \bar{U}$ and $A \cap \bar{W} \cap \bar{V}$ are disjoint relatively clopen subsets covering $A \cap \bar{W}$ (as $\bar{W} \cap \bar{U} \cap \bar{V} \cap A = \emptyset$). Let f be the characteristic function of $\bar{W} \cap \bar{U} \cap A$. This is a bounded continuous function on $A \cap \bar{W}$. By Proposition 1.9 this function can be continuously extended on $\bar{W}$. But the point x belongs both to $\overline{\bar{W} \cap \bar{U} \cap A}$ and to $\overline{\bar{W} \cap \bar{V} \cap A}$, which is impossible.

The proof continues in several steps. Firstly, it can be proved by induction that the assertion of the lemma does hold if $G = \bar{U}_1 \cap \dots \cap \bar{U}_n$ with $U_1, \dots ,U_n$ open and pairwise disjoint. The next step is to show that, if $G = \bar{V}_1 \cap \dots \cap \bar{V}_n$ and V_i are open, not necessarily disjoint, then G can be expressed as the union of finitely many sets of the form $\bar{U}_1 \cap \dots \cap \bar{U}_k$ with $U_1, \dots ,U_k$ open and pairwise disjoint. Finally, to show the general version, we can use countable compactness of A. ∎

We finish this section with the following results on those compact spaces which are continuous images of Valdivia compact spaces.

Lemma 1.16. *Let K be a compact space. Then K is a continuous image of a Valdivia compact space if and only if K has a dense subset which is continuous image of a countably compact subset of $\Sigma(\Gamma)$.*

Proof. The 'only if' part is trivial. To prove the 'if' part let B be a countably compact subset of $\Sigma(\Gamma)$ and $f : B \to K$ be a continuous mapping with $f(B)$ dense in K. By Lemma 1.8 the compact $\bar{B}$ is Valdivia. By Proposition 1.9 we have $\bar{B} = \beta B$, hence there is $F : \bar{B} \to K$, a continuous extension of f. It follows that K is a continuous image of $\bar{B}$. ∎

Lemma 1.17. *Let K be a compact space with a countable dense set of G_δ points. If K is a continuous image of a Valdivia compact space, then K is metrizable.*

Proof. Let $L \subset \mathbb{R}^\Gamma$ be a compact space with $A = L \cap \Sigma(\Gamma)$ dense in L, and $f : L \to K$ be a continuous onto mapping. Let C be a countable dense subset of K, consisting of G_δ points. As $f(A)$ is dense in K and countably compact, we get by Lemma 1.11 that $C \subset f(A)$. Hence, we can choose a countable set $D \subset A$ with $C = f(D)$. Then $\bar{D}$ is clearly metrizable (as a separable compact subset of $\Sigma(\Gamma)$), and $f(\bar{D}) = \bar{C} = K$. It remains to use the well-known fact that metrizable compact spaces are stable to continuous images. ∎

As an immediate consequence of this lemma we get the following example.

EXAMPLE 1.18. The following compact spaces are not continuous images of Valdivia compacta.

(i) Any non-metrizable compactification of a separable completely metrizable space, e.g. $\beta\mathbb{N}$, $\beta\mathbb{R}$, the Bohr compactification of $\mathbb{R}$ (see e.g. [36]).

(ii) The double arrow space (see e.g. [17, Section 2.3]) and any its non-metrizable modification from [27].

(iii) The Kunen compact space (see [48, Section 7]).

1.3. Some remarks on κ-Corson and κ-Valdivia compact spaces

In this section we collect some basic facts on κ-Corson and κ-Valdivia compacta for κ possibly larger than $\aleph_1$. It turns out that this theory for κ regular is completely analogous to that for ordinary Corson and Valdivia compacta, as described in the previous section. However, the case of singular κ is essentially different. We will almost not give the proofs in this section, as they are either straightforward but technical or very similar to those from previous section.

We start by an analogue of Lemma 1.6.

LEMMA 1.19. *Let κ be an uncountable cardinal and Γ an arbitrary set of cardinality at least κ.*
(i) *The set $\Sigma_\kappa(\Gamma)$ is κ-closed in $\mathbb{R}^\Gamma$ if and only if κ is regular.*
(ii) *If κ is regular, then $\Sigma_\kappa(\Gamma)$ has strong tightness κ.*
(iii) *If κ is singular, then $\Sigma_\kappa(\Gamma)$ has tightness κ but not strong tightness κ.*

Proof. (Remarks on the proof) (i) This is trivial using the definition of regularity of a cardinal.

(ii) This can be proved by an induction argument similar to that used in the proof of Lemma 1.6.

(iii) The positive part follows from the point (ii) with κ^+ instead of κ. The negative part can be proved by a transfinite construction using singularity of κ. ∎

We continue by a lemma generalizing Lemma 1.7.

LEMMA 1.20. *Let K be a compact space and κ a regular uncountable cardinal. If A and B are two Σ_κ-subsets of K, and $M \subset K$ is such that $M \cap A \cap B$ is dense in M, then $A \cap M = B \cap M$.*

Proof. The proof is completely analogous to that of Lemma 1.7. ∎

Next we give an example related to Example 1.10.(i).

EXAMPLE 1.21. (i) If κ is a regular uncountable cardinal, then the ordinal segment $[0,\kappa]$ is κ-Valdivia but not κ-Corson. Moreover, $[0,\kappa)$ is its unique dense Σ_κ-subset.

(ii) If κ is a singular uncountable cardinal, then $[0,\kappa]$ is κ-Corson. Moreover, both $[0,\kappa)$ and $[0,\kappa]$ are dense Σ_κ-subsets of $[0,\kappa]$.

(iii) $[0,1]^\Gamma$ is κ-Valdivia for every κ and Γ. If κ is regular and $\operatorname{card}\Gamma \geq \kappa$, or κ is singular and $\operatorname{card}\Gamma > \kappa$, then it is not κ-Corson.

Proof. (i) This is completely analogous to Example 1.10(i).

(ii) To see that $[0,\kappa)$ is a Σ_κ-subset, it is enough to consider the embedding $h : [0,\kappa] \to [0,1]^{[0,\kappa)}$ defined by $h(\alpha) = \chi_{[0,\alpha)}$. Now let us prove that $[0,\kappa]$ is κ-Corson. As κ is singular, there is a cardinal $\lambda < \kappa$ and cardinals $\alpha_\gamma < \kappa$, $\gamma < \lambda$ such that $\kappa = \sup_{\gamma<\lambda} \alpha_\gamma$. For $\gamma < \lambda$ put $\mathcal{U}_\gamma = \{(\beta,\alpha_\gamma] : \beta < \alpha_\gamma\}$. Further, put $\mathcal{U} = \bigcup_{\gamma<\lambda} \mathcal{U}_\gamma \cup \{\{0\}\}$. Then $\mathcal{U}$ is clearly a family of clopen sets which separates the points of $[0,\kappa)$. Moreover, it is clear that $\operatorname{card}\mathcal{U}_\gamma(\beta) \leq \operatorname{card}\beta$ for every $\beta < \kappa$ and $\gamma < \lambda$, and $\mathcal{U}(\kappa) = \emptyset$. Therefore $\operatorname{card}\mathcal{U}(\beta) < \kappa$ for every $\beta \leq \kappa$. Now it is clear that $g : [0,\kappa] \to \mathbb{R}^{\mathcal{U}}$ defined by $g(\alpha)(U) = \chi_U(\alpha)$ witnesses that $[0,\kappa]$ is κ-Corson.

(iii) It follows immediately from the definitions that $[0,1]^\Gamma$ is κ-Valdivia. If κ is regular and $\operatorname{card}\Gamma \geq \kappa$, then $[0,1]^\Gamma$ has not strong tightness κ, so by Lemma 1.19 it is not κ-Corson. The assertion for κ singular follows from the previous one applied to κ^+. ∎

The above results show that there is a great difference between singular and regular cardinals. However, it seems not to be clear how great is this difference. So we formulate several questions.

QUESTION 1.22. Let κ be an uncountable singular cardinal.
(i) Has every κ-Corson compact space strong tightness κ?
(ii) Is $[0,1]^\kappa$ κ-Corson?
(iii) Is there a κ-Valdivia compactum of weight κ which is not κ-Corson?

Let us remark that positive answer to the first question would give answers to the last two ones.

Further we give a generalization of Lemma 1.11.

LEMMA 1.23. *Let K be a compact space, κ an uncountable cardinal and A a dense κ-compact subset of K. If $G \subset K$ is the intersection of strictly less than κ open sets, then $G \cap A$ is dense in G.*

Proof. (Remarks on the proof) This can be proved by refining the argument of Lemma 1.11, using transfinite induction on κ. In [33, Lemma 3] it is proved for κ being a successor cardinal. In fact, we will use only this case, but the proof of general case is exactly the same. ∎

We finish this section by the following easy lemma on continuous images and preimages of κ-closed sets. The proof is trivial and we omit it.

LEMMA 1.24. *Let $f : X \to Y$ be a continuous mapping between topological spaces and κ be an uncountable cardinal.*
(i) *$f^{-1}(B)$ is κ-closed in X whenever B is a κ-closed subset of Y.*
(ii) *If f is moreover a closed mapping, then $f(A)$ is κ-closed in Y whenever A is κ-closed in X.*

2. POL TYPE CHARACTERIZATION OF VALDIVIA COMPACTA

In this chapter we give a characterization of Valdivia compact spaces, which generalizes Pol's characterization of Corson compact spaces [59], [3, Section IV.3]. Our exposition follows [31], where this characterization is proved and used to derive some results on open continuous images of Valdivia compact spaces. Here we restrict ourselves to the characterization, applications will be given in next chapters.

We start with the following definition.

DEFINITION 2.1. If Γ is an arbitrary set, we denote by L_Γ the *one-point Lindelöfication* of the discrete space Γ, i.e. $L_\Gamma = \Gamma \cup \{\infty\}$ where each point of Γ is isolated in L_Γ and neighborhoods of ∞ are complements of countable subsets of Γ. A topological space X is called *primarily Lindelöf* if it is a continuous image of a closed subset of $(L_\Gamma)^{\mathbb{N}}$ for some set Γ.

Basic properties of the class of primarily Lindelöf spaces are summed up in the following lemma, which is proved for example in [3].

LEMMA 2.2. (i) *The class of primarily Lindelöf spaces is closed with respect to closed subspaces, countable unions, countable products and continuous images.*

(ii) *Every primarily Lindelöf space is Lindelöf.*

The fact that the class of primarily Lindelöf spaces is closely related to the Σ-product $\Sigma(\Gamma)$ is showed by the following proposition.

PROPOSITION 2.3. *Let X be a primarily Lindelöf space. Then there is an one-to-one linear continuous mapping of $C_p(X)$, the space of real continuous functions on X endowed with the topology of pointwise convergence, into $\Sigma(\Gamma)$ for a set Γ.*

This result, which can be viewed as a generalization of a remark of Corson [10, Proposition 5], is due to S.P. Gul'ko. The proof is done by transfinite induction, using long sequences of retractions on closed subsets of $(L_\Gamma)^{\mathbb{N}}$. It can be found in [59, Proposition 1.4] or [3, Proposition IV.3.10]. A nice exposition on duality of spaces like $\Sigma(\Gamma)$ and primarily Lindelöf spaces is given in a recent survey paper by S.P. Gul'ko [23].

We will need a modification of the topology of pointwise convergence. This topology was used for example by M. Valdivia in [65]. We define also an analogical modification of the weak topology of a Banach space.

DEFINITION 2.4. (i) Let K be a compact space and A be a subset of K. By τ_A we denote the *topology of the pointwise convergence on A*, i.e. τ_A is the weakest topology on $C(K)$ such that $f \mapsto f(a)$ is continuous for every $a \in A$.

(ii) Let X be a Banach space and $A \subset X^*$. Then we denote by w_A the weakest topology on X such that each functional from A is continuous.

Let us remark that both τ_A and w_A are locally convex topologies, and that τ_A is Hausdorff provided A is dense in K, and w_A is Hausdorff whenever the linear span of A is weak* dense in X^*. This will be always the case in the applications.

Now we are ready to formulate the announced characterization. These theorems are proved in [31, Section 2]. We sketch the proofs later.

THEOREM 2.5. *Let K be a compact space and A be a dense subset of K. Then the following two conditions are equivalent.*

(i) *A is a Σ-subset of K.*

(ii) *A is countably compact and $(C(K), \tau_A)$ is primarily Lindelöf.*

As an immediate consequence we get the following result generalizing Pol's characterization to completely regular countably compact spaces.

COROLLARY 2.6. *Let X be a completely regular countably compact space. Then X is homeomorphic to a subset of $\Sigma(\Gamma)$ for some set Γ if and only if $C_p(X)$, the space of continuous functions on X with the topology of pointwise convergence, is primarily Lindelöf.*

Proof. As X is countably compact, every continuous function on X is bounded, and hence the space $C_p(X)$ is canonically homeomorphic to $(C(\beta X), \tau_X)$. Further, it follows from Proposition 1.9 that X is homeomorphic to a subset of $\Sigma(\Gamma)$ if and only if X is a Σ-subset of βX. Now it is enough to use Theorem 2.5. ∎

We continue by the following theorem generalizing [4, Proposition 1.2].

THEOREM 2.7. *Let X be a Banach space and A be a weak* dense subset of the dual unit ball B_{X^*}. Then the following three conditions are equivalent.*

(i) *There is a linear weak* continuous one-to-one mapping $T : X^* \to \mathbb{R}^\Gamma$ such that $A = B_{X^*} \cap T^{-1}(\Sigma(\Gamma))$.*

(ii) *A is a convex symmetric Σ-subset of (B_{X^*}, w^*).*

(iii) *A is weak* countably compact and (X, w_A) is primarily Lindelöf.*

The proof of Theorem 2.5 uses Pol's methods (see [3, Section IV.3]), together with some additional ideas. We formulate two of them as lemmas, since they can be of use also elsewhere. The first of them is the following simple result on closedness of certain spaces of continuous functions. This follows [31, Lemma 2.9]. The formulation is due to P. Holický.

LEMMA 2.8. *Let $\varphi : K \to L$ be a continuous surjection between compact spaces and A be a dense subset of K. Put $E = \{(x, y) \in K \times K : \varphi(x) = \varphi(y)\}$. If $E \cap (A \times A)$ is dense in E, then the space $\varphi^*(C(L)) = \{f \circ \varphi : f \in C(L)\}$ is τ_A-closed in $C(K)$.*

Proof. Let $f_\nu \xrightarrow{\tau_A} f$, where $f_\nu \in \varphi^*(C(L))$ and $f \in C(K)$. First we will show that $f(x) = f(y)$ whenever $\varphi(x) = \varphi(y)$. So let $\varphi(x) = \varphi(y)$, which means $(x, y) \in E$. By the assumptions there is a net $(x_\mu, y_\mu) \in E \cap (A \times A)$ converging to (x, y). As $f_\nu \in \varphi^*(C(L))$, we have $f_\nu(x_\mu) = f_\nu(y_\mu)$ for every ν and μ. Since $x_\mu, y_\mu \in A$, we get $f_\nu(x_\mu) \xrightarrow{\nu} f(x_\mu)$ and $f_\nu(y_\mu) \xrightarrow{\nu} f(y_\mu)$, so $f(x_\mu) = f(y_\mu)$ for every μ. Now it follows from the continuity of f that $f(x) = f(y)$.

So there is a function $g : L \to \mathbb{R}$ such that $f = g \circ \varphi$. As f is continuous, φ closed and hence a quotient mapping, it follows that g is continuous as well (see [16, Theorems 2.4.2 and 2.4.3]). It means $f \in \varphi^*(C(L))$. ∎

The second lemma is the following result on Lindelöf property of the topology τ_A. It was proved in [31, Proposition 2.13]. The proof we present here is a bit more elementary.

LEMMA 2.9. *Let K be a compact space and $A \subset K$ be a countably closed dense subset. If $(C(K), \tau_A)$ is Lindelöf, then $K = \beta A$.*

Proof. Put $K' = \beta A$ and let $\psi : K' \to K$ be the continuous extension of the identity mapping of A. It is enough to show that $\psi(u) \neq \psi(v)$ whenever $u, v \in K' \setminus A$ are distinct.

Suppose that $u, v \in K' \setminus A$ are distinct such that $\psi(u) = \psi(v) = p$. Clearly $p \in K \setminus A$. Denote by $\mathcal{G}$ the set of open neighborhoods of p, ordered by the inverse inclusion. Choose U and V open neighborhoods of u and v, respectively, such that $\bar{U} \cap \bar{V} = \emptyset$. For every $G \in \mathcal{G}$ choose $u_G \in U \cap \psi^{-1}(G) \cap A$ and $v_G \in V \cap \psi^{-1}(G) \cap A$. This is possible by density of A in K'. Further put

$$W_G = \{f \in C(K) \mid |f(\psi(u_G)) - f(\psi(v_G))| < 1\}, \qquad G \in \mathcal{G}.$$

Clearly each W_G is a τ_A-open set in $C(K)$. Moreover these sets cover $C(K)$. Indeed, it is enough to observe that $\psi(u_G) \to p$ and $\psi(v_G) \to p$ as well. As $(C(K), \tau_A)$ is Lindelöf, there is a sequence G_n, $n \in \mathbb{N}$ such that $C(K) = \bigcup_{n\in\mathbb{N}} W_{G_n}$. Put $H = \{\psi(u_{G_n}) \mid n \in \mathbb{N}\}$ and $L = \{\psi(v_{G_n}) \mid n \in \mathbb{N}\}$. As A is countably closed, we have $\bar{H} \subset A$ and $\bar{L} \subset A$. Since $\psi^{-1}(H) \subset U$, $\psi^{-1}(L) \subset V$ and $\psi \restriction A$ is a homeomorphism, we have $\bar{H} \cap \bar{L} = \emptyset$. So there is, by Urysohn's lemma [16, Theorem 1.5.5], $f \in C(K)$ with $f \restriction L = 0$ and $f \restriction H = 1$. Then f belongs to no W_{G_n}, which is a contradiction completing the proof. ∎

Proof. (Sketch of the proof of Theorem 2.5) (i)⇒(ii) Let A be a dense Σ-subset of K. It follows from Lemma 1.6 that A is countably compact. It remains to show that $(C(K),\tau_A)$ is primarily Lindelöf. This can be done in three steps.

`Step 1.` *Reduction to the zero-dimensional case.* In this step we show that it is enough to consider $K\subset\{0,1\}^\Gamma$ with $A=K\cap\Sigma(\Gamma)$. The proof will follow the proof of [31, Proposition 2.10], which refines [3, Proposition IV.3.9].

If A is a dense Σ-subset of K, we can suppose that $K\subset[0,1]^\Gamma$ and $A=K\cap\Sigma(\Gamma)$. Let $\psi:\{0,1\}^{\mathbb N}\to[0,1]$ be a continuous surjection such that $\psi^{-1}(0)=\{0\}$. Such a mapping exists, one can take for example $\psi(x)=\sum_{n\in\mathbb N}\frac{x_n}{2^n}$. We can define the mapping $\Psi:\left(\{0,1\}^{\mathbb N}\right)^\Gamma\to[0,1]^\Gamma$ by the formula $\Psi(x)(\gamma)=\psi(x(\gamma))$. It is clear that Ψ is a continuous surjection satisfying the condition

$$x\in\Sigma(\mathbb N\times\Gamma)\Leftrightarrow\Psi(x)\in\Sigma(\Gamma)\tag{1}$$

Put $L=\Psi^{-1}(K)$, $B=L\cap\Sigma(\mathbb N\times\Gamma)$ and $\varphi=\Psi\restriction L$. By (1) we get that $A=\varphi(B)$, hence clearly $(C(K),\tau_A)$ is homeomorphic to $(\varphi^*C(K),\tau_B)$ (where $\varphi^*f=f\circ\varphi$, see Lemma 2.8). The proof of Step 1 will be finished if we prove that B is dense in L and $\varphi^*C(K)$ is τ_B-closed in $C(L)$, due to Lemma 2.2. To this end it is enough to prove the following claim.

$$\begin{aligned}&a,b\in L,\ \varphi(a)=\varphi(b)\Rightarrow\\&\qquad\exists\text{ a net }(a_\nu,b_\nu)\in B\times B,\varphi(a_\nu)=\varphi(b_\nu),(a_\nu,b_\nu)\to(a,b).\end{aligned}\tag{2}$$

Indeed, to show the density of B it suffices to take $a=b$, and to prove that $\varphi^*C(K)$ is τ_B-closed in $C(L)$ it is enough to use Lemma 2.8. So let us prove (2).

Put $c=\varphi(a)=\varphi(b)$. Let $\mathcal G$ denote the family of all G_δ subsets of K containing c, ordered by the inverse inclusion. For any $G\in\mathcal G$ choose some $c_G\in A\cap G$. This is possible due to Lemma 1.11. The net c_G converges to c in a strong sense, i.e.

$$\forall\gamma\in\Gamma\ \exists G\in\mathcal G\ \forall H\in\mathcal G\ H\subset G\ c_H(\gamma)=c(\gamma).\tag{3}$$

Indeed, it suffices to take $G=\{x\in K:x(\gamma)=c(\gamma)\}$. Now we will construct a_G and b_G in the following manner. If $c_G(\gamma)=c(\gamma)$ put $a_G(\gamma)=a(\gamma)$ and $b_G(\gamma)=b(\gamma)$, otherwise choose $a_G(\gamma),b_G(\gamma)\in\psi^{-1}(c_G(\gamma))$ arbitrary. It is clear that $\varphi(a_G)=\varphi(b_G)=c_G$, by (1) we have $a_G,b_G\in B$ and it follows easily from (3) that $a_G\to a$ and $b_G\to b$. This completes the proof of (2).

`Step 2.` *If K is zero-dimensional, $A\subset K$ dense, and there is a primarily Lindelöf set $M\subset(C(K,\{0,1\}),\tau_A)$ which separates the points of K, then*

$(C(K),\tau_A)$ is primarily Lindelöf. This is proved in a series of lemmas in [31], and the proof is completely analogical to the proofs of the respective lemmas in [3, Section IV.3]. In fact it suffices to check that the arguments of [3], given for the pointwise topology, work also for τ_A.

`Step 3.` *If $K \subset \{0,1\}^\Gamma$ such that $A = K \cap \Sigma(\Gamma)$ is dense in K, then $(C(K,\{0,1\}),\tau_A)$ contains a primarily Lindelöf space separating the points of K.* Let us define $\phi : L_\Gamma \to C(K,\{0,1\})$ by putting

$$\phi(\gamma) = \begin{cases} \pi_\gamma \restriction K & \gamma \in \Gamma, \\ 0 & \gamma = \infty, \end{cases}$$

where π_γ is the projection of $\{0,1\}^\Gamma$ onto the γ-th coordinate. It is clear that ϕ maps L_Γ to $C(K,\{0,1\})$. Let us show that ϕ is continuous to τ_A. Suppose that γ_ν is a net in L_Γ converging to some $\gamma \in L_\Gamma$. If $\gamma \in \Gamma$, then there is ν_0 such that for every $\nu > \nu_0$ we have $\gamma_\nu = \gamma$, and hence $\phi(\gamma_\nu) = \phi(\gamma)$. If $\gamma = \infty$, we will prove that $\phi(\gamma_\nu) \to 0$ in τ_A. Indeed, if $x \in A$ then $\operatorname{supp} x$ is countable, thus $U = L_\Gamma \setminus \operatorname{supp} x$ is a neighborhood of ∞. Hence there is some ν_0 such that $\gamma_\nu \in U$ for $\nu > \nu_0$. Therefore we have, for $\nu > \nu_0$, $\phi(\gamma_\nu)(x) = 0$, so $\phi(\gamma_\nu)(x) \to 0 = \phi(\gamma)(x)$.

By the definition $\phi(L_\Gamma)$ is primarily Lindelöf, and it is clear that it separates points of K.

(ii)⇒(i) By Proposition 2.3 there is an one-to-one continuous mapping $T : C_p(C(K),\tau_A) \to \Sigma(\Gamma)$ for some Γ. Further, the evaluation mapping $e : A \to C_p(C(K),\tau_A)$ defined by $e(a)(f) = f(a)$, is clearly continuous and one-to-one, hence $T \circ e$ is a continuous one-to-one mapping of A into $\Sigma(\Gamma)$. As A is countably compact, each closed subset of A is countably compact as well, and therefore by Lemma 1.8 the mapping $T \circ e$ is closed. It follows that A is homeomorphic to a countably compact subset of $\Sigma(\Gamma)$. Now, clearly A is countably closed in K, and hence $K = \beta A$ by Lemma 2.9. It remains to conclude by Proposition 1.9. ∎

Proof. (Remarks on the proof of Theorem 2.7) $1 \Rightarrow 2$ This is trivial.

(ii)⇒(iii) Let us consider the canonical embedding $e : X \to C(B_{X^*}, w^*)$ defined by $e(x)(f) = f(x)$. It is clear from definitions that the mapping e is a $w_A \to \tau_A$ homeomorphism. By Theorem 2.5 the space $(C(B_{X^*}),\tau_A)$ is primarily Lindelöf, so it is enough to show that $e(X)$ is τ_A-closed in $C(B_{X^*})$. Let f be in the τ_A-closure of $e(X)$. As A is convex and symmetric, it follows that $f \restriction A$ is a symmetric affine function. As A is weak* dense in B_{X^*} and f continuous, we get that f is the restriction to B_{X^*} of a linear functional on

X^*. Since f is continuous on B_{X^*}, it follows from Banach-Dieudonné theorem (see [24, Corollary 224]) that $f \in e(X)$.

(iii)⇒(i) By Proposition 2.3 there is a linear one-to-one continuous mapping $T_0 : C_p(X, w_A) \to \Sigma(\Gamma)$ for some Γ. It is clear from the definition of w_A that $\operatorname{span} A \subset C_p(X, w_A)$. By continuity of T_0 it follows that $T_0(A)$ is dense in $T_0(\operatorname{span} A \cap B_{X^*})$. But in the same time $T_0(A)$ is closed in $\Sigma(\Gamma)$ by Lemma 1.8. Therefore, $A = \operatorname{span} A \cap B_{X^*}$.Similarly as in the proof of Theorem 2.5 we get that A is homeomorphic to a countably compact subset of $\Sigma(\Gamma)$. Further, we have $B_{X^*} = \beta A$. This is proved in [31], using an analogue of Lemma 2.9 ([31, Lemma 2.18]). The proof is based on a technical construction of certain linear space endowed with a topology, to be able to interpret the extension of the identity $A \to A$ to $\beta A \to B_{X^*}$ as a linear mapping.

Hence there is a linear mapping $T : X^* \to \mathbb{R}^\Gamma$ extending T_0 such that $T \restriction B_{X^*}$ is weak* continuous. It follows by Banach-Dieudonné theorem [24, Corollary 224] that T is weak* continuous. By Lemma 1.8 and Proposition 1.9 we get that $T(B_{X^*})$ is a Čech-Stone compactification of $T(A)$, and so $T \restriction B_{X^*}$ is one-to-one, therefore T is one-to-one as well. The argument can be completed by Lemma 1.7. ∎

We finish this chapter by discussing necessity of the conditions in Theorems 2.5 and 2.7. It turns out that they cannot be improved in any obvious way, as it is clear from the following example.

Example 2.10. (i) Neither in condition (ii) of Theorem 2.5 nor in condition (iii) of Theorem 2.7 the assumption that the respective space is primarily Lindelöf can be dropped.

(ii) If K is a compact space and M a dense subset of K which is contained in a dense Σ-subset, then $(C(K), \tau_M)$ is primarily Lindelöf. So the assumption on countable compactness cannot be dropped. The analogous statement hold in case of Banach spaces as well.

(iii) The converse of the previous point does not hold. Namely, if $K = \beta\mathbb{N}$ and $A = \mathbb{N}$, then $(C(K), \tau_A)$ is primarily Lindelöf while A is contained in no dense Σ-subset of K. Moreover, $B_{C(K)^*}$ is not Valdivia, but there is a weak* dense convex symmetric subset $B \subset B_{C(K)^*}$ such that $(C(K), w_B)$ is primarily Lindelöf.

(iv) If $X = \ell_1(\Gamma)$ with uncountable Γ, then there are $A_1, \dots, A_4$, dense Σ-subsets of B_{X^*}, such that A_1 is convex symmetric, A_2 is convex non-symmetric, A_3 is symmetric non-convex and A_4 is neither convex nor symmetric.

Proof. (i) This follows from the well-known obvious facts that every compact space is countably compact and not every compact space is Corson (for example $K = A = [0, \omega_1]$, or the dual unit ball of $C[0, \omega_1]$).

(ii) If $M \subset A \subset K$, then $(C(K), \tau_M)$ is a continuous image of $(C(K), \tau_A)$. Similarly, (X, w_M) is a continuous image of (X, w_A) whenever $M \subset A \subset X^*$. It remains to use the fact that primarily Lindelöf spaces are closed with respect to continuous images (Lemma 2.2).

(iii) It is well-known and easily follows from the properties of Čech-Stone compactification that $C(\beta\mathbb{N})$ can be canonically identified with ℓ_∞. The topology $\tau_{\mathbb{N}}$ is then just the pointwise convergence topology on ℓ_∞. Thus ℓ_∞ with this topology naturally embeds to $\mathbb{R}^{\mathbb{N}}$, being there F_σ. Hence it is an F_σ subset of a separable completely metrizable space, so it is primarily Lindelöf by a classical theorem [37, Theorem 7.9] and Lemma 2.2. However, $\mathbb{N}$ is contained in no dense Σ-subset of $\beta\mathbb{N}$, as $\beta\mathbb{N}$ is not a Valdivia compactum by Example 1.18.

Further, it follows from Example 1.18 and Theorem 5.3 below that the dual unit ball of ℓ_∞ is not a Valdivia compactum. But if we put $B = \ell_1 \cap B_{\ell_\infty^*}$, then B is convex, symmetric and weak* dense in $B_{\ell_\infty^*}$ (by Goldstine theorem [24, Theorem 64]). Further, it is clear that $w_B = w^*$. As ℓ_1 is separable, the unit ball B_{ℓ_∞}, in the w^* topology, is a metrizable compact space [24, Proposition 62]. Hence (B_{ℓ_∞}, w_*) is primarily Lindelöf by [37, Theorem 7.9]. Moreover, by Lemma 2.2 primarily Lindelöf spaces are stable to countable unions, hence $(\ell_\infty, w_B) = (\ell_\infty, w^*)$ is primarily Lindelöf.

(iv) The dual unit ball of $\ell_1(\Gamma)$ is canonically homeomorphic with $[-1, 1]^\Gamma$. Choose $\gamma_0 \in \Gamma$. Then we have the following.

- $A_1 = \{x \in [-1, 1]^\Gamma : \operatorname{supp} x \text{ is countable}\}$ is a dense convex symmetric Σ-subset.
- $A_2 = \{x \in [-1, 1]^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 1\} \text{ is countable}\}$ is a dense convex non-symmetric Σ-subset.
- $A_3 = \{x \in [-1, 1]^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq x(\gamma_0)^3\} \text{ is countable}\}$ is a dense symmetric non-convex Σ-subset.
- $A_4 = \{x \in [-1, 1]^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq x(\gamma_0)^2\} \text{ is countable}\}$ is a dense Σ-subset which is neither convex nor symmetric.

The density of each A_i follows immediately from the definition of product topology. The assertions on A_1 and A_2 are obvious. It is clear that A_3 is symmetric and not convex (for example the constants 0 and 1 belong to A_3 but the constant $\frac{1}{2}$ does not). Similarly, A_4 is neither convex nor symmetric.

It remains to show that A_3 and A_4 are Σ-subsets. To this end it is enough to consider the following mappings:

$$h_3(x)(\gamma) = \begin{cases} x(\gamma) - x(\gamma_0)^3 & \gamma \in \Gamma \setminus \{\gamma_0\}, \\ x(\gamma_0) & \gamma = \gamma_0; \end{cases}$$

$$h_4(x)(\gamma) = \begin{cases} x(\gamma) - x(\gamma_0)^2 & \gamma \in \Gamma \setminus \{\gamma_0\}, \\ x(\gamma_0) & \gamma = \gamma_0. \end{cases}$$

∎

3. Topological properties of Valdivia compacta

This chapter is devoted to study namely the permanence (and non-permanence) properties of Valdivia compact spaces with respect to topological operations - like taking subsets, continuous images, products, sums and unions.

3.1. Some general topological properties of Valdivia compacta

In this section we collect some easy useful topological properties of Valdivia compact spaces. We begin by the following observation.

Theorem 3.1. *Every infinite Valdivia compact space contains a one-to-one convergent sequence.*

Proof. Let K be an infinite Valdivia compact space, and A be a dense Σ-subset of K. Then A is clearly infinite, and since it is countably compact (Lemma 1.6), it has an accumulation point a. Now, a belongs to the closure of $A \setminus \{a\}$, and therefore, by Lemma 1.6, there is a sequence $a_n \in A \setminus \{a\}$ converging to a. It is clear that this sequence can be chosen one-to-one. ∎

As a corollary we get the following example.

Example 3.2. Every Valdivia compact space which is homeomorphic to a subset of $\beta\mathbb{N}$ is finite.

Proof. By [16, Theorem 3.5.4] the space $\beta\mathbb{N}$ contains no one-to-one convergent sequence. It remains to use Theorem 3.1. ∎

We continue by the following theorem which is a partial converse of Corollary 1.12. In the same time it is a generalization of a known fact that every Corson compactum has a dense set of G_δ points. A related result in the framework of Banach spaces is given in [4, Corollary 1.12].

THEOREM 3.3. *Let K be a compact space such that there is a Σ-subset A of K with $K \setminus A$ being of first category in K. Then K has a dense set of G_δ points.*

Proof. At first we prove the following statement.

(4) Every nonempty Corson compact space has at least one G_δ point.

Let $H \subset \Sigma(\Gamma)$ be compact. Let us introduce on H the following order.

$$x \leq y \Leftrightarrow (\forall \gamma \in \Gamma)(x(\gamma) \neq 0 \Rightarrow x(\gamma) = y(\gamma)).$$

This is a partial order and it is clear from compactness of H that any subset of H totally ordered by this relation has an upper bound. So by Zorn's lemma, there is a maximal element x_m of H. It is clear that

$$\{x_m\} = \{y \in H \mid y(\gamma) = x_m(\gamma) \text{ for all } \gamma \in \operatorname{supp} x_m\},$$

which is a G_δ set as $\operatorname{supp} x_m$ is countable. This completes the proof of (4).

Now, let A be a residual Σ-subset of K and $U \subset K$ a nonempty open set. It follows easily from the regularity of K that there is a nonempty closed G_δ set $H \subset A \cap U$. As H is a Corson compactum, by (4) it has a G_δ point, which is also a G_δ point of K contained in U. This completes the proof. ∎

We finish this section by some observations on density and weight of Valdivia compacta. Let us recall that the *density* of a topological space is the minimal cardinality of a dense subset, and the *weight* is the minimal cardinality of a basis of open sets. It is obvious that density is always at most equal to the weight, while the equality need not hold.

LEMMA 3.4. *Let K be a compact space and A a dense Σ-subset of K. Then the density of A is equal to the weight of K.*

Proof. If A is finite, the assertion is obvious, so let us suppose that A is infinite. It is obvious that the density of A is less than or equal to the weight of K. Hence it is enough to prove the inverse inequality. Suppose that $K \subset \mathbb{R}^\Gamma$ with $A = K \cap \Sigma(\Gamma)$. Let D be a dense subset of A. Put

$$I = \bigcup \{\operatorname{supp} x : x \in D\}.$$

Then $\operatorname{card} I = \operatorname{card} D$ (as D is infinite and $\operatorname{supp} x$ is countable for every $x \in D$). As D is dense also in K, it follows that $\operatorname{supp} x \subset I$ for every $x \in K$. Hence K is homeomorphic to a subset of $\mathbb{R}^I$, and therefore has weight at most $\operatorname{card} I = \operatorname{card} D$. This completes the proof. ∎

THEOREM 3.5. (i) *If K is a Valdivia compactum with a dense set of isolated points, then the weight of K is equal to the density of K.*

(ii) *If K is a Valdivia compactum with a dense set M of G_δ points, then the weight of K is equal to the density of M.*

(iii) *There is a separable super-Valdivia compact space, which has not countable weight.*

Proof. The assertions (i) and (ii) follows immediately from Lemma 3.4 and Lemma 1.11. To prove the assertion (iii) it is enough to consider $K = [0,1]^\Gamma$ with $\aleph_0 < \operatorname{card} \Gamma \leq 2^{\aleph_0}$. This compactum is separable by [16, Theorem 2.3.7], it is super-Valdivia for example by Corollary 3.30 below, and it is obviously not Fréchet-Urysohn. ∎

3.2. SUBSETS OF VALDIVIA COMPACTA It is well-known that a closed subset of a Valdivia compact space need not be Valdivia. In this section we collect some strong non-stability results, as well as several stability properties. We begin by the following well-known theorem.

THEOREM 3.6. *Every compact space is homeomorphic to a closed subset of a super-Valdivia compact space.*

Proof. It is well-known [16, Theorem 3.2.5] that every compact space is homeomorphic to a closed subset of $[0,1]^\Gamma$ for a set Γ. And it is easy to see (and follows from Corollary 3.30) that $[0,1]^\Gamma$ is super-Valdivia compact space for every Γ. ∎

We continue by a positive result.

THEOREM 3.7. (i) *Every closed subset of a Corson compactum is again Corson. In particular Corson compacta are hereditarily Valdivia.*

(ii) *If $\alpha < \omega_2$, then every closed subset of $[0,\alpha]$ is a Valdivia compactum.*

Proof. (i) This is trivial.

(ii) At first let us remark that any nonempty closed subset of $[0,\alpha]$ is homeomorphic to $[0,\beta]$ for some $\beta \leq \alpha$. So, to prove that every closed subset of $[0,\alpha]$ is Valdivia it is enough to prove that $[0,\beta]$ is Valdivia for $\beta \leq \alpha$. We will prove this by transfinite induction, using the Rosenthal type characterization given in Proposition 1.9.

The space $[0,0] = \{0\}$ is clearly Valdivia (even metrizable). Further, it is clear that $[0,\beta+1]$ is Valdivia whenever $[0,\beta]$ is Valdivia. Suppose that

$\alpha < \omega_2$ is limit and that $[0,\beta]$ is Valdivia for every $\beta < \alpha$. Then cofinality of α is either countable or ω_1. Suppose the latter takes place. There is an increasing transfinite sequence of ordinals $\beta_\gamma < \alpha$, $\gamma < \omega_1$, such that $\beta_0 = 0$, $\alpha = \sup_{\gamma<\omega_1} \beta_\gamma$ and, moreover, for every limit $\lambda < \omega_1$ we have $\beta_\lambda = \sup_{\gamma<\lambda} \beta_\gamma$. By the induction hypothesis $(\beta_\gamma, \beta_{\gamma+1}]$ is Valdivia for every $\gamma < \omega_1$. Let $\mathcal{U}_\gamma$ be the separating family from Proposition 1.9. Put

$$\mathcal{U} = \bigcup_{\gamma<\omega_1} \mathcal{U}_\gamma \cup \{(\beta_\gamma, \alpha] : \gamma < \omega_1\}.$$

It is easy to check that this family witnesses (in sense of Proposition 1.9) that $[0,\alpha]$ is Valdivia. If α has countable cofinality, the proof is similar and easier. ∎

The following theorem follows immediately from Lemma 1.11 and Lemma 1.15.

THEOREM 3.8. (i) *Every subset of a Valdivia (super-Valdivia) compact space which is the closure of an arbitrary union of G_δ sets is again Valdivia (super-Valdivia).*

(ii) *If K is Valdivia (super-Valdivia) and $G = \bigcap_{n\in\mathbb{N}} \bar{U}_n$ with U_n open in K, then G is Valdivia (super-Valdivia) as well.*

To formulate next positive result we need to recall the notion of derived set.

DEFINITION 3.9. Let X be a topological space.

(i) The *derived set* of X (denoted by X^d) is the set of all non-isolated points of X.

(ii) If α is an arbitrary ordinal, we define the *α-th derived set* of X (denoted by $X^{(\alpha)}$) by transfinite induction using the following rules.

- $X^{(0)} = X$;
- $X^{(\alpha+1)} = (X^{(\alpha)})^d$;
- $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ for λ limit.

LEMMA 3.10. *Let K be a compact space and A be a dense countably compact subset of K. Then $K^{(\alpha)} \cap A$ is dense in $K^{(\alpha)}$ for every $\alpha < \omega_1$.*

Proof. Due to an induction argument, it is enough to prove the following two claims.

(5) K compact, $A \subset K$ dense countably compact $\Rightarrow A \cap K^d$ is dense in K^d

$$(6) \quad \left.\begin{array}{l} K \text{ compact, } A \subset K \text{ countably compact,} \\ F_n \subset K \text{ closed, } F_n \cap A \text{ dense in } F_n,\ F_n \searrow F \end{array}\right\} \Rightarrow F \cap A \text{ dense in } F$$

Let us prove (5). Let $x \in K^d$ and U be an open neighborhood of X. By regularity of K there is an open set V with $x \in V \subset \bar{V} \subset U$. Then $A \cap V$ is dense in V. As x is not isolated, V is infinite, hence $A \cap V$ is infinite as well. By countable compactness of A there is a, an accumulation point of $A \cap V$ in A. In particular, $a \in K^d \cap A \cap U$.

The claim (6) follows by a standard argument using countable compactness of A. ∎

THEOREM 3.11. (i) *If K is a Valdivia compact space and $\alpha < \omega_1$, then $K^{(\alpha)}$ is Valdivia as well.*

(ii) *If K is a super-Valdivia compact space and α an arbitrary ordinal, then $K^{(\alpha)}$ is super-Valdivia.*

(iii) *There is a Valdivia compact space K such that $K^{(\omega_1)}$ is not Valdivia.*

Proof. (i) Let A be a dense Σ-subset of K. It follows from Lemma 3.10 that $K^{(\alpha)} \cap A$ is dense in $K^{(\alpha)}$, so $K^{(\alpha)}$ is Valdivia.

(ii) Let P denote the set of all isolated points of K. Then $\bar{P}$ is super-Valdivia by Theorem 3.8. But by Lemma 1.7 the space $\bar{P}$ has at most one dense Σ-subset. It follows that $\bar{P}$ is Corson.

Let α be an arbitrary ordinal and $x \in K^{(\alpha)}$. There is a dense Σ-subset A of K with $x \in A$. As clearly $P \subset A$, $\bar{P}$ is Fréchet Urysohn and A countably closed (Lemma 1.6), we get $\bar{P} \subset A$. Moreover, clearly $(K \setminus \bar{P}) \cap A$ is dense in $K \setminus \bar{P}$. Remark that $(K \setminus \bar{P}) \subset K^{(\alpha)}$, hence $K^{(\alpha)} \cap A$ is dense in $K^{(\alpha)}$.

(iii) Put $K = [0, \omega_1] \times [0, \omega_1]$. It is easy to check and follows from Theorem 3.29 that K is a Valdivia compactum. Further, it is not hard to show that $K^{(\omega_1)}$ is homeomorphic to the collated double-interval ω_1 from Example 1.10. ∎

Now we give the fundamental embedding result. We formulate it for weak* compact sets in dual Banach spaces. This is no restriction as any compactum K is homeomorphic to a weak* compact subset of $C(K)^*$. This proposition puts together ideas of [33, Theorem 2] and [32, Proposition 3, Step 2].

Proposition 3.12. *Let X be a Banach space and $K \subset X^*$ weak* compact. Suppose there is a homeomorphic embedding $h : K \to \mathbb{R}^\Gamma$ with $A = h^{-1}(\Sigma(\Gamma))$ dense in K. If $\xi \in K$ is such that* $\operatorname{card}\operatorname{supp} h(\xi) = \kappa > \aleph_0$, *then there is a homeomorphic embedding $\varphi : [0,\kappa] \to K$ such that the following conditions are fulfilled.*

(i) $\operatorname{supp} h(\varphi(\alpha)) \subset \operatorname{supp} h(\varphi(\beta)) \subset \operatorname{supp} h(\xi)$ *for every* $\alpha \leq \beta < \kappa$;

(ii) $\varphi(\kappa) = \xi$;

(iii) $\operatorname{card}\operatorname{supp} h(\varphi(\alpha)) = \max(\operatorname{card}\alpha, \aleph_0)$ *for every* $\alpha < \kappa$;

(iv) $\overline{\operatorname{conv}\varphi([\alpha+1,\kappa])}^{w^*} \cap \overline{\operatorname{conv}\varphi([0,\alpha])}^{w^*} = \emptyset$ *for every* $\alpha < \kappa$.

Proof. Put $I = \operatorname{supp} h(\xi)$ and fix an enumeration $I = \{i_\alpha : \alpha < \kappa\}$. We construct by transfinite induction $\xi_\alpha \in K$, $x_\alpha \in X$ and $J_\alpha \subset I$ such that the following conditions are satisfied.

(a) $i_\alpha \in J_{\alpha+1}$, $\bigcup_{\beta<\alpha} \operatorname{supp} h(\xi_\beta) \subset J_\alpha$, $\operatorname{card} J_\alpha \leq \max(\operatorname{card}\alpha, \aleph_0)$;

(b) $J_\alpha \subset J_{\alpha+1}$, $\operatorname{supp} h(\xi_\alpha) \cap I \subsetneq J_{\alpha+1} \cap I$;

(c) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is limit;

(d) $h(\xi_\alpha)(i) = h(\xi)(i)$ for $i \in J_\alpha$;

(e) $\xi_\alpha(x_\beta) = \xi(x_\beta)$ for $\beta < \alpha$;

(f) $\xi_\alpha = \lim_{\beta<\alpha} \xi_\beta$ if α is limit;

(g) $\xi(x_\alpha) > \sup_{\beta\leq\alpha} \xi_\beta(y_\alpha)$.

The construction uses Lemma 1.23 and Hahn-Banach separation theorem. It is a straightforward generalization of the construction performed in [32, Proposition 3, Step 2].

Put $\xi_\kappa = \xi$. It is clear from conditions (a), (c), (d), (f) that $\operatorname{supp} h(\xi_\alpha) \subset \operatorname{supp} h(\xi)$ for each limit $\alpha < \kappa$ and $\xi = \lim_{\alpha<\kappa} \xi_\alpha$. If we denote by L the set of all limit ordinals from $[0,\kappa]$, there is clearly a bijective increasing mapping $\psi : [0,\kappa] \to L$. Let us define $\varphi : [0,\kappa] \to K$ by the formula $\varphi(\alpha) = \xi_{\psi(\alpha)}$. This mapping is one-to-one by the condition (b) and the continuity follows easily from the condition (f) and the definition of ξ_κ. The condition (iv) follows easily from (e) and (g). This completes the proof. ∎

Remarks 3.13. (i) As a consequence of the previous proposition we get the result of [11, Proposition III-2], that a Valdivia non-Corson compactum contains a copy of $[0,\omega_1]$. In case that this compactum is contained in a dual Banach space, we get a stronger result.

(ii) An analogous statement holds for τ^+-Valdivia compacta with an infinite cardinal τ. Only in condition (iii) the cardinal $\aleph_0$ should be replaced by τ. For τ-Valdivia compacta with regular τ the formulation is a bit more complicated - in condition (iii) there should be $\operatorname{card}\operatorname{supp} h(\xi_\alpha) < \max(\tau, (\operatorname{card}\alpha)^+)$.

(iii) It follows from the condition (iv) of the previous proposition that $\varphi(\alpha)$ is a weak* G_δ point of $\overline{\operatorname{conv}\varphi[0,\kappa]}^{w^*}$ whenever $\alpha < \kappa$ is an isolated ordinal.

In the following two theorems we sum up several results on embedding non-Valdivia compact spaces into (Valdivia) non-Corson compact spaces.

THEOREM 3.14. *Let K be a non-Corson Valdivia compact space.*

(i) *If K is not $\aleph_2$-Corson, then K contains uncountably many pairwise disjoint nowhere dense closed subsets such that one of them is homeomorphic to $[0,\omega_2]$, uncountably many of them are homeomorphic to the collated double interval ω_1, and uncountably many of them are homeomorphic to the interval $[0,\omega_1]$ with collated sequence (cf. Example 1.10).*

(ii) *If K has at least two distinct dense Σ-subsets, then K contains uncountably many pairwise disjoint nowhere dense copies of the interval $[0,\omega_1]$ with collated sequence.*

(iii) *If K can be expressed as a product of two infinite compact spaces, then K contains a closed subset which is not Valdivia compactum.*

Proof. (i) Suppose that $K \subset \mathbb{R}^\Gamma$ with $A = K \cap \Sigma(\Gamma)$ dense in K. Put $B = K \cap \Sigma_{\aleph_2}(\Gamma)$. By Lemma 3.10 we have that $K^d \cap A$ is dense in K^d. By the assumption $K \setminus B \neq \emptyset$, so clearly $K^d \setminus B \neq \emptyset$. It follows from Proposition 3.12 that there is a homeomorphic injection $\varphi : [0,\omega_2] \to K^d$ with $\varphi([0,\omega_1)) \subset \Sigma(\Gamma)$ and $\varphi([0,\omega_2)) \subset \Sigma_{\aleph_2}(\Gamma)$. Now clearly $\varphi([\omega_1,\omega_2)) \subset B \setminus A$. So it is clear that there are pairwise disjoint compact sets M^1, M^2_α, M^3_α for $\alpha < \omega_1$ such that M^1 is homeomorphic to $[0,\omega_2]$ and is contained in $K^d \setminus A$, all M^2_α are homeomorphic to $[0,\omega]$ and are contained in $B \setminus A$, and all M^3_α are homeomorphic to $[0,\omega_1]$ and are contained in $B \setminus A$. Let us denote by m^2_α the "point ω" of M^2_α, and by m^3_α the "point ω_1" of M^3_α. We will construct by the transfinite induction N^2_α, N^3_α, $\alpha < \omega_1$ such that the following conditions are fulfilled.

(a) $N^2_\alpha \subset K^d \setminus \bigcup_{\beta<\alpha}(N^2_\beta \cup N^3_\beta)$, $N^3_\alpha \subset K^d \setminus \left(N^2_\alpha \cup \bigcup_{\beta<\alpha}(N^2_\beta \cup N^3_\beta)\right)$;

(b) There is a continuous bijection $\phi : [0, \omega_1] \to N^i_\alpha$ such that $\phi([0, \omega_1)) \subset K^d \cap A$ and $\phi(\omega_1) = m^i_\alpha$, $i = 2, 3$.

This can be done using Proposition 3.12 together with Lemma 1.11 and the following easy observation which holds in every regular space.

$$\text{(7)} \qquad \begin{array}{l} \text{If } G \text{ is a } G_\delta \text{ set such that } x \in G, \\ \text{then there is a closed } G_\delta \text{ set } F \text{ with } x \in F \subset G \end{array}$$

Finally, the required subsets are M^1, $M^i_\alpha \cup N^i_\alpha$, $\alpha < \omega_1$, $i = 2, 3$.

(ii) Due to (i) we can suppose that K is $\aleph_2$-Corson. Let A and B be two distinct dense Σ-subsets. By Lemma 1.7 it follows that there is a nonempty open set U such that $\bar{U} \cap A \cap B = \emptyset$. We can construct by a transfinite induction $M_\alpha \subset K \cap \bar{U}$, $m_\alpha \in M_\alpha$, $y^n_\alpha \in K \cap \bar{U}$, $n \in \mathbb{N}$, $\alpha < \omega_1$, such that

(a) $M_\alpha \subset K \setminus \bigcup_{\beta<\alpha} \left(M_\beta \cup \{y^n_\beta : n \in \mathbb{N}\} \right)$;

(b) there is a continuous bijection $\phi_\alpha : [0, \omega_1] \to M_\alpha$ with $\phi_\alpha([0, \omega_1)) \subset A$ and $\phi_\alpha(\omega_1) = m_\alpha$, and $m_\alpha \in B$;

(c) $y^n_\alpha \in B \cap \left(K \setminus \left(M_\alpha \cup \bigcup_{\beta<\alpha} \left(M_\beta \cup \{y^n_\beta : n \in \mathbb{N}\} \right) \right) \right)$ and $y^n_\alpha \xrightarrow{n} m_\alpha$.

This can be done using Proposition 3.12, the observation (7), Lemma 1.11 and Lemma 1.6. Then $M_\alpha \cup \{y^n_\alpha : n \in \mathbb{N}\}$, $\alpha < \omega_1$ are the required subsets.

(iii) Let $K = L \times H$ with both H and L infinite. As K contains a homeomorphic copies of L and H, we can suppose that both L and H are Valdivia. If both of them are Corson, then K is Corson as well (Theorem 3.31 below). Suppose that L is not Corson. By Proposition 3.12 it contains a copy of $[0, \omega_1]$. Further, H is an infinite Valdivia compactum, so it contains a nontrivial convergent sequence, i.e. a copy of $[0, \omega]$ (Theorem 3.1). Hence K contains a copy of $[0, \omega_1] \times [0, \omega]$, which has a subset homeomorphic to the interval $[0, \omega_1]$ with collated sequence (Example 1.10). ∎

Remark 3.15. We do not know whether there is a reasonable characterization of hereditarily Valdivia compact spaces. From the previous theorem it follows that hereditarily Valdivia compactum has exactly one dense Σ-subset, is $\aleph_2$-Corson and contains no non-Corson product of infinite compact spaces. However, by Theorem 3.7 there are non-Corson hereditarily Valdivia compacta. The next theorem shows that within convex weak* compact subsets of dual Banach spaces the situation is more clear.

THEOREM 3.16. *Let K be a convex weak* compact subset of a dual Banach space X^*. If K is not Corson, there is a convex compact subset of K which is not Valdivia.*

Proof. (Sketch) This theorem is proved in [32, Proposition 3]. The proof is rather long and technical, we give only main ideas. If K is not Valdivia, there is nothing to prove. Suppose that K is Valdivia and that $A \subset K$ is a dense Σ-subset. Let $h : K \to \mathbb{R}^\Gamma$ be a homeomorphic injection with $h(A) = h(K) \cap \Sigma(\Gamma)$. By Proposition 3.12 there is $g \in K$ such that $\operatorname{card} \operatorname{supp} h(g) = \aleph_1$.

The first step consists in constructing $f_n \in K$ and $x_n \in X$ satisfying the following conditions.

(i) $\|f_n - g\| < \frac{1}{n}$;

(ii) f_n does not belong to the affine envelope of $\{g\} \cup \{f_i : 1 \leq i < n\}$;

(iii) $g(x_k) < f_n(x_k) < f_1(x_k)$ for $1 \leq k < n$;

(iv) $g(x_n) < f_1(x_n) = \cdots = f_{n-1}(x_n) < f_n(x_n)$.

The construction given in [32] strongly uses the convexity of K. In the second step we put $G = \{\xi \in K : \forall n \in \mathbb{N}\ \xi(x_n) = g(x_n)\}$ and find g_α, $\alpha \leq \omega_1$, a copy of $[0, \omega_1]$ in G, with the properties from Proposition 3.12 such that $g_{\omega_1} = g$.

Finally put $L = \{f_n : n \in \mathbb{N}\} \cup \{g_\alpha : \alpha \leq \omega_1\}$ and $H = \overline{\operatorname{conv} L}^{w^*}$.

It follows from the construction that each f_n and each g_α for $\alpha < \omega_1$ isolated is a weak* G_δ point of H. So, if H were Valdivia, L would be Valdivia as well by Theorem 3.8. But L is clearly homeomorphic to the interval $[0, \omega_1]$ with collated sequence (Example 1.10). ∎

Remark 3.17. By the same method a more general statement can be proved. Namely, it can be shown that a convex weak* non-κ-Corson compact set contains a convex weak* compact subset which is not κ-Valdivia, provided κ is a regular cardinal.

3.3. CONTINUOUS IMAGES OF VALDIVIA COMPACTA In this section we give several results on continuous images of Valdivia compacta. We begin by the strong non-stability result from [30], a positive result on open continuous images of Valdivia compact spaces [31] will follow. We will finish by some properties that continuous images of Valdivia compact spaces share with Valdivia compacta [33].

We start by the following auxiliary Lemma [30, Lemma 2.8]. In fact, we will need only a special case of it, but we formulate it in a more general setting.

LEMMA 3.18. *Let K be a compact space and $F \subset K$ be a metrizable closed subset. Put $L = K \setminus F \cup \{F\}$ endowed with the quotient topology induced by the mapping $Q : K \to L$ defined by*

$$Q(x) = \begin{cases} x & x \notin F, \\ F & x \in F. \end{cases}$$

If A is a Σ-subset of L, then $Q^{-1}(A)$ is a Σ-subset of K.

Proof. Let $h_0 : L \to \mathbb{R}^\Gamma$ be a homeomorphic injection such that $h_0(A) = h_0(L) \cap \Sigma(\Gamma)$, and $h_1' : F \to \mathbb{R}^\mathbb{N}$ be any homeomorphic injection. For $n \in \mathbb{N}$ let $h_1(n)$ be a continuous extension of $h_1'(n)$ on K. Define $h : K \to \mathbb{R}^{\Gamma \cup \mathbb{N}}$ by the formula

$$\begin{aligned} h(x)(\gamma) &= h_0(Q(x))(\gamma), \qquad & \gamma \in \Gamma, \\ h(x)(n) &= h_1(x)(n), \qquad & n \in \mathbb{N}. \end{aligned}$$

Now it is obvious that h is a homeomorphic injection and $h(Q^{-1}(A)) = h(K) \cap \Sigma(\Gamma \cup \mathbb{N})$. ∎

PROPOSITION 3.19. *Let K be a compact space and $a, b \in K$ be two distinct non-isolated points of K such that at least one of them is contained in no dense Σ-subset. Let L be the quotient space made from K by identifying a and b. Then L is not Valdivia.*

Proof. Denote by Q the canonical quotient mapping and put $p = Q(a) = Q(b)$. Choose open sets $U, V \subset K$ such that $a \in U$, $b \in V$ and $\bar{U} \cap \bar{V} = \emptyset$. Put $U' = Q(U \setminus \{a\})$ and $V' = Q(V \setminus \{b\})$. It follows that U' and V' are open in L and $\overline{U'} \cap \overline{V'} = \{p\}$. Hence, if A is a dense Σ-subset of L, then $p \in A$ by Lemma 1.15. By Lemma 3.18 the set $Q^{-1}(A)$ is a Σ-subset of K. Clearly it is dense and contains both a and b. This is a contradiction. ∎

PROPOSITION 3.20. *Let K be a compact space, M, N be two disjoint closed nowhere dense mutually homeomorphic subsets of K, which are not*

Valdivia compacta. Let $h : M \to N$ be a homeomorphism, put $L = K \setminus M$ with the quotient topology induced by the mapping

$$Q(x) = \begin{cases} x & x \notin M, \\ h(x) & x \in M. \end{cases}$$

Then L is not a Valdivia compactum.

Proof. Suppose that L is Valdivia. As K is normal, there are open (in K) sets $U \supset M$, $V \supset N$ such that $\bar{U} \cap \bar{V} = \emptyset$. The sets $U' = Q(U \setminus M)$ and $V' = Q(V \setminus N)$ are clearly disjoint open sets in L with $\overline{U'} \cap \overline{V'} = N$. Thus, by Theorem 3.8 the compactum N is Valdivia, a contradiction. ∎

THEOREM 3.21. *Let K be a compact space. Then the following assertions are equivalent.*

(i) *Every continuous image of K is a Valdivia compactum.*

(ii) *Every at most two-to-one continuous image of K is a Valdivia compactum.*

(iii) *K is a Corson compactum.*

Proof. (iii) $\Rightarrow$ (i) This implication follows from the well-known result of Gul'ko, Michael and Rudin, which says that Corson compact spaces are stable to continuous images. A proof is given for example in [3, Section IV.3]. It also follows from Theorem 3.22 below.

(i) $\Rightarrow$ (ii) This is trivial.

(ii) $\Rightarrow$ (iii) Let K be a Valdivia compactum which is not Corson. If K is not super-Valdivia, use Proposition 3.19. If K is super-Valdivia, then it contains at least two distinct dense Σ-subsets. Hence, by Theorem 3.14, K contains two disjoint nowhere dense copies of the interval $[0, \omega_1]$ with collated sequence. These subsets are not Valdivia by Example 1.10. It remains to use Proposition 3.20. ∎

The previous theorem gives the complete answer to a question posed in [12]. This question was whether Valdivia compact spaces are stable to continuous images. A counterexample was found by M. Valdivia [66]. It was just the interval $[0, \omega_1]$ with collated sequence from Example 1.10. M. Valdivia asked whether there is a non-Corson Valdivia compact space whose every continuous image is Valdivia. The above theorem, proved first in [30], answers this question in the negative.

Now we give a theorem from [31] which gives a finer idea on when continuous image of a Valdivia compactum is again Valdivia.

THEOREM 3.22. *Let $\varphi : K \to L$ be a continuous surjection between compact spaces, $A \subset K$ be a dense Σ-subset and $B = \varphi(A)$. Then the following assertions are equivalent.*

(i) *B is a Σ-subset of L.*

(ii) *$\varphi^* C(L) = \{f \circ \varphi : f \in C(L)\}$ is τ_A-closed in $C(K)$.*

(iii) *$L = \beta B$ and $\varphi \restriction A$ is a quotient mapping of A onto B.*

Proof. (i) $\Rightarrow$ (iii) If B is a Σ-subset of L, then $L = \beta B$ by Proposition 1.9. Moreover, B is homeomorphic to a countably compact subset of $\Sigma(\Gamma)$ for some set Γ. So it is easy to see, using Lemma 1.8, that $\varphi \restriction A$ is closed, and therefore a quotient mapping.

(iii) $\Rightarrow$ (ii) Let $f_\nu \in \varphi^* C(L)$ and $f \in C(K)$ be such that $f_\nu \xrightarrow{\tau_A} f$. We will prove that $f \in \varphi^* C(L)$. It follows from the definition of τ_A that f is constant on $\varphi^{-1}(l) \cap A$ for every $l \in L$. Hence there is a function $g : B \to \mathbb{R}$ such that $f \restriction A = g \circ (\varphi \restriction A)$. As f is continuous and $\varphi \restriction A$ is a quotient mapping, we get that g is continuous as well. Now, since $L = \beta B$ and g is bounded on B (as B is countably compact), there is a continuous extension $\tilde{g}$ of g onto L. It follows that $f = \tilde{g} \circ \varphi$ which completes the argument.

(ii) $\Rightarrow$ (i) By Theorem 2.5 we have that A is countably compact and $(C(K), \tau_A)$ is primarily Lindelöf. Hence B is countably compact (as a continuous image of A). Moreover, $(C(L), \tau_B)$ is homeomorphic to $(\varphi^* C(L), \tau_A)$, which primarily Lindelöf by Lemma 2.2. It follows from Theorem 2.5 that B is a dense Σ-subset of L. ∎

The above theorem follows [31, Theorem 2.20]. Although the formulation is due to [31], some implications had been essentially proved before. For example, the implication 2 $\Rightarrow$ 1 follows also from [65, Theorem 2], together with [29, Lemma 3] (and Lemma 1.7). The implication 3 $\Rightarrow$ 1 follows also from [22] and Proposition 1.9.

The previous theorem can be used to get some results on open continuous images of Valdivia compacta. We follow [31, Section 4].

LEMMA 3.23. *Let $\varphi : K \to L$ be an open continuous surjection between compact spaces. If L has a dense set of G_δ points and A is a dense Σ-subset of K, then $\varphi(A)$ is a Σ-subset of L.*

Proof. By Theorem 3.22 it is enough to show that $\varphi^* C(L)$ is τ_A-closed in $C(K)$. To prove this we will use Lemma 2.8. Put $E = \{(x, y) \in K \times K \mid \varphi(x) = \varphi(y)\}$. We will show that $E \cap (A \times A)$ is dense in E.

Choose an arbitrary pair $(u, v) \in E$, and U, V open neighborhoods of u, v, respectively. Put $z = \varphi(u) = \varphi(v)$. Then $\varphi(U)$ and $\varphi(V)$ are open neighborhoods of z, since φ is open. Therefore $W = \varphi(U) \cap \varphi(V)$ is a nonempty open set, so there is $g \in W$, a G_δ point of L. The set $\varphi^{-1}(g)$ is G_δ in K, and hence $\varphi^{-1}(g) \cap A$ is dense in $\varphi^{-1}(g)$ by Lemma 1.11. It follows that we can choose $x \in \varphi^{-1}(g) \cap A \cap U$ and $y \in \varphi^{-1}(g) \cap A \cap V$. Then $(x, y) \in (U \times V) \cap E \cap (A \times A)$, which completes the proof. ∎

As an immediate consequence we get the following theorem.

THEOREM 3.24. *Let $\varphi : K \to L$ be an open continuous surjection between compact spaces. Suppose that L has a dense set of G_δ points. Then the following holds.*

(i) *If K is Valdivia, then so is L.*

(ii) *If K is super-Valdivia, then L is Corson.*

Remarks 3.25. (i) It is easy to check (see e.g. [31, Lemma 4.3]) that any compact space which is an open continuous image of a compact space with a dense set of G_δ points, has again this property. So we get that any open continuous image of a Valdivia compactum with a dense set of G_δ points is again Valdivia. In particular, any open continuous image of $[0, \omega_1]$ (or even $[0, \omega_1]^{\mathbb{N}}$ - cf. Theorem 3.29 below) is Valdivia.

(ii) The assertion of Lemma 3.23 does not hold without assumptions on L. Take $L = [0, 1]^\Gamma$ with uncountable Γ, $K = L \times \{0, 1\}$ with φ being the canonical projection of K onto L. If we put

$$A = ((L \cap \Sigma(\Gamma)) \times \{0\}) \cup (\{x \in L : \{\gamma : x(\gamma) \neq 1\} \text{ is countable}\} \times \{1\}),$$

then A is a dense Σ-subset of K and $\varphi(A)$ is not a dense Σ-subset of L (even though L is Valdivia).

(iii) We do not know whether the assertion of Theorem 3.24 holds without additional assumptions on L. So we can formulate a question.

QUESTION 3.26. Is every open continuous image of a Valdivia (super-Valdivia) compact space again Valdivia (super-Valdivia)?

We finish this section with a theorem of [33], which shows that, in a sense, the class of continuous images of Valdivia compacta is not so far from the class of Valdivia compacta. The first point of this theorem generalizes [11, Proposition III-2], the second one generalizes Theorem 3.1.

Theorem 3.27. (i) *If K is a non-Corson continuous image of a Valdivia compactum, then K contains a homeomorphic copy of* $[0,\omega_1]$.

(ii) *Every infinite compact space which is a continuous image of a Valdivia compactum contains an one-to-one convergent sequence.*

Proof. Let L be a Valdivia compactum, $f : L \to K$ a continuous surjection, and A a dense Σ-subset of L. Suppose that K contains no homeomorphic copy of $[0,\omega_1]$. We will show that $f \restriction A$ is a closed (into K) mapping.

Choose $F \subset A$ a relatively closed subset. Then clearly F is a dense Σ-subset of $\bar{F}$. Let $h : \bar{F} \to \mathbb{R}^\Gamma$ be a homeomorphic embedding such that $h(F) = h(\bar{F}) \cap \Sigma(\Gamma)$. We will show that $f(F) = f(\bar{F})$. Suppose it is not the case. Let $\kappa = \min\{\operatorname{card}\operatorname{supp} h(x) \mid x \in \bar{F} \ \&\ f(x) \notin f(F)\}$ and let $x \in \bar{F}$ be such that $\operatorname{card} h(x) = \kappa$ and $f(x) \notin f(F)$. It is clear that κ is uncountable. Let $\varphi : [0,\kappa] \to \bar{F}$ be the mapping satisfying the conditions of Proposition 3.12.

If κ is singular, then there is an infinite cardinal $\lambda < \kappa$ and cardinals $(\tau_\gamma)_{\gamma<\lambda}$ with $\tau_\gamma < \kappa$ for $\gamma < \lambda$ and $\kappa = \sup_{\gamma<\lambda} \tau_\gamma$. By the definition of κ we have $f(\varphi(\tau_\gamma)) \in f(F)$. Moreover, $f(F) = f(\bar{F} \cap h^{-1}(\Sigma_{\lambda^+}(\Gamma)))$, so $f(F)$ is λ^+-closed in K (by Lemma 1.19 and Lemma 1.24). In particular, $f(x) \in f(F)$, as it is in the closure of the set $\{f(\varphi(\tau_\gamma)) \mid \gamma < \lambda\}$. This is a contradiction.

Hence κ is a regular cardinal. Put $g = f \circ \varphi$. At first let us note that $\operatorname{card} g((\alpha,\kappa)) = \kappa$ for every $\alpha < \kappa$. Otherwise by regularity of κ, the set $g^{-1}(l)$ is unbounded in $[0,\kappa)$ for some $l \in f(F)$. But then $x = g(\kappa) \in f(F)$, as it is equal to l by continuity of g.

So we can choose by transfinite induction ordinals $\eta_\alpha < \kappa$ for $\alpha < \kappa$ such that

(a) $\eta_{\alpha+1} > \eta_\alpha$;

(b) $g(\eta_{\alpha+1}) \notin \{g(\eta_\beta) \mid \beta \le \alpha\} \cup \{g(\kappa)\}$;

(c) $\eta_\alpha = \sup_{\beta<\alpha} \eta_\beta$ for $\alpha < \kappa$ limit.

Now it is clear that K contains a homeomorphic copy of $[0,\kappa]$, and therefore also that of $[0,\omega_1]$, as κ is uncountable. This is a contradiction. Hence $f \restriction A$ is closed. It follows that $f(A)$ is a dense closed subset of K, so $f(A) = K$. Therefore, K is a Σ-subset of itself by Theorem 3.22. It means that K is Corson. This completes the proof of the first point.

Let us prove the second one. Let K be an infinite continuous image of Valdivia compactum. If K is Corson, then it contains an one-to-one convergent sequence by Theorem 3.1. If K is not Corson, then it contains by the above a copy of $[0,\omega_1]$, and so also an one-to-one convergent sequence. ∎

3.4. Products of Valdivia compact spaces In this section we collect stability results from [30, Section 4] and [31, Section 4]. We begin by the following lemma on behavior of Σ-subsets with respect to products.

Lemma 3.28. *Let K_α, $\alpha \in I$, be a family of compact spaces and, for each $\alpha \in I$, let A_α be a dense Σ-subset of K_α and $x_\alpha \in A_\alpha$. Then the set*

$$A = \{y = (y_\alpha)_{\alpha\in I} \in \textstyle\prod_{\alpha\in I} A_\alpha : \{\alpha \in I : y_\alpha \neq x_\alpha\} \text{ is countable}\}$$

is a dense Σ-subset of the space $K = \prod_{\alpha\in I} K_\alpha$.

Proof. For each $\alpha \in I$ there is a homeomorphic injection $h_\alpha : K_\alpha \to \mathbb{R}^{\Gamma_\alpha}$ such that $h_\alpha(K_\alpha) \cap \Sigma(\Gamma_\alpha) = h_\alpha(A_\alpha)$. Obviously we can ensure that $h_\alpha(x_\alpha) = 0$ for each $\alpha \in I$. Let us define $h : K \to \prod_{\alpha\in I} \mathbb{R}^{\Gamma_\alpha} = \mathbb{R}^\Gamma$, where $\Gamma = \{(\gamma,\alpha) \mid \gamma \in \Gamma_\alpha, \alpha \in I\}$, by the formula

$$h((y_\alpha)_{\alpha\in I})(\gamma,\beta) = h_\beta(y_\beta)(\gamma).$$

It is obvious that h is a homeomorphic injection, and that $h(A) = h(K)\cap\Sigma(\Gamma)$, so A is a Σ-subset of K. The density of A in K follows easily from the definition of the product topology. ∎

Now the following stability theorem easily follows.

Theorem 3.29. *The product of an arbitrary family of Valdivia (super-Valdivia) compact spaces is again Valdivia (super-Valdivia, respectively).*

Corollary 3.30. *Arbitrary product of Corson compact spaces is super-Valdivia. In particular, $[-1,1]^\Gamma$, $[0,1]^\Gamma$ and $\{0,1\}^\Gamma$ are super-Valdivia compact spaces for any set Γ.*

It follows from Lemma 3.28 that Corson compact spaces are closed with respect to countable products. In fact, a converse holds as well.

Theorem 3.31. *Let K_α, $\alpha \in I$ be a family of nonempty compact spaces and $K = \prod_{\alpha\in I} K_\alpha$. Then K is Corson if and only if each K_α is Corson and all but countably many K_α's are one-point spaces.*

Proof. The 'if' part follows immediately from Lemma 3.28. To show the converse suppose that K is Corson. Each K_α is homeomorphic to a subset of K, hence it is Corson by Theorem 3.7. If uncountably many of K_α's have at least two points, then the dense Σ-subset of K defined in Lemma 3.28 is a proper subset of K, so K is not Corson by Lemma 1.7. ∎

The question whether given compact spaces are Valdivia provided their product has this property seems to be more difficult. In fact, we do not know the complete answer. The following results of [31] give a partial answer.

LEMMA 3.32. *Let K and L be nonempty compact spaces such that L has a dense set of G_δ points.*
(i) *If $K \times L$ is Valdivia, then both K and L are Valdivia as well.*
(ii) *If $K \times L$ is super-Valdivia, then K is super-Valdivia and L is Corson.*

Proof. Let us first show the assertion on K. Pick $l \in L$, a G_δ point of L. Then $K \times \{l\}$ is a closed G_δ subset of $K \times L$, hence we can use Theorem 3.8. To show the assertion on L, it is enough to observe that the projection of $K \times L$ onto L is an open continuous mapping, and use Theorem 3.24. ∎

The following theorem now follows immediately.

THEOREM 3.33. *Let K_α, $\alpha \in I$ be a family of nonempty compact spaces, such that each K_α has a dense set of G_δ points. Put $K = \prod_{\alpha \in I} K_\alpha$.*
(i) *K is Valdivia if and only if each K_α is Valdivia.*
(ii) *K is super-Valdivia if and only if each K_α is Corson.*

We finish this section by the following question.

QUESTION 3.34. Suppose that K and L are compact spaces such that $K \times L$ is Valdivia (super-Valdivia). Are both K and L again Valdivia (super-Valdivia, respectively)?

3.5. TOPOLOGICAL SUMS AND UNIONS OF VALDIVIA COMPACTA We begin this section by the following easy result.

THEOREM 3.35. (i) *Finite topological sum of Corson (Valdivia, super-Valdivia) compact spaces has again the same property.*
(ii) *The one-point compactification of an arbitrary topological sum of Corson (Valdivia, super-Valdivia) compact spaces has again the same property.*

Proof. This easily follows from the characterization of Σ-subsets in terms of separating families given in Proposition 1.9. ∎

Now we will study stability of Valdivia compacta with respect to unions of two spaces. This is related to continuous images, as any union is a continuous image of the respective topological sum.

LEMMA 3.36. *Let K be a compact space, H and L be closed subsets of K covering K, and $M = H \cap L$ be nowhere dense both in L and H. Then K is Valdivia if and only if there are B, a dense Σ-subset of H, and C, a dense Σ-subset of L, such that $M \cap B = M \cap C$, and this set is dense in M.*

Proof. Suppose that K is Valdivia. Then there is a dense Σ-subset A of K. It follows from Lemma 1.15 that $B = A \cap H$ and $C = A \cap L$ have the required properties. This proves the 'only if' part.

To show the 'if' part, put $K' = (H \times \{0\}) \cup (L \times \{1\})$ (considered as a subset of $K \times \{0,1\}$). It is easy to check that $A' = (B \times \{0\}) \cup (C \times \{1\})$ is a dense Σ-subset of K'. Denote by φ the canonical projection of K' onto K. It is easy to verify that this mapping satisfies the assumption of Lemma 2.8, and hence it follows from Theorem 3.22 that $\varphi(A')$ is a dense Σ-subset of K. ∎

Remarks 3.37. (i) It is clear from the proof of the previous lemma that the 'if' part holds also without assumption that M is nowhere dense in H and in L.

(ii) The 'only if' part of the previous lemma does not hold without assumption that M is nowhere dense both in L and H. This takes place for example if $K = H$ is Valdivia and L is a non-Valdivia subset of K, or if $K = H = [0, \omega_1]$ and $L = \{\omega_1\}$.

Now we are ready to formulate the following negative result.

THEOREM 3.38. *Let K and L be infinite compact spaces such that at least one of the following conditions holds true.*

(i) *Either K or L is not super-Valdivia.*

(ii) *Neither K nor L is Corson.*

Then there is a non-Valdivia compactum which can be represented as a union of two subsets, one homeomorphic to K, the second homeomorphic to L.

Proof. First suppose that L is not super-Valdivia. Then there is $a \in L$ which is contained in no dense Σ-subset of L. Choose $b \in K$ any non-isolated point. The required union will be the quotient space made from $K \oplus L$ by identifying a and b (see Proposition 3.19).

Now suppose that both K and L are super-Valdivia but not Corson. Then both K and L contain a nowhere dense copy of the interval $[0, \omega_1]$ with collated sequence by Theorem 3.14. We can conclude by Proposition 3.20. ∎

A very partial positive result, given in the following proposition, follows immediately from Lemma 3.36.

PROPOSITION 3.39. *Let $K = H \cup L$ such that both H and L are super-Valdivia compact spaces, and $H \cap L$ contains at most one point. Then K is Valdivia.*

We do not know whether, under the assumptions of the previous proposition, K is necessarily super-Valdivia. Also, we do not know the answer to the following question.

QUESTION 3.40. Suppose that $K = H \cup L$ such that H is super-Valdivia and L is Corson. Is then K Valdivia (super-Valdivia)?

We finish this section by a proposition which relates the previous question with other problems.

PROPOSITION 3.41. *Let K be a compact space. The following assertions are equivalent.*

(i) *For every Corson compactum L, any compactum which can be represented as a union of K and L is Valdivia.*

(ii) *For every Corson compactum L, any compactum which can be represented as a union of K and L is super-Valdivia.*

(iii) *Every closed subset of K which is a Corson compactum, is contained in a dense Σ-subset of K.*

Proof. (ii) $\Rightarrow$ (i) This is trivial.

(i) $\Rightarrow$ (iii) Let H be a closed subset of K which is Corson and is contained in no dense Σ-subset. As the interior of H is clearly contained in every dense Σ-subset, we can suppose without loss of generality that H is nowhere dense. Then $L = H \times [0, \omega]$ is Corson (Theorem 3.31). Put $K' = K \cup (H \times [0, \omega))$

endowed with the quotient topology induced by the mapping $Q : K \oplus L \to K'$ defined by

$$Q(x) = \begin{cases} x & x \in K \cup (H \times [0, \omega)), \\ h & x = (h, \omega) \in H \times \{\omega\}. \end{cases}$$

Then K' can be expressed as a union of K and L, and it easily follows from Lemma 3.36 that K' is not Valdivia.

(iii) $\Rightarrow$ (ii) Let $K' = K \cup L$ with L being Corson and $a \in K'$ be arbitrary. Then $(K \cap L) \cup \{a\}$ is Corson, and hence by the assumptions, it is contained in a dense Σ-subset B of K. Now it is easy to check, using Theorem 3.22, similarly as in the proof of Lemma 3.36, that $A = B \cup L$ is a dense Σ-subset of K' which contains a. ▮

3.6. Retractions on Valdivia compacta In this section we give some basic facts on retractions on Valdivia compact spaces. The construction of such retractions on some subspaces of the space $\Sigma(\Gamma)$ goes back to [22]. These results for Valdivia compact spaces were probably first formulated in [5]. A nice recent survey on families of retractions is [23]. First we give the following lemma.

Lemma 3.42. *Let K be a compact space and A be a dense Σ-subset of K. If M is an infinite subset of A, then there is a retraction $R : K \to K$ such that*

(i) *$M \subset R(K)$;*
(ii) *$R(K) \cap A$ is dense in $R(K)$;*
(iii) *The weight of $R(K)$ is at most* $\operatorname{card} M$.

Using this lemma, one can prove by a standard transfinite induction the following theorem.

Theorem 3.43. *Let K be a compact space of weight $\kappa > \aleph_0$ and A be a dense Σ-subset of K. Then there is a family of retractions $(R_\alpha : \omega \le \alpha \le \kappa)$ on K such that the following conditions are fulfilled.*

(i) *$R_\kappa = \mathrm{Id}_K$;*

(ii) *$R_\alpha R_\beta = R_\beta R_\alpha = R_\alpha$ for $\omega \le \alpha \le \beta \le \kappa$;*

(iii) *The weight of $R_\alpha K$ is at most* $\operatorname{card} \alpha$*;*

(iv) *$R_\alpha K \cap A$ is dense in $R_\alpha K$;*

(v) $R_\lambda K = \overline{\bigcup_{\omega\leq\alpha<\lambda} R_\alpha K}$ *whenever* $\lambda \in (\omega, \kappa]$ *is limit.*

4. Valdivia type Banach spaces

This chapter is devoted to the study of Banach spaces with Valdivia type properties. The first section has an auxiliary character and deals with projectional resolutions and projectional generators. In the second section we introduce and characterize some classes of Banach spaces whose duals have a Valdivia type property. The third section contains a result of [32] concerning strong non-stability of such spaces with respect to renormings. In the fourth section we discuss stability with respect to various types of products. The fifth section is devoted to study of subspaces and quotients.

Recently some results on spaces with Valdivia type biduals were obtained in [35]. We do not give a systematic exposition but we include some examples in the last chapter.

We will use some standard notation from Banach space theory. For a subset A of a Banach space we will mean by $\bar{A}$, $\overline{A}^{w}$, $\overline{A}^{w^*}$ its closure with respect to the norm, to the weak topology, to the weak* topology (if the respective Banach space is dual), respectively.

Further, if X is Banach space, we put

$$A^\perp = \{\xi \in X^* : (\forall a \in A)(\xi(a) = 0)\}, \qquad \text{for } A \subset X$$
$$B_\perp = \{x \in X : (\forall b \in B)(b(x) = 0)\}, \qquad \text{for } B \subset X^*$$

By $\mathbb{Q}$ we denote the field of rational numbers. By $\mathbb{Q}$-linear subspace of a Banach space we mean a subset which is closed with respect to rational linear combinations.

4.1. Projectional generators and projectional resolutions of the identity

In this auxiliary section we sketch a method of constructing projectional resolutions of the identity in certain Banach spaces. This goes back to J. Orihuela and M. Valdivia [52], a finer approach is given in [17, Chapter 6].

We start with the following definition.

Definition 4.1. Let X be a Banach space and $S \subset X^*$ be an arbitrary subset.

(i) We say that S is *1-norming* if $\|x\| = \sup\{\xi(x) : \xi \in S, \|\xi\| \leq 1\}$ for any $x \in X$.

(ii) We say that S is *norming* if the formula $|x| = \sup\{\xi(x) : \xi \in S, \|\xi\| \leq 1\}$, $x \in X$, defines an equivalent norm on X.

Remarks 4.2. (i) It is obvious that $S \subset X^*$ is norming if and only if there is an equivalent norm on X such that S is 1-norming with respect to this norm.

(ii) If $S \subset X^*$ is linear (or even only $\mathbb{Q}$-linear), then it easily follows from the Hahn-Banach separation theorem that S is 1-norming if and only if $S \cap B_{X^*}$ is weak* dense in B_{X^*}.

We continue by the definition of the projectional generator. There are several different definitions in the literature ([52], [17]). We choose an easier one as it is sufficient for applications.

DEFINITION 4.3. Let X be a Banach space. *Projectional generator* on X is a pair (S, Φ) such that

(a) S is a 1-norming $\mathbb{Q}$-linear subspace of X^*;

(b) $\Phi : S \to X$ is a countably valued mapping;

(c) $(\Phi(B))^\perp \cap \overline{B}^{w^*} = \{0\}$ whenever $B \subset S$ is $\mathbb{Q}$-linear.

Now we give the definition of the projectional resolution of the identity which goes back to [43, 44].

DEFINITION 4.4. Let X be a non-separable Banach space with $\operatorname{dens} X = \mu$. By *projectional resolution of the identity* (shortly *PRI*) on X we mean a family $(P_\alpha : \omega \leq \alpha \leq \mu)$ of projections on X such that the following conditions are satisfied.

(i) $P_\omega = 0$, $P_\mu = \mathrm{Id}_X$;

(ii) $\|P_\alpha\| = 1$ for $\omega < \alpha \leq \mu$;

(iii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\omega \leq \alpha \leq \beta \leq \mu$;

(iv) $\operatorname{dens} P_\alpha X \leq \operatorname{card} \alpha$ for $\omega < \alpha \leq \mu$;

(v) $\bigcup_{\beta<\alpha} P_\beta X$ is dense in $P_\alpha X$ whenever $\alpha \leq \mu$ is a limit ordinal.

The previous definition is a standard one but it seems that the formulation of condition (iv) is not the best one. It turns out that in some cases it is too strong and in some cases it seems to be too weak. So we introduce the following modifications of the notion of PRI.

DEFINITION 4.5. Let X be a non-separable Banach space with $\operatorname{dens} X = \mu$. By *weak projectional resolution of the identity* (shortly *weak PRI*) on X we mean a family $(P_\alpha : \omega \leq \alpha \leq \mu)$ of projections on X which satisfies the same conditions as those in the definition of PRI except for the condition (iv) which is replaced by

(iv') $\operatorname{dens} P_\alpha X < \mu$ for every $\omega < \alpha < \mu$.

By *strong projectional resolution of the identity* (shortly *strong PRI*) on X we mean a family $(P_\alpha : \omega \leq \alpha \leq \mu)$ of projections on X which satisfies the same conditions as those in the definition of PRI except for the condition (iv) which is replaced by

(iv") $\operatorname{dens} P_\alpha X = \operatorname{card} \alpha$ for every $\omega < \alpha < \mu$.

The rest of this section is devoted to constructing a PRI from a projectional generator. We follow the ideas of [17, Chapter 6]. However, we refine the construction, and so we give the proofs of some steps.

We begin the following lemma on the "norming pair". The proof is easy and standard, it is given for example in [17, Lemma 6.1.1].

LEMMA 4.6. *Let X be a Banach space, $A \subset X$ and $B \subset X^*$ be such that*

(a) *$\bar{A}$ and $\bar{B}$ are linear spaces;*

(b) *$\|x\| = \sup\{\xi(x) : \xi \in B, \|\xi\| \leq 1\}$ for every $x \in A$;*

(c) *$A^\perp \cap \overline{B}^{w^*} = \{0\}$.*

Then there is a projection P on X of norm 1 such that $PX = \bar{A}$, $P^{-1}(0) = B_\perp$, $P^* X^* = \overline{B}^{w^*}$.

The following lemma follows from [17, Lemma 6.1.3]. In fact, the proof given there is not completely correct, but it works in our setting. The lemma formulated in [17] is also valid but requires a finer proof.

LEMMA 4.7. *Let X be a Banach space, $S \subset X^*$ be $\mathbb{Q}$-linear, $\Phi : S \to X$ and $\Psi : X \to S$ be countably valued (set-valued) mappings. If κ is an infinite cardinal and $A_0 \subset X$, $B_0 \subset S$ are such that* $\operatorname{card} A_0 \le \kappa$ *and* $\operatorname{card} B_0 \le \kappa$, *then there are A, B such that*

(i) $A_0 \subset A \subset X$ *and* $B_0 \subset B \subset S$;

(ii) A *and* B *are* $\mathbb{Q}$*-linear spaces;*

(iii) $\operatorname{card} A \le \kappa$, $\operatorname{card} B \le \kappa$;

(iv) $\Phi(B) \subset A$ *and* $\Psi(A) \subset B$.

The following proposition refines a bit a special case of [17, Proposition 6.1.4].

PROPOSITION 4.8. *Let X, S, Φ, Ψ be like in Lemma 4.7. Suppose that X is nonseparable and $\mu = \operatorname{dens} X$. Then there are families $(A_\alpha : \omega < \alpha \le \mu)$ and $(B_\alpha : \omega < \alpha \le \mu)$, such that the following conditions are fulfilled.*

(a) $A_\alpha \subset X$, $B_\alpha \subset S$, $\bar{A}_\mu = X$;

(b) A_α *and* B_α *are* $\mathbb{Q}$*-linear;*

(c) $\operatorname{dens} A_\alpha = \operatorname{card} A_\alpha = \operatorname{card} \alpha$, $\operatorname{card} B_\alpha \le \operatorname{card} \alpha$;

(d) $\Phi(B_\alpha) \subset A_\alpha$, $\Psi(A_\alpha) \subset B_\alpha$;

(e) $A_\beta \subset A_\alpha$ *and* $B_\beta \subset B_\alpha$ *if* $\beta \le \alpha$, $\bar{A}_\beta \subsetneq \bar{A}_\alpha$ *if* $\beta < \alpha$;

(f) $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ *and* $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ *if α is a limit ordinal.*

Proof. Let $\{x_\alpha : \omega \le \alpha < \mu\}$ be a dense subset of X not containing 0. Let $A_{\omega+1}$ and $B_{\omega+1}$ be the sets A, B from Lemma 4.7 applied to $A_0 = \{x_0\}$ and $B_0 = \emptyset$. Then the conditions (a), (b), (d) are fulfilled by Lemma 4.7. The condition (c) is satisfied due to Lemma 4.7 together with the obvious fact that $A_{\omega+1}$ is infinite as $x_0 \ne 0$.

Suppose we have A_α and B_α for some $\alpha < \mu$. Let $\gamma = \min\{\delta : x_\delta \notin \bar{A}_\alpha\}$. This γ exists, as the set on the right-hand side is nonempty, since $\operatorname{dens} X = \mu > \operatorname{card} \alpha = \operatorname{card} A_\alpha$ (by (c)). Let $A_{\alpha+1}$ and $B_{\alpha+1}$ be the sets A, B obtained by Lemma 4.7 applied to $A_0 = A_\alpha \cup \{x_\gamma\}$, $B_0 = B_\gamma$. Then the conditions (a)–(e) clearly remain valid.

Let $\alpha \le \mu$ be a limit ordinal such that we have constructed A_β and B_β for each $\beta < \alpha$. Put $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. Then the conditions (a), (b), (d), (e), (f) are clearly satisfied. (The validity of (a) in case $\alpha = \mu$ follows by the construction, as $\bar{A}_\mu$ contains x_γ for every $\gamma < \mu$.) It remains to prove the condition (c). It is clear, by the induction hypothesis, that $\operatorname{dens} A_\alpha \le \operatorname{card} A_\alpha \le \operatorname{card} \alpha$ and $\operatorname{card} B_\alpha \le \operatorname{card} \alpha$. Now it suffices to prove that $\operatorname{dens} A_\alpha \ge \operatorname{card} \alpha$. If α either is not a cardinal or is a limit cardinal, then $\operatorname{card} \alpha = \sup_{\beta<\alpha} \operatorname{card} \beta$, hence the required inequality follows immediately from the induction hypothesis. If α is a successor cardinal, say $\alpha = \kappa^+$, and $\operatorname{dens} A_\alpha < \alpha$, then $\operatorname{dens} A_\alpha \le \kappa$. Let D be a dense subset of A_α such that $\operatorname{card} D \le \kappa$. Then it follows from (f) that there is $\beta < \alpha$ with $D \subset A_\beta$. But then $\bar{A}_\beta = \bar{A}_\alpha$, which contradicts (e).

This completes the proof. ∎

We finish this section by the following proposition on the existence of PRI. This is a slight generalization of [17, Proposition 6.1.7].

Proposition 4.9. *Let X be a non-separable space which admits a projectional generator (S, Φ), and $M \subset X$ be a set such that for every $s \in S$ the set $\operatorname{supp} s = \{x \in M : s(x) \ne 0\}$ is at most countable. Let μ denote the density of X. Then X has a strong PRI $(P_\alpha : \omega \le \alpha \le \mu)$, such that $M \subset \bigcup_{\omega\le\alpha<\mu} (P_{\alpha+1} - P_\alpha)X$.*

Proof. At first define $\Phi'(s) = \Phi(s) \cup \operatorname{supp} s$. It is easy to check that (S, Φ') is again a projectional generator. As S is 1-norming, we can choose for every $x \in X$ a countable set $\Psi(x) \subset S \cap B_{X^*}$, such that $\|x\| = \sup\{s(x) : s \in \Psi(x)\}$. Now apply Proposition 4.8 (with Φ' instead of Φ) to get the long sequences $(A_\alpha : \omega < \alpha \le \mu)$ and $(B_\alpha : \omega < \alpha \le \mu)$ satisfying the conditions of the mentioned proposition.

It is easy to see that A_α and B_α fulfil the assumptions of the Lemma 4.6, and so there is P_α, a norm one projection on X such that $P_\alpha X = \bar{A}_\alpha$, $P_\alpha^* X^* = \overline{B_\alpha}^{w^*}$ and $(P_\alpha)^{-1}(0) = (B_\alpha)_\perp$. It is easy to verify (cf. [17]) that these P_α together with $P_\omega = 0$ form a strong PRI.

Next we are going to prove that $M \subset A_\mu \cup \{0\}$. Suppose it is not the case and choose $0 \ne m \in M \setminus A_\mu$. As $\Phi'(B_\mu) \subset A_\mu$, we get $m \notin \Phi'(B_\mu)$. By the definition of Φ' it follows that $s(m) = 0$ for every $s \in B_\mu$, hence $m \in (B_\mu)_\perp$. But $(B_\mu)_\perp = \{0\}$ by the previous paragraph. This is a contradiction.

So, for any $0 \ne m \in M$ there is a minimal α with $m \in A_\alpha$. Due to the condition (f) of Proposition 4.8 it is of the form $\alpha = \beta + 1$. As $m \notin A_\beta$, we get $m \notin \Phi'(B_\beta)$, and hence $m \in (B_\beta)_\perp$. It follows that $m \in \bar{A}_{\beta+1} \cap (B_\beta)_\perp = (P_{\beta+1} - P_\beta)X$. This completes the proof. ∎

Let us remark, that by choosing $M = \emptyset$ we get that every nonseparable Banach space with projectional generator admits a (strong) PRI [17, Proposition 6.1.7].

Remark 4.10. Similarly as we introduced the notion of weak PRI, we can introduce the notion of *weak projectional generator* on a Banach space X. This would be a pair (S, Φ) with the same properties as projectional generator (see Definition 4.3), only the condition (b) would be replaced by the following one.

(b') $\Phi : S \to X$ is a multivalued mapping such that $\operatorname{card} \Phi(s) < \operatorname{dens} X$ for every $s \in S$.

Then it is easy to show (following the methods used in this section) that the existence of a weak projectional generator implies the existence of a weak PRI provided $\operatorname{dens} X$ is a regular cardinal.

4.2. Some classes of Banach spaces with Valdivia type duals

In this section we give definition and some characterizations of certain classes of Banach spaces associated with Valdivia compacta.

Definition 4.11. Let X be a Banach space.

(i) We say that $S \subset X^*$ is a *Σ-subspace* of X^* if there is a linear one-to-one weak* continuous mapping $T : X^* \to \mathbb{R}^\Gamma$, for a set Γ, such that $S = T^{-1}(\Sigma(\Gamma))$.

(ii) We say that X is *weakly Lindelöf determined* (shortly *WLD*) if X^* is a Σ-subspace of itself.

(iii) We say that X is a *Plichko* space if X^* has a norming Σ-subspace.

(iv) We say that X is a *1-Plichko* space if X^* has a 1-norming Σ-subspace.

The class of WLD spaces was investigated probably first in [63], the name WLD was given in [4]. We introduce the notions of Plichko and 1-Plichko spaces. These classes were, with a different definition, thoroughly studied by A. Plichko [53, 54, 55, 56]. It turns out that the class of 1-Plichko spaces coincides with the class $\mathcal{V}$ of J. Orihuela [50], studied also by M. Valdivia in [65].

Remark 4.12. By replacing $\Sigma(\Gamma)$ with $\Sigma_\kappa(\Gamma)$ we can obviously define notions of *Σ_κ-subspace, κ-WLD space, κ-Plichko* and *1-κ-Plichko space*. These notions can be sometimes useful.

The above introduced classes of Banach spaces can be characterized in terms of certain Markushevich bases. In fact, this was the definition used by A. Plichko. Let us now give the definition of these bases.

DEFINITION 4.13. Let X be a Banach space.

(i) A *Markushevich basis* (shortly *M-basis*) of X is a family $(x_\alpha, f_\alpha)_{\alpha\in\Lambda}$ of elements of $X \times X^*$ such that the following conditions are fulfilled.

- $\overline{\operatorname{span}\{x_\alpha : \alpha \in \Lambda\}} = X$;
- $f_\alpha(x_\alpha) = 1$, $f_\alpha(x_\beta) = 0$ if $\alpha \neq \beta$;
- For any $x \in X \setminus \{0\}$ there is $\alpha \in \Lambda$ with $f_\alpha(x) \neq 0$.

(ii) An M-basis $(x_\alpha, f_\alpha)_{\alpha\in\Lambda}$ is called *norming* (*1-norming*) if the space

$$\operatorname{span}\{f_\alpha : \alpha \in \Lambda\} = \{f \in X^* : \{\alpha \in \Lambda : f(x_\alpha) \neq 0\} \text{ is finite}\}$$

is a norming (1-norming, respectively) subspace of X^*.

(iii) An M-basis $(x_\alpha, f_\alpha)_{\alpha\in\Lambda}$ is called *countably norming* (*countably 1-norming*) if the space

$$\{f \in X^* : \{\alpha \in \Lambda : f(x_\alpha) \neq 0\} \text{ is countable}\}$$

is a norming (1-norming, respectively) subspace of X^*.

Now we formulate the main theorems of this section. The proofs, together with some auxiliary results will then form the rest of this section.

THEOREM 4.14. *Let X be a nonseparable Banach space and let μ denote the density of X.*

(i) *If X is 1-Plichko, then there is a strong PRI $(P_\alpha : \omega \leq \alpha \leq \mu)$ such that $(P_{\alpha+1} - P_\alpha)X$ is 1-Plichko for every $\alpha \in [\omega, \mu)$.*

(ii) *If X has a weak PRI, then X is 1-μ-Plichko.*

(iii) *If μ is a regular cardinal and X is 1-μ-Plichko, then X has a weak PRI.*

THEOREM 4.15. *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X is 1-Plichko.*
2. *There is a set $M \subset X$ such that $\overline{\operatorname{span} M} = X$ and that*

$$\{f \in X^* : \{m \in M : f(x) \neq 0\} \text{ is countable}\}$$

is a 1-norming subspace of X^*.

3. *X has a countably 1-norming M-basis.*
4. *(B_{X^*}, w^*) has a dense convex symmetric Σ-subset.*

If, moreover, $\operatorname{dens} X = \aleph_1$, then the previous conditions are also equivalent with the following one.

5. *X has a PRI.*

THEOREM 4.16. *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X is Plichko.*
2. *There is an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ is 1-Plichko.*
3. *X has a countably norming M-basis.*

THEOREM 4.17. *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X is weakly Lindelöf determined.*
2. *There is an M-basis $(x_\alpha, f_\alpha)_{\alpha\in\Lambda}$ of X such that the set $\{\alpha \in \Lambda : f(x_\alpha) \neq 0\}$ is countable for every $f \in X^*$.*
3. *(B_{X^*}, w^*) is a Corson compactum.*
4. *(X, w) is primarily Lindelöf.*

To prove these theorems we will use several lemmas. The first one will be a characterization of 1-norming Σ-subspaces.

LEMMA 4.18. *Let X be a Banach space and $S \subset X^*$ a 1-norming linear subspace. Then the following assertions are equivalent.*

(a) *S is a Σ-subspace of X^*.*

(b) $S \cap B_{X^*}$ *is a* Σ*-subset of* (B_{X^*}, w^*).

(c) *There is a dense convex symmetric* Σ*-subset* A *of* (B_{X^*}, w^*) *such that* $S = \operatorname{span} A$.

(d) $S \cap B_{X^*}$ *is weak* countably compact and* (X, w_S) *is primarily Lindelöf.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious.

(c)⇒(d) Suppose that the condition (c) holds. Then (X, w_A) is primarily Lindelöf by Theorem 2.7. And it is obvious that $w_A = w_S$, as $S = \operatorname{span} A$. Further, by Theorem 2.7 (the implication (ii)⇒(i)) we have $A = B_{X^*} \cap \operatorname{span} A$, so $S \cap A$ is weak* countably compact by Lemma 1.6.

(d)⇒(a) This follows from Theorem 2.7 (the implication (iii)⇒(i)). ∎

LEMMA 4.19. *Let* X *be a Banach space and* $M \subset X$ *be such that* $\overline{\operatorname{span} M} = X$ *and that*

$$S = \{f \in X^* : \{m \in M : f(m) \neq 0\} \text{ is countable}\}$$

is a 1-norming subspace of X^*. *Then the following hold.*

(i) *If* $\operatorname{dens} X = \mu > \aleph_0$, *then there is a strong PRI* $(P_\alpha : \omega \leq \alpha \leq \mu)$ *such that the following conditions are fulfilled.*

- $M \subset \bigcup_{\omega \leq \alpha < \mu} (P_{\alpha+1} - P_\alpha)X$;
- $\overline{\operatorname{span}(M \cap (P_{\alpha+1} - P_\alpha)X)} = (P_{\alpha+1} - P_\alpha)X$ *for every* $\omega \leq \alpha < \mu$;
- $S_\alpha = \{s \restriction (P_{\alpha+1} - P_\alpha)X : s \in S\}$ *is a 1-norming subspace of* $((P_{\alpha+1} - P_\alpha)X)^*$ *for every* $\omega \leq \alpha < \mu$.

(ii) *There is an M-basis* $(x_\alpha, f_\alpha)_{\alpha \in \Lambda}$, *such that all* x_α *belong to* $\operatorname{span} M$ *and for every* $s \in S$ *the set* $\{\alpha \in \Lambda : s(x_\alpha) \neq 0\}$ *is countable.*

Proof. (i) For every $s \in S$ put $\Phi(s) = \{m \in M : s(m) \neq 0\}$. By the assumptions this is a countably valued mapping, and it is easy to check that (S, Φ) is projectional generator on X. By Proposition 4.9 there is a strong PRI on X satisfying the first condition. The second condition follows easily from the fact that $\operatorname{span} M$ is dense in X. The third one follows immediately from the fact that S is 1-norming.

(ii) We will prove this by transfinite induction on $\operatorname{dens} X$. If X is separable, this follows from a classical theorem of Markushevich (see e.g. [24, Theorem

272]). Suppose that X has density $\mu > \aleph_0$ and that we have proved the statement for spaces with density strictly less than μ. Let $(P_\alpha : \omega \leq \alpha \leq \mu)$ be a PRI on X satisfying the conditions from (i) . For $\alpha \in [\omega, \mu)$ put $M_\alpha = M \cap (P_{\alpha+1} - P_\alpha)X$ and $S_\alpha = \{s \restriction (P_{\alpha+1} - P_\alpha)X : s \in S\}$. By (i) and induction hypothesis there is an M-basis $(x_\gamma, f_\gamma)_{\gamma \in \Lambda_\alpha}$ of $(P_{\alpha+1} - P_\alpha)X$ such that all x_γ are in $\operatorname{span} M_\alpha$ and for every $s \in S_\alpha$ the set $\{\gamma \in \Lambda_\alpha : s(x_\gamma) \neq 0\}$ is countable. Every f_γ can be extended to a continuous functional $\tilde{f}_\gamma$ on X such that $\tilde{f}_\gamma$ is zero on $P_\alpha X \oplus (\mathrm{Id} - P_{\alpha+1})X$. Then, putting together all these M-bases, one get an M-basis of X with the required properties. ∎

Now we are ready to prove the main theorems of this section.

Proof. (of Theorem 4.15) The equivalence $1 \Leftrightarrow 4$ follows from Lemma 4.18.

$1 \Rightarrow 2$ If X is 1-Plichko, there is a 1-norming Σ-subspace S of X^*. It means there is a linear one-to-one weak* continuous mapping $T : X^* \to \mathbb{R}^\Gamma$ such that $S = T^{-1}(\Sigma(\Gamma))$. For every $\gamma \in \Gamma$ the functional $f \mapsto T(f)(\gamma)$ is weak* continuous, and hence is represented by some $m_\gamma \in X$ (see e.g. [24, Theorem 55]). Put $M = \{m_\gamma : \gamma \in \Gamma\}$.

$2 \Rightarrow 3$ This follows from Lemma 4.19(ii) .

$3 \Rightarrow 1$ Let $(x_\alpha, f_\alpha)_{\alpha \in \Lambda}$ be a countably 1-norming M-basis of X. Define $T : X^* \to \mathbb{R}^\Lambda$ by putting $T(f)(\alpha) = f(x_\alpha)$. This mapping clearly witnesses that X is 1-Plichko.

If $\operatorname{dens} X = \aleph_1$, then $1 \Leftrightarrow 5$ follows from Theorem 4.14 which we are going to prove in a while. (Note, that the implication $1 \Rightarrow 5$ holds regardless of density of X.) ∎

Proof. (of Theorem 4.14) The assertion (i) follows from Lemma 4.19(i) together with the equivalence $1 \Leftrightarrow 2$ of Theorem 4.15.

(ii) Let $(P_\alpha : \omega \leq \alpha \leq \mu)$ be a weak PRI on X. Put $\kappa_\alpha = \operatorname{dens} P_{\alpha+1}X$. Then $\kappa_\alpha < \mu$ for every $\alpha \in [\omega, \mu)$ and, moreover, κ_α form a non-decreasing long sequence. For every $\alpha \in [\omega, \mu)$ let I_α be a dense subset of $(P_{\alpha+1} - P_\alpha)X$ such that $\operatorname{card} I_\alpha \leq \kappa_\alpha$. Put $I = \bigcup_{\omega \leq \alpha < \mu} I_\alpha$ and define $T : X^* \to \mathbb{R}^I$ by the formula $T(f)(i) = f(i)$. It is easy to check that $S = \bigcup_{\omega \leq \alpha < \mu} P_\alpha^* X^*$ is 1-norming and $T(S) \subset \Sigma_\mu(\Gamma)$. It follows that X is 1-μ-Plichko.

(iii) Let S be a 1-norming Σ_μ-subspace of X^* and $T : X^* \to \mathbb{R}^\Gamma$ a linear one-to-one weak* continuous mapping such that $T(S) \subset \Sigma_\mu(\Gamma)$. For every $\gamma \in \Gamma$ the linear functional $T_\gamma(f) = T(f)(\gamma)$ is weak* continuous, and hence is represented by an element of X. If we put $\Phi(f) = \{T_\gamma : T_\gamma(f) \neq 0\}$, then the pair (S, Φ) is clearly a weak projectional generator. It remains to use Remark 4.10. ∎

Proof. (of Theorem 4.16) This theorem follows immediately from Theorem 4.15 using Remark 4.2. ∎

Proof. (of Theorem 4.17) The equivalences $1 \Leftrightarrow 3 \Leftrightarrow 4$ follow from Lemma 4.18.

The implication $1 \Rightarrow 2$ follows from Lemma 4.19 using the same idea as in the proof of Theorem 4.15, $1 \Rightarrow 2$.

Finally, $2 \Rightarrow 1$ can be proved in the exactly same way as the implication $3 \Rightarrow 1$ of Theorem 4.15. ∎

Remarks 4.20. (i) It follows from Theorem 4.15 that the dual unit ball of a 1-Plichko Banach space in its weak* topology is a Valdivia compactum. We do not know whether the converse holds true. We formulate it below as a question. A partial positive answer will be given in Theorem 5.3 below. An example in the negative direction will be given in Example 6.8.

(ii) The idea of proof of Theorem 4.14(ii) comes from [18]. Note, that we cannot expect that the space in question would be 1-Plichko. It can be easily seen that the space $C[0, \omega_2]$ has even strong PRI, but the dual unit ball is not Valdivia (Example 1.10 and Theorem 5.3).

(iii) The theorem 4.17 was proved, in a bit different form, in [4]. The implication $3 \Rightarrow 5$ of Theorem 4.15 was proved first by A. Plichko [54]. The implications $4 \Rightarrow 3$ and $4 \Rightarrow 5$ follow also from results of [65]. The validity of $3 \Rightarrow 4$ was observed in [29, Lemma 3].

QUESTION 4.21. Let X be a Banach space such that (B_{X^*}, w^*) is Valdivia. Is then X 1-Plichko?

4.3. NON-STABILITY WITH RESPECT TO EQUIVALENT NORMS While it is clear that Plichko spaces are stable with respect to equivalent norms, for 1-Plichko spaces it is not the case. This was shown for example in [29] and [18]. In fact, a stronger statement holds. The following theorem was proved in [32].

THEOREM 4.22. *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X is weakly Lindelöf determined.*
2. *X is 1-Plichko with respect to every equivalent norm on X.*

3. $B_{(X,|\cdot|)^*}$ *is a Valdivia compactum in the weak* topology, for every equivalent norm* $|\cdot|$ *on* X.

If, moreover, $\operatorname{dens} X = \aleph_1$*, then the previous conditions are equivalent also to the following one.*

4. *X has a projectional resolution of the identity with respect to every equivalent norm.*

Before proving this theorem we collect several interesting consequences.

COROLLARY 4.23. *If* Γ *is an uncountable set, then there is an equivalent norm on* $\ell_1(\Gamma)$ *such that the respective dual unit ball (with the weak* topology) is not a Valdivia compactum.*

COROLLARY 4.24. *There is an equivalent norm on* $\ell_1([0,\omega_1))$*, such that the space* $\ell_1([0,\omega_1))$ *has no PRI with respect to this norm.*

Proof. (of Corollaries 4.23 and 4.24) It is enough to observe that $\ell_1(\Gamma)$ is 1-Plichko for every Γ and that it is not WLD if Γ is uncountable; and use Theorem 4.22. ∎

A quantitative version of Corollary 4.24 was obtained in [58].

Theorem 4.22 also yields the following answer to a question of [46, §4, p.517].

COROLLARY 4.25. *Under continuum hypothesis there is a Corson compact space* K*, and an equivalent norm on* $C(K)$ *such that* $C(K)$ *has no PRI with respect to this norm.*

Proof. Assume continuum hypothesis. By [5] there is a Corson compactum K of weight $\aleph_1$ such that $C(K)$ is not WLD. Then it suffices to use Theorem 4.22. ∎

To prove Theorem 4.22 we will use the following two easy lemmas from [32].

LEMMA 4.26. *Let* X *be a Banach space and* $K \subset S_{X^*}$ *be a convex weak* compact set. Then there is a convex weak* compact set* L *which is weak** G_δ *in* B_{X^*} *and* $K \subset L \subset S_{X^*}$.

Proof. Let $n \in \mathbb{N}$. The sets $(1-\frac{1}{n})B_{X^*}$ and K are two disjoint convex weak* compact sets, so there is, by Hahn-Banach theorem, $x_n \in X$ and $c_n \in \mathbb{R}$ such that

$$\sup_{f\in(1-\frac{1}{n})B_{X^*}} f(x_n) < c_n < \inf_{f\in K} f(x_n).$$

It is enough to put

$$L = \{f \in B_{X^*} : (\forall n \in \mathbb{N})(f(x_n) \geq c_n)\}.$$

∎

LEMMA 4.27. *Let $(X, \|\cdot\|)$ be a Banach space such that there is a weak* compact convex set $K \subset S_{X^*}$ which is not a Valdivia compactum. Then there is, for any $\varepsilon \in (0,1)$, an equivalent norm $|\cdot|$ on X such that $(1-\varepsilon)\|\cdot\| \leq |\cdot| \leq \|\cdot\|$ and that $B_{(X,|\cdot|)^*}$ is not Valdivia.*

Proof. Let L be a convex weak* compact set, weak* G_δ in B_{X^*} such that $K \subset L \subset S_{X^*}$. Such L exists due to Lemma 4.26. Put

$$B = \operatorname{conv}\left(K \cup (-K) \cup (1-\varepsilon)B_{X^*}\right).$$

Then B is a convex symmetric weak* compact set such that $(1-\varepsilon)B_{X^*} \subset B \subset B_{X^*}$, so there is an equivalent norm $|\cdot|$ on X such that B is its dual unit ball. It remains to show that B is not Valdivia. To see it we will prove that $K = L \cap B$. Choose $f \in L \cap B$. Then there are $s, t \geq 0$, $s + t \leq 1$ and $k_1, k_2 \in K$, $b \in (1-\varepsilon)B_X^*$ such that $f = sb + tk_1 + (1-s-t)(-k_2)$. We have

$$1 = \|f\| \leq s\|b\| + t\|k_1\| + (1-s-t)\|-k_2\| \leq s(1-\varepsilon) + t + 1 - s - t = 1 - s\varepsilon,$$

hence $s = 0$. So $f = tk_1 + (1-t)(-k_2)$. As $k_2 \in K \subset L$, we get $\frac{1}{2}(f + k_2) \in L$, but $\frac{1}{2}(f + k_2) = \frac{t}{2}(k_1 + k_2)$, so $\|\frac{1}{2}(f + k_2)\| = t$, hence $t = 1$.

So $K = L \cap B$ and therefore K is weak* G_δ in B. If B was Valdivia, K would be Valdivia as well by Theorem 3.8. It follows that B is not Valdivia which completes the proof. ∎

Proof. (of Theorem 4.22) The implication $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 3$ This follows from Theorem 4.15.

$3 \Rightarrow 1$ Assume X is not WLD. Then (B_{X^*}, w^*) is not Corson by Theorem 4.17. If it is not Valdivia, there is nothing to prove, hence suppose that (B_{X^*}, w^*) is Valdivia. Let A be a dense Σ-subset. If $S_{X^*} \subset A$, then by Lemma 1.6 and a corollary to Josefson-Nissenzweig theorem [13, Chapter

XII, Exercise 2(i)] we would get $B_{X^*} \subset A$, and so B_{X^*} would be Corson. So there is $f \in S_{X^*} \setminus A$. Apply Lemma 4.26 to get a convex weak* compact set $L \subset S_{X^*}$ which is weak* G_δ in B_{X^*} and contains f. By Lemma 1.11 we have that $L \cap A$ is dense in L, so L is Valdivia, and as $f \in L \setminus A$, L is not Corson. By Theorem 3.16 there is $K \subset L$ convex weak* compact, non-Valdivia. Finally by Lemma 4.27 there is an equivalent norm on X such that the corresponding dual unit ball is not Valdivia. This completes the proof.

$1 \Rightarrow 4$ This follows from Theorem 4.14.

If $\operatorname{dens} X = \aleph_1$, then $4 \Rightarrow 2$ follows from Theorem 4.15. ∎

For spaces of density greater than $\aleph_1$ we have the following theorem.

THEOREM 4.28. *Let X be a Banach space of density μ such that μ is a regular cardinal. Then the following assertions are equivalent.*

(a) *X is μ-WLD.*

(b) *(B_{X^*}, w^*) is a μ-Corson compact space.*

(c) *X is 1-μ-Plichko with respect to every equivalent norm.*

(d) *$(B_{(X,|\cdot|)^*}, w^*)$ is a μ-Valdivia compactum for every equivalent norm $|\cdot|$ on X.*

(e) *X has a weak PRI with respect to every equivalent norm.*

(f) *X has a weak PRI $(P_\alpha : \omega \le \alpha \le \mu)$ such that $\bigcup_{\alpha<\mu} P_\alpha^* X^* = X^*$.*

Proof. (Remarks on the proof) The implication (a)⇒(b) is trivial.

(b)⇒ (f) This can be proved following the proof [17, Proposition 8.3.1], and then using Remark 4.10.

(b)⇒ (e) This follows from the previous implication, using the obvious fact that the dual unit ball is μ-Corson with respect to every equivalent norm.

The implication $(e) \Rightarrow (c)$ follows from Theorem 4.14.

(c)⇒ (d) This is trivial.

(d)⇒ (b) This can be proved in the same way as the implication $3 \Rightarrow 1$ of Theorem 4.22, using Remark 3.17 instead of Theorem 3.16.

(f)⇒ (a) This follows from the proof of Theorem 4.14(ii) . ∎

Let us remark that the existence of a weak PRI implies the existence of a PRI provided the density of the Banach space in question is a successor cardinal, so we have the following example.

EXAMPLE 4.29. Let X be the c_0 sum of $\aleph_2$ copies of the space $\ell_1([0,\omega_1))$. Then X is $\aleph_2$-WLD, not WLD and has PRI with respect to every equivalent norm.

This shows that WLD spaces of large density cannot be characterized using the notion of PRI. However, the following question seems to be open.

QUESTION 4.30. Let X be a Banach space such that X has a strong PRI with respect to each equivalent norm. Is then X necessarily WLD?

4.4. STABILITY TO PRODUCTS In this section we collect some results on stability and non-stability of Valdivia type classes of Banach space with respect to taking products. We begin by naming the theorems.

THEOREM 4.31. (i) *Let $X_1,\dots,X_n$ be 1-Plichko spaces and N be a norm on $\mathbb{R}^n$ such that $N(t_1,\dots,t_n) = N(|t_1|,\dots,|t_n|)$ for any $t_1,\dots,t_n \in \mathbb{R}$. Equip the product $X = X_1 \times \dots \times X_n$ with the norm $\|(x_1,\dots,x_n)\| = N(\|x_1\|,\dots,\|x_n\|)$. Then $(X,\|\cdot\|)$ is 1-Plichko.*

(ii) *Let X, Y be two non-trivial Banach spaces such that at least one of them is not WLD and at least one of them has Valdivia dual unit ball. Then there is a norm $\|\cdot\|$ on $X\times Y$ such that $\max(\|x\|,\|y\|) \le \|(x,y)\| \le \|x\|+\|y\|$ and that the respective dual unit ball is not Valdivia.*

(iii) *The c_0-sum, as well as the ℓ_p-sum for any $p \in [1,\infty)$, of an arbitrary family of 1-Plichko spaces is again 1-Plichko.*

THEOREM 4.32. (a) *The product of a finite number of Plichko spaces is again Plichko.*

(b) *There exists a sequence $(X_n : n \in \mathbb{N})$ of Plichko spaces such that neither the c_0-sum nor the ℓ_p-sum for any $p \in [1,\infty)$ of these spaces is Plichko.*

THEOREM 4.33. (i) *The product of a finite number of WLD spaces is again WLD.*

(ii) *The c_0-sum, as well as the ℓ_p-sum for any $p \in (1,\infty)$, of an arbitrary family of WLD spaces is again WLD.*

(iii) *The ℓ_1-sum of a countable family of WLD spaces is WLD. The ℓ_1-sum of an arbitrary family of WLD spaces is 1-Plichko.*

Before proving the theorems we give the following lemma, which can be viewed as a Banach space analogue of Lemma 3.28.

LEMMA 4.34. *Let $(X_a : a \in \Lambda)$ be a family of Banach spaces, S_a be a Σ-subspace of X_a^* for every $a \in \Lambda$, $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p < \infty$, let X be the ℓ_p-sum of all X_a's, if $p = \infty$, let X be the c_0-sum of all X_a's. Then the dual X^* is canonically isometric to the ℓ_q-sum of all X_a^*'s, and the set*

$$S = \{(\xi_a)_{a\in\Lambda} \in X^* : (\forall a \in \Lambda)(\xi_a \in S_a) \ \&\ \{a \in \Lambda : \xi_a \neq 0\} \text{ is countable}\}$$

is a Σ-subspace of X^*.

Proof. Let $T_a : X_a^* \to \mathbb{R}^{\Gamma_a}$ be a linear mapping witnessing that S_a is a Σ-subspace of X_a^*. Put

$$T((\xi_b)_{b\in\Lambda})(a, \gamma) = T_a(\xi_a)(\gamma), \qquad a \in \Lambda,\ \gamma \in \Gamma_a.$$

Then this mapping witnesses that S defined above is a Σ-subspace of X^*. ∎

Proof. (of Theorem 4.31) (i) By N^* denote the dual norm on $\mathbb{R}^n$, i.e. $N^*(s_1, \dots, s_n) = \sup\{|s_1t_1 + \dots + s_nt_n| : N(t_1, \dots, t_n) \le 1\}$. Then it is easy to check that X^* is canonically isometric to $X_1^* \times \dots \times X_n^*$ with the norm $\|(\xi_1, \dots, \xi_n)\| = N^*(\|\xi_1\|, \dots, \|\xi_n\|)$. Let S_j be a 1-norming Σ-subspace of X_j^* for $j = 1, \dots, n$. It follows from Lemma 4.34 that $S = S_1 \times \dots \times S_n$ is a Σ-subspace of X^*. It remains to prove that it is 1-norming.

Let $x = (x_1, \dots, x_n) \in X$ and $\varepsilon > 0$. There are $s_1, \dots, s_n \in [0, \infty)$ such that $N^*(s_1, \dots, s_n) = 1$ and $s_1\|x_1\| + \dots + s_n\|x_n\| = N(\|x_1\|, \dots, \|x_n\|) = \|(x_1, \dots, x_n)\|$. For every $j = 1, \dots, n$ choose some $\xi_j \in S_j$, $\|\xi_j\| = s_j$ such that $\xi_j(x_j) > s_j\|x_j\| - \frac{\varepsilon}{n}$. Then $\xi = (\xi_1, \dots, \xi_n) \in S$, $\|\xi\| = 1$, and $\xi(x) > \|x\| - \varepsilon$. This completes the proof.

(ii) Suppose without loss of generality that X is not WLD. First assume that B_{X^*} is Valdivia. Choose $y_0 \in Y$ with $\|y_0\| = 1$ and put $L = \{g \in B_{Y^*} : g(y_0) = 1\}$. This is a convex weak* compact subset of the sphere S_{Y^*}.

Further, B_{X^*} is Valdivia and not Corson, and hence there is a non-Valdivia weak* compact $K \subset B_{X^*}$ which is convex, contains 0, and has a dense set of G_δ points. This can be shown by a minor modification in the proof of Theorem 3.16. Namely, either $g = 0$, or we can choose $f_1 = 0$, using the notation of Theorem 3.16. Hence we can arrange that $0 \in K$. The fact that K has a dense set of G_δ points follows from the fact that K is the weak* closed convex hull of a weak* compact scattered space (by [32, Lemma 4] the space K is a continuous image of a Radon-Nikodým compactum, and hence has a dense set of G_δ points).

Now put

$$B = \mathrm{conv}((B_{X^*} \times \{0\}) \cup (\{0\} \times B_{Y^*}) \cup (K \times L) \cup ((-K) \times (-L))),$$

all considered in the space $X^* \times Y^* = (X \times Y)^*$. then B is a convex symmetric weak* compact set, such that $\mathrm{conv}((B_{X^*} \times \{0\}) \cup (\{0\} \times B_{Y^*})) \subset B \subset B_{X^*} \times B_{Y^*}$, hence B is the dual unit ball of an equivalent norm $\|\cdot\|$ on $X \times Y$ such that $\max(\|x\|, \|y\|) \leq \|(x, y)\| \leq \|x\| + \|y\|$ for every $(x, y) \in X \times Y$. We claim that B is not a Valdivia compactum.

Let us first show that $K \times L = \{(f, g) \in B : (f, g)(0, y_0) = 1\}$. The inclusion "$\subset$" immediately follows from the choice of L and B. Let us prove the inclusion "$\supset$". Choose $(f, g) \in B$ such that $g(y_0) = (f, g)(0, y_0) = 1$. By the definition of B there are $s, t, u \geq 0$ with $s + t + u \leq 1$ and $a \in B_{X^*}$, $b \in B_{Y^*}$, $k_1, k_2 \in K$ and $l_1, l_2 \in L$ such that $(f, g) = s \cdot (a, 0) + t \cdot (0, b) + u \cdot (k_1, l_1) + (1 - s - t - u) \cdot (-k_2, -l_2)$, in particular

$$\begin{aligned} 1 = g(y_0) &= tb(y_0) + ul_1(y_0) + (1 - s - t - u)(-l_2)(y_0) \\ &= tb(y_0) + u - (1 - s - t - u) \leq t + u \leq s + t + u \leq 1, \end{aligned}$$

hence the equalities take place, therefore $s = 0$, $t + u = 1$ and $b(y_0) = 1$. It follows that $b \in L$ and $(f, g) = t(0, b) + (1-t)(k_1, l_1) = ((1-t)k_1, tb + (1-t)l_1) \in K \times L$, as K and L are convex and K contains 0.

Let us suppose that B is a Valdivia compactum. Then $K \times L$ is also Valdivia, due to Theorem 3.8. Now, as K has a dense set of G_δ points, K is Valdivia by Lemma 3.32. This is a contradiction.

Now suppose that B_{X^*} is not Valdivia. Then, by the assumptions, B_{Y^*} is Valdivia. If Y is not WLD, then we can do the same as above interchanging the roles of X and Y. If Y is WLD, then we can choose $\|(x, y)\| = \|x\| + \|y\|$. In this case the respective dual unit ball is weak* homeomorphic to $B_{X^*} \times B_{Y^*}$, and so it is not Valdivia by Theorem 3.3 and Lemma 3.32.

(iii) This follows from Lemma 4.34, as it can be easily seen that S is 1-norming whenever each S_a is 1-norming. ∎

Proof. (of Theorem 4.32) (a) This is trivial using Lemma 4.34.

(b) This was proved by A. Plichko and D. Yost in [58, Section 7]. Let us indicate the idea.

Let $(X, \|\cdot\|)$ be Plichko. Then there is an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ is 1-Plichko, and thus there is a family of projections $(P_\alpha : \omega \leq P_\alpha \leq \mathrm{dens}\, X)$ which form a PRI for $(X, |\cdot|)$ (Theorem 4.14). This family of projections, considered on the space $(X, \|\cdot\|)$, has all the properties of a PRI except for condition (ii) which is replaced by (ii') $\sup_{\omega\le\alpha\le\operatorname{dens} X}\|P_\alpha\|<\infty$.

Let us call such a family of projections *bounded projectional resolution*, and the constant $\sup_{\omega\le\alpha\le\operatorname{dens} X}\|P_\alpha\|<\infty$ be called the *projection constant* (cf. [58, Section 7]).

The space $\ell_1([0,\omega_1))$ with the standard norm is 1-Plichko, and so it is Plichko in every equivalent norm. By [58, Section 7, Theorem] there is, for each $n\in\mathbb{N}$, an equivalent norm $\|\cdot\|_n$ on $\ell_1([0,\omega_1))$ such that the projection constant of any bounded projectional resolution in $(\ell_1([0,\omega_1)),\|\cdot\|_n)$ is at least n.

Put $X_n=(\ell_1([0,\omega_1)),\|\cdot\|_n)$ and let X be either the c_0-sum or the ℓ_p-sum for some $p\in[1,\infty)$ of X_n's. Then X_n is Plichko for any $n\in\mathbb{N}$ but X is not Plichko. Although it is very natural, the proof is not completely trivial, it is given in [58] for the case of ℓ_1-sum, but the same argument works for all other cases. ∎

Proof. (of Theorem 4.33) All assertions easily follow from Lemma 4.34. ∎

Remark 4.35. It is claimed in [50] that 1-Plichko spaces are stable to finite products. However, it is not clear which norm on $X\times Y$ one takes for a product norm. The assertions (i) and (ii) of Theorem 4.31 clarify the situation. As for the assertion (ii), we do not know what happens if neither X nor Y has Valdivia dual unit ball (cf. Lemma 4.41 and Question 4.45).

4.5. SUBSPACES AND QUOTIENTS In this section we collect some results on stability and non-stability of Valdivia type classes of Banach spaces with respect to taking subspaces and quotients.

The situation concerning quotients is easy to describe.

THEOREM 4.36. (i) *If X is WLD and there is a bounded linear operator $T:X\to Y$ with TX dense in Y, then Y is WLD as well.*

(ii) *Every Banach space is isometric to a quotient of a 1-Plichko space.*

(iii) *If X is 1-Plichko (Plichko) and $Y\subset X$ a separable subspace, then X/Y is 1-Plichko (Plichko, respectively).*

Proof. (i) If TX is dense in Y, then $T^*:Y^*\to X^*$ is one-to-one. Obviously T^* is weak* to weak* continuous. Let $F:X^*\to\Sigma(\Gamma)$ witnesses that X is WLD. Then $F\circ T^*$ witnesses that Y is WLD.

(ii) It is well-known (cf. [24, p. 71]) that every Banach space is isometric to a quotient of $\ell_1(\Gamma)$ for a set Γ. And $\ell_1(\Gamma)$ is obviously 1-Plichko (for example by Theorem 4.31).

(iii) Suppose that X is 1-Plichko and that S is a 1-norming Σ-subspace of X^*. The dual $(X/Y)^*$ is canonically isometric to $Y^\perp$ and the latter is a weak* closed weak* G_δ subspace of X^*. In particular, $S \cap B_{X^*} \cap Y^\perp$ is weak* dense in $B_{X^*} \cap Y^\perp$ by Lemma 1.11. It follows from Lemma 4.18 that X/Y is 1-Plichko.

If X is Plichko, then there is an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ is 1-Plichko. By the previous paragraph the quotient $(X, |\cdot|)/Y$ is 1-Plichko. Finally, X/Y is isomorphic to that space and so it is Plichko. ∎

As for subspaces, there are more open question than results. We begin by the following easy example of [29].

EXAMPLE 4.37. The space $X = C[0, \omega_1]$ of continuous functions on the ordinal segment $[0, \omega_1]$ with the supremum norm is 1-Plichko, while the dual unit ball of its hyperplane $Y = \{f \in X : f(\omega_1) = 0\}$ is not a Valdivia compactum.

Proof. (Sketch) As $[0, \omega_1]$ is Valdivia (Example 1.10), the space $X = C[0, \omega_1]$ is 1-Plichko by Theorem 5.2 below. Suppose that B_{Y^*} is Valdivia. Let A be a dense Σ-subset. For every $\alpha < \omega_1$ the functional δ_α (the Dirac measure supported by α) is a weak* G_δ point of B_{Y^*}, hence $\delta_\alpha \in A$. Similarly, $\frac{1}{2}(\delta_n - \delta_{n+1})$ is a weak* G_δ point of B_{Y^*}, hence it belongs to A. It is easy to check that the set $\{\delta_\alpha : \alpha < \omega_1\} \cup \{0\} \cup \{\frac{1}{2}(\delta_n - \delta_{n+1}) : n \in \mathbb{N}\}$ is weak* homeomorphic to the interval $[0, \omega_1]$ with collated sequence from Example 1.10, so it is not Valdivia. But we have just proved that it is Valdivia, a contradiction. ∎

We continue by the following analogue of Theorem 3.22.

THEOREM 4.38. *Let X be a Banach space, Y a subspace of X, and S a 1-norming Σ-subspace of X^*. Then the following assertions are equivalent.*

1. *$\{s \restriction Y : s \in S\}$ is a Σ-subspace of Y^*.*
2. *Y is w_S-closed in X.*
3. *The mapping $Q : s \mapsto s \restriction Y$ is a quotient mapping of $(S \cap B_{X^*}, w^*)$ onto its image, and $(B_{Y^*}, w^*) = \beta(Q(S \cap B_{X^*}), w^*)$.*

Proof. The proof is completely analogous to that of Theorem 3.22, due to Lemma 4.18. ∎

As a corollary we obtain the following example.

EXAMPLE 4.39. (a) Every subspace of a WLD space is again WLD.
(b) Every hyperplane of $\ell_1(\Gamma)$ is 1-Plichko.

Proof. (a) This immediately follows from Theorem 4.38.
(b) It will be shown in Example 6.9 that for every $x \in \ell_1(\Gamma)^* = \ell_\infty(\Gamma)$ there is a 1-norming Σ-subspace of $\ell_\infty(\Gamma)$ containing x. Let Y be a hyperplane of $\ell_1(\Gamma)$. Then there is $x \in \ell_\infty(\Gamma)$ such that $Y = \{x\}_\perp$. Let S be a 1-norming Σ-subspace containing x. Then Y is w_S closed, and hence we can conclude by Theorem 4.38.
Let us remark that this assertion can be also proved in an elementary way, explicitly determining the 1-norming Σ-subspace of Y^*. ∎

Another positive result on subspaces is the following consequence of Theorem 4.36(iii).

THEOREM 4.40. *Let X be a Banach space and Y a subspace of X such that X/Y is separable.*

(i) *If Y is complemented in X, then Y is Plichko if and only if X is Plichko.*

(ii) *If Y is 1-complemented, then Y is 1-Plichko whenever X is 1-Plichko. The converse does not hold.*

Proof. The 'if' part of both cases follows from Theorem 4.36(iii).
The converse for Plichko case follows from Theorem 4.32. The non-validity of the converse for the 1-Plichko case follows from Theorem 4.31(ii). ∎

LEMMA 4.41. *Let X and Y be Banach spaces such that the dual ball B_{X^*} has at least one weak* G_δ point and that $X \oplus_1 Y$ is 1-Plichko. Then X is 1-Plichko and (B_{Y^*}, w^*) has a dense convex Σ-subset.*

Proof. Let S be a 1-norming Σ-subspace of $(X \oplus_1 Y)^*$. The dual unit ball of $X \oplus_1 Y$ is canonically weak* homeomorphic to $B_{X^*} \times B_{Y^*}$. Let f be a G_δ point of B_{X^*}. Then $(\{f\} \times B_{Y^*}) \cap S$ is dense in $\{f\} \times B_{Y^*}$, due to Lemma 1.11. Hence B_{Y^*} has a dense convex Σ-subset.

To show that X is 1-Plichko, it is enough to prove that X is w_S-closed in $X \oplus Y$ (by Theorem 4.38). Suppose that a net $(x_\alpha, 0)$ converges in w_S to a point (x, y). We will show that $y = 0$. By the above the set $A = \{g \in B_{Y^*} : (f, g) \in S\}$ is weak* dense in B_{Y^*}. For any $g \in A$ we have, by the definition of w_S, that

$$f(x_\alpha) = (f, g)(x_\alpha, 0) \xrightarrow{\alpha} (f, g)(x, y) = f(x) + g(y).$$

As the left-hand side does not depend on g, neither does the right-hand side. It follows that the mapping $g \mapsto g(y)$ is constant on A. Therefore, by density of A, it is constant also on B_{Y^*}, and thus $y = 0$, which completes the proof. ∎

As an immediate consequence of the previous lemma we get the following theorem.

THEOREM 4.42. *Let $(X_\alpha : \alpha \in \Lambda)$ be a family of Banach spaces such that $B_{X_\alpha^*}$ has a weak* G_δ point for each $\alpha \in \Lambda$. Then the ℓ_1 sum of all X_α's is 1-Plichko, if and only if each X_α is 1-Plichko.*

The just named results are, up to our knowledge, almost all what is known about stability or non-stability with respect to subspaces (except for Theorems 5.13 and 5.14 below). Hence there are many open questions in this area.

QUESTION 4.43. Is a Banach space X WLD provided one of the following conditions hold?

(i) The dual unit ball of every subspace of X is Valdivia.

(ii) Every subspace of X is 1-Plichko.

(iii) Every subspace of X admits a (weak) PRI.

A partial positive answer, within certain $C(K)$ spaces, is given in [34]. We reproduce it in Section 5.2. In particular, even the following question seems to be open.

QUESTION 4.44. Is every subspace of $\ell_1(\Gamma)$ 1-Plichko (or at least Plichko)?

The following questions are natural to ask, in view of Theorem 4.40 and Lemma 4.41.

QUESTION 4.45. (i) Is every subspace of a Plichko space again Plichko?

(ii) Is every 1-complemented subspace of a 1-Plichko subspace again 1-Plichko?

(iii) Let $X \oplus_1 Y$ be 1-Plichko. Are X and Y 1-Plichko as well?

Another natural question concerning quotients and subspaces is whether the classes of Banach spaces in question have so called three-space property. It is the following question: Let X be a Banach space with a subspace Y such that both Y and X/Y belong to some class of spaces. Does X necessarily belong to this class as well? There are some well-known examples of spaces witnessing that for our classes it is not the case, even in a very strong sense.

EXAMPLE 4.46. There are pairs of Banach spaces X, Y with Y being a subspace of X with the following properties.

(i) Y is isometric to c_0, X/Y is isometric to $\ell_2([0,\omega_1))$ and X is not Plichko.

(ii) Y is isometric to c_0, X/Y is isometric to $c_0([0,\omega_1))$ and X is not Plichko.

(iii) Y is isometric to $C[0,1]$, X/Y is isometric to $c_0(A)$ (where A is an arbitrary uncountable subset of $(0,1)$) and X is not isomorphic to any subspace of a Banach space with Valdivia dual unit ball.

Proof. The conditions from (i) and (ii) are satisfied by the well-known examples of Johnson and Lindenstrauss [26]. The fact that these spaces are not Plichko follows from [8, Claim on p. 139], where it is proved that these spaces have PRI with respect to no equivalent norm.

Let us prove the assertion (iii) . Let $A \subset (0,1)$ be an arbitrary uncountable set and K_A be the modification of the double arrow space described in [27] and X be the space $C(K_A)$. Then there is a subspace $Y \subset X$ with the required properties (cf. e.g. [17, Section 2.3] or [8, Section 5.6]). By Example 1.18 the space K_A is not Valdivia, in particular X is not WLD. It follows from [28, Proposition 7] that the space $P(K_A)$ of Radon probability measures on K_A is weak* first countable, and hence so is $P(K_A) \times P(K_A) \times [0,1]$. The dual unit ball $B_{C(K_A)^*}$ is a continuous image of the latter space and hence it is Fréchet-Urysohn (it is easy to check that Fréchet-Urysohn spaces are preserved by closed continuous maps). In particular, it contains no copy of $[0,\omega_1]$. Clearly the same is true for the dual unit ball with respect to any equivalent norm. By Theorem 3.27 we get that the dual unit ball is not a continuous image of a Valdivia compactum, for any equivalent norm on X. The result now follows immediately. ∎

However, similarly as WCG spaces (see [26] or [8, Proposition 4.10.f]), the class of WLD spaces satisfies a weak version of the three space property. The analogous statement for 1-Plichko spaces is not true.

THEOREM 4.47. (a) *Let X be a Banach space such that there is $Y \subset X$ which is WLD and such that X/Y is separable. Then X is WLD.*

(b) *There is a Banach space X which is not 1-Plichko such that there exists (even 1-complemented) 1-Plichko subspace $Y \subset X$ with X/Y separable.*

Proof. (a) As Y is WLD, there is a linear one-to-one weak*-continuous mapping $T_1 : Y^* \to \Sigma(\Gamma)$ for a set Γ. Since X/Y is separable, there is a linear one-to-one weak* continuous mapping $\tilde{T}_2 : Y^\perp = (X/Y)^* \to \mathbb{R}^{\mathbb{N}}$. By Hahn-Banach theorem this mapping can be linearly weak* continuously extended on a mapping $T_2 : X^* \to \mathbb{R}^{\mathbb{N}}$. Let us define the mapping $T : X \to \Sigma(\Gamma \cup \mathbb{N})$ by the formula:

$$T(f)(\gamma) = T_1(f \restriction Y)(\gamma), \qquad \gamma \in \Gamma$$
$$T(f)(n) = T_2(f)(n), \qquad n \in \mathbb{N}.$$

This is clearly a linear one-to-one weak* continuous mapping, and hence X is WLD.

(b) This follows from Theorem 4.31(ii). ∎

The following question seems to be open.

QUESTION 4.48. Let X be a Banach space such that there is a Plichko subspace $Y \subset X$ with X/Y separable. Is then X necessarily Plichko?

5. VALDIVIA TYPE $C(K)$ SPACES

In this chapter we will study Banach spaces of the form $C(K)$, where K is a compact space and Valdivia type properties of their duals. It turns out that these properties are related with Valdivia properties of K, but the exact relationship remains to be an open question. First we fix some notation.

If K is a compact space, we denote by $C(K)$ the Banach space of all continuous functions on K, endowed with the standard max-norm. The dual space $C(K)^*$ can be, due to Riesz theorem, identified with the space of finite signed Radon measures on K, the norm of a measure μ being its total variation.

A special role is sometimes played by the space of Radon probability measures on K, we will denote in $P(K)$ and always consider it with the weak* topology. We will often use also the following standard equality

$$P(K) = \{\mu \in C(K)^* : \|\mu\| \leq 1 \ \& \ \langle \mu, 1_K \rangle = 1\},$$

from which it follows that $P(K)$ is a convex weak* closed weak* G_δ subset of $B_{C(K)^*}$.

If μ is an element of $C(K)^*$, we denote by μ^+ the positive part of μ and by μ^- the negative part. If μ is a non-negative measure, we denote by $\operatorname{supp}\mu$ the support of the measure μ, i.e. the set of those points $x \in K$ such that each neighborhood of x has positive μ-measure. The support of a signed measure μ is the union of $\operatorname{supp}\mu^+$ and $\operatorname{supp}\mu^-$. It is well-known and easy to see that every signed Radon measure is supported by its support, i.e. every Borel set disjoint with the support has measure zero.

There is also a canonical embedding of K into $C(K)^*$. This embedding we will denote by δ, and it assigns to a point $x \in K$ the Dirac measure δ_x. It is well-known that this δ is a homeomorphic embedding.

5.1. Valdivia compacta and duality In this section we study relations between Valdivia type properties of a compactum K, the space of probability measures $P(K)$ and the dual unit ball $B_{C(K)^*}$. We start by the following key proposition. One of the ideas used in the proof goes back to [50, Corollary 5].

Proposition 5.1. *Let K be a compact space and A be a dense Σ-subset of K. Then the set*

$$S = \{\mu \in C(K)^* : \operatorname{supp}\mu \text{ is a separable subset of } A\} \tag{8}$$

is a 1-norming Σ-subspace of $C(K)^*$.

Proof. Let $h : K \to \mathbb{R}^\Gamma$ be a homeomorphic injection with $h(A) = h(K) \cap \Sigma(\Gamma)$. For $\gamma \in \Gamma$ let $f_\gamma = \pi_\gamma \circ h$, where π_γ denotes the projection of $\mathbb{R}^\Gamma$ onto the γ-th coordinate. It is clear that the family $(f_\gamma \mid \gamma \in \Gamma)$ separates the points of K and that

$$A = \{x \in K \mid \{\gamma \in \Gamma \mid f_\gamma(x) \neq 0\} \text{ is countable}\}.$$

Let $\tilde{\Gamma}$ be the set of all (possibly empty) finite sequences of elements of Γ. For $\tilde{\gamma} \in \tilde{\Gamma}$ let us define

$$g_{\tilde{\gamma}} = \begin{cases} 1_K & \text{if } \tilde{\gamma} = \emptyset, \\ f_{\gamma_1} \cdot \ldots \cdot f_{\gamma_n} & \text{if } \tilde{\gamma} = (\gamma_1, \ldots, \gamma_n). \end{cases}$$

It follows from Stone-Weierstrass theorem, that $\overline{\operatorname{span}\{g_{\tilde\gamma} \mid \tilde\gamma \in \tilde\Gamma\}} = C(K)$, hence the family $\left(g_{\tilde\gamma} \mid \tilde\gamma \in \tilde\Gamma\right)$ separates points of $C(K)^*$. Therefore, if we define the mapping $\tilde h : C(K)^* \to \mathbb{R}^{\tilde\Gamma}$ by the formula

$$\tilde h(\mu)(\tilde\gamma) = \langle \mu, g_{\tilde\gamma}\rangle,$$

it is a linear weak* continuous injection. Put

$$S = \tilde h^{-1}\left(\Sigma(\tilde\Gamma)\right).$$

This S is clearly a Σ-subspace of $C(K)^*$. Moreover, S contains the Dirac measure δ_x for every $x \in A$. Indeed, if $x \in A$ and $\tilde\gamma \in \tilde\Gamma$ with $g_{\tilde\gamma}(x) \neq 0$, then either $\tilde\gamma = \emptyset$ or $\tilde\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i(x) \neq 0$, $i = 1, \dots, n$. So clearly $\{\tilde\gamma \in \tilde\Gamma \mid g_{\tilde\gamma}(x) \neq 0\}$ is countable, and therefore $\delta_x \in S$. It follows that S is 1-norming.

It remains to prove the equality (8). At first let us prove the inclusion "$\supset$". Suppose that $\mu \in C(K)^*$ is such that $\operatorname{supp}\mu$ is a separable subset of A. Hence $\operatorname{supp}\mu$ is a separable Corson compactum, and therefore metrizable. Further, we have that $\operatorname{supp}\mu = \operatorname{supp}\mu^+ \cup \operatorname{supp}\mu^-$, thus both $\operatorname{supp}\mu^+$ and $\operatorname{supp}\mu^-$ are separable. If we knew that μ^+ and μ^- belong to S, then $\mu = \mu^+ - \mu^-$ would belong to S as well. It follows that we can suppose without loss of generality that μ is non-negative. Moreover, as $0 \in S$, we can suppose that $\mu \in P(K)$.

Put $F = \operatorname{supp}\mu$. Then it is well-known and easy to see that $P(F)$ is a topological subspace of $P(K)$. By the above we have $\delta_x \in S$ for every $x \in F$, and hence also measures supported by a finite subset of F belong to S. It is a standard fact that such measures are weak* dense in $P(F)$, and as F is metrizable, $P(F)$ is also metrizable, and hence μ is the limit of a sequence of elements of $S \cap P(K)$. It follows that $\mu \in S$ by Lemma 1.6.

It remains to prove the inclusion "$\subset$". Put $S' = \operatorname{span}\{\delta_x : x \in A\}$. It is clear that $S' \subset S$ and that S' is 1-norming. It follows that $S' \cap B_{C(K)^*}$ is weak* dense in $B_{C(K)^*}$. In particular, every $\mu \in S \cap B_{C(K)^*}$ belongs to the weak* closure of $S' \cap B_{C(K)^*}$. Hence, by Lemma 1.6, there is a sequence $\mu_n \in S' \cap B_{C(K)^*}$ weak* converging to μ. Let C be the set of all $x \in K$ such that $\mu_n(\{x\}) \neq 0$ for some n. It is clear that C is a countable subset of A and that μ is supported by $\bar C$. As $\bar C \subset A$ (Lemma 1.6), $\bar C$ is a separable Corson compactum, and hence metrizable. It follows that $\operatorname{supp}\mu$ is separable, too, as a subset of $\bar C$. This completes the proof. ∎

Now we are going to formulate duality results on Valdivia type properties.

THEOREM 5.2. *Let K be a compact space. Consider the following conditions*

1. *K is a Valdivia compactum.*
2. *$C(K)$ is a 1-Plichko space.*
3. *$\left(B_{C(K)^*}, w^*\right)$ has a dense convex Σ-subset.*
4. *$P(K)$ has a dense convex Σ-subset.*
5. *$\left(B_{C(K)^*}, w^*\right)$ is a Valdivia compactum.*
6. *$P(K)$ is a Valdivia compactum.*

Then the following implications hold true.

$$1 \Rightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Rightarrow 5 \Rightarrow 6$$

THEOREM 5.3. *Let K be a compact space with a dense set of G_δ points. Then all conditions 1–6 of Theorem 5.2 are equivalent.*

THEOREM 5.4. *Let K be a compact space. Then the following assertions are equivalent.*

(a) *K is a Corson compact space such that the support of every Radon probability on K is separable (i.e., K has so called property (M)).*

(b) *$C(K)$ is WLD.*

(c) *$P(K)$ is a Corson compactum.*

Now we proceed to proofs of the just stated theorems.

Proof. (of Theorem 5.2) The implication $1 \Rightarrow 2$ follows from Proposition 5.1.

$2 \Rightarrow 3$ This is trivial, as the dual unit ball of a 1-Plichko space has even a convex symmetric dense Σ-subset (cf. Theorem 4.15).

$3 \Rightarrow 4$ This follows from Lemma 1.11, as $P(K)$ is a weak* closed weak* G_δ convex subset of $B_{C(K)^*}$.

$4 \Rightarrow 2$ Let A be a dense convex Σ-subset of $P(K)$. Then $(C(P(K)), \tau_A)$ is primarily Lindelöf by Theorem 2.5. Let us consider the injection $T : C(K) \to C(P(K))$ defined by the formula $T(f)(\mu) = \langle \mu, f \rangle$, $\mu \in P(K)$, $f \in C(K)$. We claim that $F \in C(P(K))$ belongs to $T(C(K))$ if and only if F is affine.

The 'only if' part is obvious, so let us prove the 'if' part. Let $F \in C(P(K))$ be affine. Put $f = (F \restriction \delta(K)) \circ \delta$. Then $f \in C(K)$. And it easily follows from the facts that F is continuous and affine that $F = T(f)$.

Further let us notice that $T(C(K))$ is τ_A-closed in $C(P(K))$. Indeed, let $F_\nu \xrightarrow{\tau_A} F$, where $F_\nu \in T(C(K))$ and $F \in C(P(K))$. Each F_ν is affine and A is convex, so $F \restriction A$ is affine. As A is dense and F continuous, we get that F is affine, hence $F \in T(C(K))$.

So $T(C(K))$ is τ_A-primarily Lindelöf, and clearly $(C(K), w_A)$ is homeomorphic to $(T(C(K)), \tau_A)$, therefore $(C(K), w_A)$ is primarily Lindelöf. Put $\tilde{A} = \operatorname{conv}(A \cup (-A))$. Then clearly $w_A = w_{\tilde{A}}$, so $(C(K), w_{\tilde{A}})$ is primarily Lindelöf. In view of Theorem 2.7 it is enough to show that $\tilde{A}$ is weak* countably compact and weak* dense in $B_{C(K)^*}$.

By Lemma 3.28 we have that $A \times A \times [0,1]$ is a dense Σ-subset of $P(K) \times P(K) \times [0,1]$, and hence it is countably compact by Lemma 1.6. Consider the mapping

$$\psi : P(K) \times P(K) \times [0,1] \to B_{C(K)^*}$$

defined by the formula

$$\psi(\mu, \nu, t) = t\mu - (1-t)\nu.$$

Then ψ is continuous and onto, and $\tilde{A} = \psi(A \times A \times [0,1])$, so clearly $\tilde{A}$ is weak* countably compact and weak* dense. This completes the proof.

The implication $3 \Rightarrow 5$ is trivial.

$5 \Rightarrow 6$ This follows from Theorem 3.8. ∎

To prove Theorem 5.3 we need the following lemma. In fact, we need only a special case, but the present formulation will be of use later.

LEMMA 5.5. *Let K be a compact Hausdorff space, $x_1, \ldots, x_n$ be G_δ points of K and $t_1, \ldots, t_n \geq 0$ with $t_1 + \cdots + t_n = 1$. Then $t_1\delta_{x_1} + \cdots + t_n\delta_{x_n}$ is a G_δ point of $P(K)$.*

Proof. At first let us show that the set of probability measures on K supported by the set $F = \{x_1, \ldots, x_n\}$ is G_δ. As F is clearly G_δ, it is enough to show the following assertion.

(9) $\quad F \subset K$ is closed and $G_\delta \Rightarrow \{\mu \in P(K) \mid \mu(F) = 1\}$ is G_δ in $P(K)$.

Let $f : K \to [0,1]$ be continuous with $f^{-1}(1) = F$. Then it is clear that $\mu(F) = 1$ if and only if $\langle \mu, f \rangle = 1$, which proves (9).

Finally observe, that $\{\mu \in P(K) \mid \mu(F) = 1\}$ is homeomorphic to $P(F)$, which is metrizable whenever F is finite. So every point of $P(F)$ is G_δ in $P(F)$, and the assertion of the lemma follows. ∎

Proof. (of Theorem 5.3) It is enough to prove the implication 6 $\Rightarrow$ 1. Suppose that $P(K)$ is a Valdivia compactum and let A be a dense Σ-subset of $P(K)$. If k is a G_δ point of K, then δ_k is, due to Lemma 5.5 a G_δ point of $P(K)$, and hence $\delta_k \in A$ by Lemma 1.11. So $\delta(K) \cap A$ is dense in $\delta(K)$, and hence K is a Valdivia compactum. ∎

It can be easily checked that the exactly same argument as in the previous paragraph gives the following lemma, which will be useful in the next section.

LEMMA 5.6. *Let K be a compact space such that $P(K)$ is Valdivia. Let M denote the set of all G_δ points of K. Then $\bar{M}$ is Valdivia as well.*

Proof. (of Theorem 5.4) (a)$\Rightarrow$(b) If K is Corson, then K is a (dense) Σ-subset of itself, and hence we can use Proposition 5.1. By the assumptions on K it follows that the Σ-subspace S defined in Proposition 5.1 is the whole space $C(K)^*$, so $C(K)$ is WLD.

(b)$\Rightarrow$(a) If $C(K)$ is WLD, then K is Corson (as a subset of the Corson compactum $\left(B_{C(K)^*}, w^*\right)$, and hence K is a (dense) Σ-subset of itself. Let S be the 1-norming Σ-subspace defined in Proposition 5.1. As $C(K)$ is WLD, the whole space $C(K)^*$ is a Σ-subspace of itself, and it follows easily from Lemma 1.7 that it is the unique 1-norming Σ-subspace. Hence $S = C(K)^*$, so K has the property (M).

(b)$\Rightarrow$(c) This follows immediately from the fact that $P(K)$ is a closed subset of the Corson compactum $\left(B_{C(K)^*}, w^*\right)$.

(c)$\Rightarrow$(b) Suppose that $P(K)$ is a Corson compactum. Then $P(K) \times P(K) \times [0,1]$ is Corson as well (it follows e.g. from Lemma 3.28). Moreover, the ball $\left(B_{C(K)^*}, w^*\right)$ is a continuous image of the space $P(K) \times P(K) \times [0,1]$ (by the mapping $(\mu, \nu, t) \mapsto t\mu - (1-t)\nu$), hence it is Corson, too (as Corson compacta are stable to continuous images, cf. e.g. Theorem 3.22). Finally, $C(K)$ is WLD by Theorem 4.17. ∎

It is proved in [5, Theorem 3.12] that, under continuum hypothesis there is a Corson compact space K without property (M). Hence the respective dual unit ball of $C(K)$ is not Corson. The following example strengthens a bit this result.

Example 5.7. Under continuum hypothesis there is a Corson compact space K such that neither $P(K)$ nor $(B_{C(K)^*}, w^*)$ is a super-Valdivia compactum.

Proof. As said above, under continuum hypothesis there is a Corson compactum K such that $(B_{C(K)^*}, w^*)$ is not Corson. We will prove that it is not even super-Valdivia. Suppose on the contrary that $(B_{C(K)^*}, w^*)$ is a super-Valdivia compactum. Then $P(K)$ is super-Valdivia as well, due to Theorem 3.8. Further, K has a dense set of G_δ points by Theorem 3.3. It is a standard fact that finitely supported probability measures are dense in $P(K)$, and therefore it follows from Lemma 5.5 that $P(K)$ has a dense set of G_δ points, too. By Corollary 1.12 we get that $P(K)$ has at most one dense Σ-subset. So it is Corson, and thus $(B_{C(K)^*}, w^*)$ is also Corson by Theorem 5.4. This is a contradiction. ∎

The previous example answers negatively the natural question whether $(B_{C(K)^*}, w^*)$ is super-Valdivia whenever K has this property. However, under Martin's axiom and negation of the continuum hypothesis every Corson compactum has the property (M) (see e.g. [5, Remark 3.2(3)] or [9, p.205]). Hence it is natural to ask the following question.

Question 5.8. Suppose Martin's axiom and negation of the continuum hypothesis hold. Is then $(B_{C(K)^*}, w^*)$ super-Valdivia whenever K is super-Valdivia?

In fact, we do not know the answer to the following natural question.

Question 5.9. Is $(B_{C(K)^*}, w^*)$ super-Valdivia if $K = [0,1]^\Gamma$ or $K = \{0,1\}^\Gamma$?

Another questions concern the remaining implications of Theorem 5.2.

Question 5.10. Are all conditions of Theorem 5.2 equivalent? In particular,

(i) is K Valdivia whenever $C(K)$ is 1-Plichko?

(ii) is $C(K)$ 1-Plichko whenever $(B_{C(K)^*}, w^*)$ is Valdivia?

(iii) is $(B_{C(K)^*}, w^*)$ Valdivia whenever $P(K)$ is Valdivia?

5.2. Non-stability to subspaces In this section we sketch a partial answer to Question 4.43 given in [34]. Let us start with the definition of a class of compact spaces.

Definition 5.11. Let K be a compact space. We say that K belongs to the *class* $\mathcal{G}\Omega$ if, for every nonempty open set $U \subset K$ either U contains a G_δ point or the one-point compactification of U contains a copy of the ordinal segment $[0, \omega_1]$.

This definition is natural in view of the below theorems. The following examples show that it is a wide class but does not contain all compact spaces.

Example 5.12.

(a) Every compact space with a dense set of G_δ points belongs to $\mathcal{G}\Omega$.

(b) Every continuous image of a Valdivia compact space belongs to $\mathcal{G}\Omega$.

(c) The space $\beta\mathbb{N} \setminus \mathbb{N}$, the remainder of $\mathbb{N}$ in its Čech-Stone compactification, does not belong to $\mathcal{G}\Omega$.

Proof. The assertion (a) is trivial, the assertion (b) easily follows from Theorem 3.3 and Theorem 3.27. To prove (c) it suffices to observe that $\beta\mathbb{N}\setminus\mathbb{N}$ contains no one-to-one convergent sequence [16, Theorem 3.5.4] and has no isolated points. Therefore it has no G_δ points and contains no copy of $[0, \omega_1]$. ∎

Now we are ready to formulate the theorems, which are main results of [34].

Theorem 5.13. *Let K be a compact space from the class $\mathcal{G}\Omega$. Then the following assertions are equivalent.*

1. *K is Corson.*
2. *For any L, continuous image of K, the space $C(L)$ is 1-Plichko.*
3. *For any L, continuous image of K, the dual unit ball $(B_{C(L)^*}, w^*)$ is Valdivia.*
4. *For any L, continuous image of K, the compact space $P(L)$ is Valdivia.*

In particular, the assumptions of this theorem are satisfied if K is a continuous image of a Valdivia compactum.

THEOREM 5.14. *Let K be a compact space from the class $\mathcal{G}\Omega$. Then the following assertions are equivalent.*

1. *K is a Corson compactum with the property (M).*
2. *Any subspace of $C(K)$ is 1-Plichko.*
3. *For any subspace $Y \subset C(K)$ the dual unit ball (B_{Y^*}, w^*) is Valdivia.*

In particular, the assumptions of this theorem are satisfied if K is a continuous image of a Valdivia compactum.

Now we are going to sketch the proofs by giving the main ideas.

Proof. (Sketch of the proof of Theorem 5.13) The implication $1 \Rightarrow 2$ follows from the fact that Corson compact spaces are stable to continuous images ([3, Section IV.3], or Theorem 3.22), together with Theorem 5.2. The implication $2 \Rightarrow 3$ is trivial and $3 \Rightarrow 4$ follows from Theorem 5.2.

It remains to prove $4 \Rightarrow 1$. Let K be a non-Corson compactum from the class $\mathcal{G}\Omega$. Let M be the set of all G_δ points of K. If $\bar{M}$ is not Valdivia, then $P(K)$ is not Valdivia by Lemma 5.6. If $\bar{M}$ is Valdivia but not Corson, then it has exactly one dense Σ-subset A (by Corollary 1.12). Choose $a \in A$ non-isolated and $b \in \bar{M} \setminus A$. Let L be the quotient space made from K by identifying the points a and b and Q be the quotient mapping. By Proposition 3.19 the space $Q(\bar{M})$ is not Valdivia. Further, it is easy to check that $Q(\bar{M})$ is the closure of the set of all G_δ points of L, hence $P(L)$ is not Valdivia by Lemma 5.6.

If $\bar{M}$ is Corson, then $K \setminus \bar{M}$ is an uncountable open set without isolated points. Hence there are four nonempty open sets $U_1, \dots, U_4 \subset K \setminus \bar{M}$ with pairwise disjoint closures. It easily follows from the definition of the class $\mathcal{G}\Omega$ that a suitable continuous image K' of K contains four pairwise disjoint nowhere dense copies of $[0, \omega_1]$. Now clearly, a further continuous image K'' contains two pairwise disjoint nowhere dense copies of the collated double interval ω_1 from Example 1.10. Let us collate these two sets in the way described in Proposition 3.20 and call the resulting quotient space by L. Using the idea of Proposition 3.20 we get that there is $N \subset L$, a nowhere dense copy of the collated double interval ω_1, and two disjoint open set $U, V \subset L$ with $\bar{U} \cap \bar{V} = N$. Consider the space $P(N)$ canonically embedded in $P(L)$. It is clear that $P(N) = P(\bar{U}) \cap P(\bar{V})$. For any open $W \subset K$ we have the following series of inclusions.

$$\begin{aligned} P(\bar{W}) = \overline{P(W)} &= \overline{\bigcap_{n\in\mathbb{N}} \{\mu \in P(L) \mid \mu(W) > 1 - \frac{1}{n}\}} \\ &\subset \bigcap_{n\in\mathbb{N}} \overline{\{\mu \in P(L) \mid \mu(W) > 1 - \frac{1}{n}\}} \\ &\subset \bigcap_{n\in\mathbb{N}} \{\mu \in P(L) \mid \mu(\bar{W}) \geq 1 - \frac{1}{n}\} = P(\bar{W}) \end{aligned}$$

Hence $P(\bar{W})$ is of the form $\bigcap_{n\in\mathbb{N}} \overline{G_n}$ with G_n open. Now, if $P(L)$ is Valdivia, then $P(N)$ is Valdivia as well, due to Lemma 1.15. Further, N has a dense set of G_δ points, hence N is Valdivia by Theorem 5.3. But this contradicts Example 1.10. This completes the argument. ∎

Proof. (Sketch of the proof of Theorem 5.14) 1 ⇒ 2 If K is a Corson compactum with property (M), then $C(K)$ is WLD, and so every subspace is WLD as well (Example 4.39).

The implication 2 ⇒ 3 is trivial.

3 ⇒ 1 If K is not Corson, we can use Theorem 5.13. So suppose that K is a Corson compactum without property (M). Let A be the set of all probability measures on K with separable support. By Proposition 5.1 it is a dense Σ-subset of $P(K)$, and by Lemma 5.5 and Corollary 1.12 it is unique dense Σ-subset. There is some $\mu \in P(K) \setminus A$. As countably supported measures belong to A and A is convex, we can without loss of generality assume that μ is continuous (i.e. all singletons have zero μ-measure). Choose $k \in \operatorname{supp} \mu$ arbitrary and put

$$Y = \{f \in C(K) : f(k) = \langle \mu, f \rangle\}.$$

We claim that B_{Y^*} is not a Valdivia compactum. Let i denote the inclusion of Y into $C(K)$ and i^* be the adjoint mapping. Then it is easy to check that $i^*(P(K)) = \{\xi \in B_{Y^*} : \langle \xi, 1 \rangle = 1\}$, so it is a weak* closed weak* G_δ subset of B_{Y^*}. If B_{Y^*} is Valdivia, then $i^*(P(K))$ is Valdivia as well, due to Theorem 3.8. Let B be a dense Σ-subset of $i^*(P(K))$. Then $(i^*)^{-1}(B) \cap P(K)$ is a Σ-subset of $P(K)$ by Lemma 5.15 below. Moreover, it can be proved that for any open set $U \subset P(K)$ the image $i^*(U)$ has nonempty interior in $i^*(P(K))$ (a technical proof is given in [34, Lemma 7]), in particular the inverse image of any dense set is again dense. Therefore $(i^*)^{-1}(B) \cap P(K)$ is dense in $P(K)$, and thus it is equal to A. However, $\delta_k \in A$ and $\mu \notin A$, while $i^*(\delta_k) = i^*(\mu)$, a contradiction. ∎

The following lemma was used in the above proof. In fact, we used it only for a hyperplane, but we formulate it in a more general setting. It can be viewed as a Banach space counterpart of Lemma 3.18.

LEMMA 5.15. *Let X be a Banach space and Y its complemented subspace such that the quotient X/Y is separable. Let i denote the canonical injection of Y into X and i^* its adjoint mapping. If K is a weak* compact set in X^* and A is a Σ-subset of $i^*(K)$, then $(i^*)^{-1}(A) \cap K$ is a Σ-subset of K.*

Proof. By our assumptions there is a closed separable subspace $Z \subset X$ such that $X = Y \oplus Z$. Denote by j the canonical injection of Z into X and by j^* the adjoint mapping.

Let $K \subset X^*$ be weak* compact and $A \subset i^*(K)$ a Σ-subset. Then there is a homeomorphic injection $h_0 : i^*(K) \to \mathbb{R}^\Gamma$ with $h_0(A) = h_0(i^*(K)) \cap \Sigma(\Gamma)$.

Further, $j^*(K)$ is a metrizable compact (as Z is separable), therefore there is a homeomorphic injection $h_1 : j^*(K) \to \mathbb{R}^{\mathbb{N}}$.

We are ready to define $h : K \to \mathbb{R}^{\Gamma \cup \mathbb{N}}$ by the formula

$$\begin{aligned} h(x)(\gamma) &= h_0(i^*(x))(\gamma), \qquad &\gamma \in \Gamma; \\ h(x)(n) &= h_1(j^*(x))(n), \qquad &n \in \mathbb{N}. \end{aligned}$$

It is clear that h is continuous. Further, h is one-to-one. Indeed, if $h(x) = h(y)$, then $i^*(x) = i^*(y)$ and $j^*(x) = j^*(y)$ (as h_0 and h_1 are one-to-one), which means $x \restriction Y = y \restriction Y$ and $x \restriction Z = y \restriction Z$, so $x = y$. Finally, it is obvious that $h(x) \in \Sigma(\Gamma \cup \mathbb{N})$ if and only if $i^*(x) \in A$. ∎

The above theorems partially answer Question 4.43. In proving them we strongly used the special structure of $C(K)$ spaces. Hence the general question remains open. Moreover, this question is not completely answered even within $C(K)$ spaces. We could drop the assumption that K belongs to $\mathcal{G}\Omega$ if the answer to Question 5.10(i) ,(ii) was positive. It seems natural to ask the following question, positive answer to which also yields the positive answer to Question 4.43 within $C(K)$ spaces.

QUESTION 5.16. (i) Let K be a compact space such that $(B_{C(K)^*}, w^*)$ is Valdivia. Does K belong to the class $\mathcal{G}\Omega$?

(ii) Let K be a compact space with at most one G_δ point, such that $C(K)$ has an equivalent locally uniformly rotund norm. Does K contain a copy of $[0, \omega_1]$?

Notice, that if $K = \beta\mathbb{N} \setminus \mathbb{N}$ (which does not belong to $\mathcal{G}\Omega$ by Example 5.12), then $B_{C(K)^*}$ is not Valdivia. (Otherwise $C(K)$ would have an equivalent locally uniformly rotund norm by [64, Corollary], which would contradict [7].)

5.3. Isomorphic $C(K)$ spaces While the questions on isomorphic stability of Banach spaces with Valdivia type duals are settled in the previous chapter, within $C(K)$ spaces there are some specific natural questions. These are questions of the following form. Let K and L be compact spaces such that $C(K)$ and $C(L)$ are isomorphic. Suppose that K has some property (P). Does L have (P), too? We will discuss these questions for Valdivia type properties of compact spaces.

We begin by the following well-known result on Corson compact spaces with property (M). The proof follows easily from Theorem 5.4 together with the obvious observation that WLD spaces are stable to isomorphisms.

Theorem 5.17. *Let K and L be compact spaces such that $C(K)$ is isomorphic to $C(L)$. If K is a Corson compact with property (M), then so is L.*

We continue by the following non-stability result.

Theorem 5.18. *Let K be a compact space which is not super-Valdivia. Then there is a non-Valdivia compactum L such that $C(L)$ is isomorphic to $C(K)$.*

To prove this theorem we need the following lemma.

Lemma 5.19. *If K is an infinite continuous image of a Valdivia compactum, then $C(K)$ is isomorphic to its hyperplane.*

Proof. By Theorem 3.27 the space K contains a one-to-one convergent sequence. Then it can be easily seen that $C(K)$ contains a complemented isometric copy of c_0. Therefore $C(K)$ is clearly isomorphic to its hyperplane. ∎

Proof. (of Theorem 5.18) If K is not Valdivia, put $L = K$. Otherwise by Proposition 3.19 there are two points $a, b \in K$ such that the quotient space L, made from K by identifying the points a and b, is not Valdivia. Clearly $C(L)$ is isometric to a hyperplane of $C(K)$, and so $C(L)$ and $C(K)$ are isomorphic by Lemma 5.19. ∎

The two above theorems are, up to our knowledge, everything which is known. Therefore there are several natural questions. The first one was asked already in [5, Problem on p. 218].

QUESTION 5.20. Let K and L be compact spaces such that $C(K)$ is isomorphic to $C(L)$. Suppose that K is Corson. Is then L Corson, too?

The following question concerns a generalization of Theorem 5.18.

QUESTION 5.21. Let K be a non-Corson compactum. Is there a non-Valdivia compactum L with $C(L)$ isomorphic to $C(K)$?

We finish by two questions on the class of continuous images of Valdivia compacta.

QUESTION 5.22. (a) Let K and L be compact spaces such that $C(K)$ is isomorphic to $C(L)$. Suppose that K is a continuous image of a Valdivia compact space. Does then L have the same property?

(b) Let K be a continuous image of a Valdivia compact space. Is there a Valdivia compactum L with $C(L)$ isomorphic to $C(K)$?

6. VARIOUS EXAMPLES OF VALDIVIA COMPACT SPACES

In this chapter we collect some illustrative examples of Valdivia compact spaces with various additional properties. The first section is devoted to topological examples, the second one to Banach spaces.

6.1. TOPOLOGICAL EXAMPLES

The first example is easy and follows from Example 1.10.

EXAMPLE 6.1. The Valdivia compact space $K = [0, \omega_1]$ has only one dense Σ-subset. It is the set $[0, \omega_1)$ which is open dense in K. The space K is not Fréchet-Urysohn with respect to open sets.

EXAMPLE 6.2. The compact space $K = [0, \omega_1]^{\mathbb{N}}$ is Valdivia. Its only dense Σ-subset is the set $[0, \omega_1)^{\mathbb{N}}$. This is a dense G_δ subset of K with empty interior. The space K is not Fréchet-Urysohn with respect to open sets.

Proof. The fact that $[0, \omega_1)^{\mathbb{N}}$ is a dense Σ-subset of K follows from Lemma 3.28. The uniqueness follows from Corollary 1.12 as K has clearly a dense set of G_δ points. Also it is obvious that the complement of $[0,\omega_1)^{\mathbb{N}}$ is dense in K. To show the last assertion it is enough to observe that if $x(1)=\omega_1$, then $x\in\bar U$ where $U=\{y:y(1)<\omega_1\}$ without being the limit of a sequence from U. ∎

EXAMPLE 6.3. Let Γ be an uncountable set. The compactum $K=[0,\omega_1]^\Gamma$ is Valdivia. A point $x\in K$ is contained in a dense Σ-subset of K if and only if $x\in[0,\omega_1)^\Gamma$. There are uncountably many pairwise disjoint dense Σ-subsets of K. The space K is not Fréchet-Urysohn with respect to open sets.

Proof. Let $x\in[0,\omega_1)^\Gamma$. Then the set

$$A_x=\{z\in[0,\omega_1)^\Gamma:\{\gamma\in\Gamma:z(\gamma)\neq x(\gamma)\}\text{ is countable}\}$$

is a dense Σ-subset of K, due to Lemma 3.28. Moreover, if $x,y\in[0,\omega_1)^\Gamma$, then either $A_x=A_y$, or A_x and A_y are disjoint. It is easy to observe that there are $\operatorname{card}\Gamma$ many pairwise disjoint A_x's.

Suppose that $x\in K$ is such that $x(\gamma_0)=\omega_1$ for some γ_0. Put $U=\{z\in K:z(\gamma_0)<\omega_1\}$. Then U is open in K, $x\in\bar U$ and x is not the limit of any sequence from U. It follows that K is not Fréchet-Urysohn with respect to open sets, and, moreover, that x is contained in no dense Σ-subset. (Let A be a dense Σ-subset of K containing x. Then $x\in\overline{U\cap A}$, and hence there are $x_n\in U\cap A$ such that $x_n\to x$ (Lemma 1.6). This is a contradiction.) ∎

EXAMPLE 6.4. The compact space $P=P[0,\omega_1]$ of all Radon probabilities on $[0,\omega_1]$ is Valdivia. The only dense Σ-subset is the set $A=\{\mu\in P:\mu(\{\omega_1\})=0\}$. The set A is dense G_δ and has empty interior. The space P is not Fréchet-Urysohn with respect to open sets.

Proof. The set A is a dense Σ-subset for example by the following Example 6.5 below and Lemma 1.11. The uniqueness follows from Lemma 1.12, as P has a dense set of G_δ-points (by Lemma 5.5). The set A is G_δ by the following formula.

$$A=\bigcap_{n\in\mathbb{N}}\bigcup_{\alpha<\omega_1}\{\mu\in P:\langle\mu,\chi_{[0,\alpha]}\rangle>1-\frac{1}{n}\}$$

The density of $P\setminus A$ easily follows from the definition of the weak* topology.

Finally, it is easy to observe that the set $U=\{\mu\in P:\mu([0,\omega_1))>\frac{1}{2}\}$ is an open dense subset of P. In particular, δ_{ω_1} belongs to $\bar U$. However, it is the limit of no sequence from U, as the mapping $\mu\mapsto\mu(\{\omega_1\})$ is easily seen to be weak* sequentially continuous. ∎

EXAMPLE 6.5. The compact space $B = (B_{C[0,\omega_1]^*}, w^*)$ is a Valdivia compactum. The only dense Σ-subset is $A = \{\mu \in B : \mu(\{\omega_1\}) = 0\}$. The set A is hereditarily almost Čech complete (i.e., every closed subset has a dense Čech complete subset) but not G_δ.

Proof. The set A is a dense Σ-subset of B by Proposition 5.1. Next we show that A is hereditarily almost Čech complete. Notice that B is a Radon-Nikodým compactum (it follows from the definition [17, Section 1.5] using [17, Theorem 1.1.13]), and hence each closed subset of B contains a dense G_δ completely metrizable subset [17, Theorem 5.1.12]. Let F be a relatively closed subset of A. Then $\overline{F}^B$ contains a dense G_δ completely metrizable subset, say M. As F is a dense Σ-subset of $\bar{F}$, we obtain by Lemma 1.11 that $F \cap M$ is dense in $\bar{F}$. As M is metrizable, and F countably closed in $\bar{F}$ (Lemma 1.6), it follows that $M \subset F$. Hence F contains a dense Čech complete subset M.

In particular A is residual in B. Hence A is the unique dense Σ-subset by Theorem 3.3 and Corollary 1.12.

To show that A is not G_δ, we prove the following claim.

Claim. If G is a G_δ subset of B containing some μ with $\|\mu\| < 1$, then $G \setminus A \neq \emptyset$

Let G be a G_δ subset of B and $\mu \in G$ with $\|\mu\| < 1$. If $\mu(\{\omega_1\}) \neq 0$, it is nothing to prove, so suppose $\mu(\{\omega_1\}) = 0$. It easily follows from the definition of the weak* topology that there is a sequence of functions $f_n \in C[0,\omega_1]$ such that

$$V = \{\nu \in B : (\forall n \in \mathbb{N})(\langle \nu, f_n \rangle = \langle \mu, f_n \rangle)\} \subset G.$$

There is some $\alpha < \omega_1$ such that all f_n's are constant on $[\alpha, \omega_1]$. Put

$$\nu = \mu - \frac{1 - \|\mu\|}{2}\delta_\alpha + \frac{1 - \|\mu\|}{2}\delta_{\omega_1}.$$

It is clear that $\nu \in V \setminus A \subset G \setminus A$. ∎

Remark 6.6. Let us remark that the previous example, namely the fact that A is not Čech-complete, answers a question posed in [25]. Using the notation of [25], the space $[0, \omega_1)$ is a Čech-complete space such that $\mathcal{M}_t^1([0, \omega_1))$ is not Čech-complete.

EXAMPLE 6.7. The compact spaces $\{0,1\}^\Gamma$ and $[0,1]^\Gamma$ are super-Valdivia. Moreover, each finite subset of these spaces is contained in a dense Σ-subset.

Proof. Let $x_1, \dots, x_n \in [0,1]^\Gamma$. Choose $\gamma \in \Gamma$ arbitrary. If none of the numbers $x_i(\gamma)$, $i = 1, \dots, n$ belongs to the open interval $(0,1)$, let h_γ be the identity mapping on $[0,1]$. Otherwise let $t_1 < \dots < t_k$ be all $x_i(\gamma)$'s which belong to $(0,1)$. Let h_γ be the piecewise linear homeomorphic mapping of $[0,1]$ onto itself such that $h_\gamma(t_j) = \frac{j}{k+1}$.

Further, let us define an equivalence relation $\sim$ on $[0,1]^n$, such that $(s_1, \dots, s_n) \sim (t_1, \dots, t_n)$ if the following conditions are fulfilled.

$$(\forall i, j \le n)((t_i < t_j \Leftrightarrow s_i < s_j) \ \& \ (t_i = t_j \Leftrightarrow s_i = s_j))$$
$$(\forall i \le n)((t_i = 0 \Leftrightarrow s_i = 0) \ \& \ (t_i = 1 \Leftrightarrow s_i = 1))$$

Clearly there are only finitely many equivalence classes. For $\gamma, \delta \in \Gamma$ we will write $\gamma \sim \delta$ if $(x_1(\gamma), \dots, x_n(\gamma)) \sim ((x_1(\delta), \dots, x_n(\delta))$. Let $\Gamma_1, \dots, \Gamma_N$ be all equivalence classes of $\sim$ on Γ. Choose $\gamma_k \in \Gamma_k$ for $k = 1, \dots, N$ and define the mapping $F : [0,1]^\Gamma \to \mathbb{R}^\Gamma$ by the formula:

$$F(x)(\gamma) = \begin{cases} h_\gamma(x(\gamma)) & \gamma \in \{\gamma_1 \dots, \gamma_k\} \\ h_\gamma(x(\gamma)) - h_{\gamma_k}(x(\gamma_k)) & \gamma \in \Gamma_k \setminus \{\gamma_k\}, k = 1, \dots, N \end{cases}$$

It follows from the construction of h_γ's that $F(x_i) \in \Sigma(\Gamma)$ (in fact, their support is finite), and it is easy to check that F is one-to-one and $F^{-1}(\Sigma(\Gamma))$ is dense in $[0,1]^\Gamma$.

The proof for $\{0,1\}^\Gamma$ is even easier. It suffices to consider the equivalence classes and there is no need to construct h_γ's. ∎

EXAMPLE 6.8. There is a Banach space X and $K \subset X^*$ convex weak* Valdivia compactum with no convex dense Σ-subset.

Proof. Put $X = C[0, \omega_1] \times \mathbb{R}$. Then X^* is canonically isomorphic to $C[0, \omega_1]^* \times \mathbb{R}$. We identify $C[0, \omega_1]^*$ with the space of finite signed Radon measures on $[0, \omega_1]$. Put

$$P = \{(\mu, x) : \mu \ge 0, \langle \mu, 1 \rangle = 1, 0 \le x \le \sqrt{\mu(\{\omega_1\})}\},$$
$$M = \{(\mu, x) : \mu \ge 0, \|\mu\| \le 1, 0 \le x \le \sqrt{\mu(\{\omega_1\})}\}.$$

Both P and M have the required properties. It is clear that both P and M are bounded. To show that they are weak* closed, remark that

$$0 \le x \le \sqrt{(\mu(\{\omega_1\}))} \Leftrightarrow (\forall \alpha < \omega_1)(\sqrt{\langle \mu, \chi_{(\alpha, \omega_1]} \rangle} \ge x \ge 0)$$

whenever $\mu \geq 0$. Hence P and M are weak* compact. Let us show their convexity. Choose any two pairs (μ, x), (ν, y) from P (or M) and any $t \in [0,1]$. As the function $z \mapsto \sqrt{z}$ is concave on $[0, \infty)$, we obtain using the assumptions

$$0 \leq tx+(1-t)y \leq t\sqrt{\mu(\{\omega_1\})}+(1-t)\sqrt{\nu(\{\omega_1\})} \leq \sqrt{t\mu(\{\omega_1\}) + (1-t)\nu(\{\omega_1\})}.$$

This completes the proof of convexity.

Consider the following mapping $h : P \to \mathbb{R}^{[-1,\omega_1]}$

$$h(\mu, x)(\alpha) = \begin{cases} \langle \mu, \chi_{[\alpha+1,\omega_1]} \rangle - x^2 & \alpha < \omega_1 \\ x & \alpha = \omega_1 \end{cases}$$

It is clear that h is weak* continuous. Moreover, it is one-to-one. Indeed, if $h(\mu, x) = h(\nu, y)$, then obviously $x = y$ and $\langle \mu, \chi_{[\alpha+1,\omega_1]} \rangle = \langle \nu, \chi_{[\alpha+1,\omega_1]} \rangle$ for every $\alpha \in [-1, \omega_1)$. It follows that $\mu = \nu$. Moreover, we have $h^{-1}(\Sigma([-1, \omega_1]))$ $= \{(\mu, x) \in P : \mu(\{\omega_1\}) = x^2\}$

It is obvious that this Σ-subset is not convex. For example, $(\delta_{\omega_1}, 1)$ and $(\delta_0, 0)$ belong to this Σ-subset while $(\frac{1}{2}(\delta_0 + \delta_{\omega_1}), \frac{1}{2})$ does not. It remains to observe that this Σ-subset is dense in P and that is the only one dense Σ-subset.

Let $(\mu, x) \in P$ and V be a weak* neighborhood of (μ, x). Then there are $f_1, \ldots, f_n \in C[0, \omega_1]$ such that

$$\{(\nu, y) \in P : y = x \ \& \ \langle \nu, f_j \rangle = \langle \mu, f_j \rangle, \ j = 1, \ldots, n\} \subset V.$$

There is $\alpha < \omega_1$ such that all f_j are constant on $[\alpha, \omega_1]$. Put $\mu' = \mu - (\mu(\{\omega_1\}) - x^2)\delta_{\omega_1} + (\mu(\{\omega_1\}) - x^2)\delta_\alpha$. Then $(\mu', x) \in V$ and belongs to the above Σ-subset.

To show the uniqueness, it is enough to observe that P is a Radon-Nikodým compactum [17, Theorems 1.1.13 and 1.1.2], and hence has a dense set of G_δ points [17, Theorem 5.1.12]. Therefore the dense Σ-subset is unique due to Corollary 1.12.

The assertions on M can be proved exactly in the same way as those on P. ∎

6.2. Examples of Banach spaces We begin this section by the following canonical example.

Example 6.9. The space $\ell_1(\Gamma)$ is 1-Plichko for any set Γ. Moreover, any element of the dual is contained in a 1-norming Σ-subspace.

Proof. Let $x \in \ell_\infty(\Gamma) = (\ell_1(\Gamma))^*$ be arbitrary nonzero element. There is a countable subset $C \subset \Gamma$ such that for any $\gamma \in \Gamma$ there is $c_\gamma \in C$ with $x(c_\gamma) \neq 0$ and $|x(\gamma)| \leq |x(c_\gamma)|$. Define $T : \ell_\infty(\Gamma) \to \mathbb{R}^\Gamma$ by the following formula.

$$T(y)(\gamma) = \begin{cases} y(\gamma) & \gamma \in C \\ y(\gamma) - \frac{x(\gamma)}{x(c_\gamma)} \cdot y(c_\gamma) & \gamma \in \Gamma \setminus C \end{cases}$$

It is easy to observe that T is a one-to-one weak* continuous linear mapping and that the Σ-subset $T^{-1}(\Sigma(\Gamma))$ is 1-norming and contains x. ∎

A large subclass of 1-Plichko spaces is formed by *abstract* L^1 spaces. A Banach lattice X is called abstract L^1 space if the norm is additive on the positive cone.

Example 6.10. Any abstract L^1 space is 1-Plichko.

Proof. Let X be an abstract L^1 space. By [42, Corollary on p.136] the space X is isometric to the ℓ_1 sum of Banach spaces, each of which has the form $L^1(\mu)$ for a finite measure μ. The space $L^1(\mu)$ with μ finite is weakly compactly generated (cf. [62, Theorem 2]) and therefore 1-Plichko. It remains to use Theorem 4.31. ∎

Corollary 6.11. (i) *The space $L^1(\mu)$ is 1-Plichko for an arbitrary measure μ.*

(ii) *The dual space $C(K)^*$ is 1-Plichko for an arbitrary compact space K.*

(iii) *Any Banach sublattice of $\ell_1(\Gamma)$ is 1-Plichko.*

Just recently the author was informed that A. Plichko [57] obtained the following result.

Example 6.12. Any order continuous Banach lattice is 1-Plichko.

Recall that a Banach lattice is order-continuous if $\|x_a\| \to 0$ whenever x_a form a downward directed family with infimum 0. In particular any Banach lattice not containing c_0 is order continuous [45, Section 1.a].

A class of dual 1-Plichko spaces was introduced in [35]. Let us say that a Banach space X belongs to the *class (T)* if X is contained in a Σ-subspace of X^{**}. Let us name some results of [35] in the following theorem.

THEOREM 6.13. (i) *If X belongs to (T), then X^* is 1-Plichko in every equivalent dual norm.*

(ii) *If X belongs to (T), then X is Asplund.*

(iii) *The class (T) is closed with respect to subspaces and quotients.*

(iv) *If $Y \subset X$ is such that Y^* is separable and X/Y belongs to (T), then X belongs to (T).*

(v) *If X is Asplund and B_{X^*} is $\aleph_2$-Corson, then X belongs to (T).*

(vi) *If $B_{X^{**}}$ is Valdivia and the norm on X is Kadec, then X belongs to (T).*

(vii) *$C[0, \omega_2]$ does not belong to (T).*

This results witness that dual 1-Plichko spaces have other behavior than non-dual ones (cf. assertion (i) of the above theorem and Theorem 4.22). This area contains many open problems not yet considered. Let us name some of them.

QUESTION 6.14. 1. Let X be a Banach space such that $B_{X^{**}}$ is Valdivia for every equivalent norm on X. Does X belong to (T)? (it can be shown that X is necessarily Asplund).

2. Let X be Asplund. Is then X^* Plichko?

3. Let X^* be Plichko (1-Plichko). What can be said on X?

4. Let X^{**} be Plichko (1-Plichko). Does X has the same property?

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