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Weakly Continuous Functions of Baire Class 1

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1. Introduction

For a compact Hausdorff space K and a Banach space X, let $WC(K,X)$ denote the space of X-valued functions defined on K, that are continuous when X has the weak topology. In this note by a simple Banach space theoretic argument, we show that given $f \in WC(K,X)$ there exists a net $\{f_\alpha\} \subset C(K,X)$ (space of norm continuous functions) such that $f_\alpha \to f$ pointwise w.r.t the norm topology. As a consequence we get that when K is a metric space, there exists a sequence $\{f_n\} \subset C(K,X)$ such that $f_n \to f$ pointwise w.r.t the norm topology on X. Such a function f is said to be of Baire class 1. This gives a purely Banach space theoretic proof of a result of Srivatsa [7, Theorem 2.1], in the special case when the domain is a compact space (see also [1]). We next show that for a compact set K the requirement that for every Banach space X, all the elements of $WC(K,X)$ are of Baire class 1 w.r.t the norm topology, is equivalent to the countable chain condition. This is related to the question of approximating weakly compact operators by sequences of finite rank operators in the strong operator topology.

We will be using results on weakly compact subsets of Banach spaces from Chapter 5 of [2]. One of the aims of this note is to give a proof of a Proposition below (a results from [1]) that is independent of Dunford, Pettis, Phillips and Grothendieck circle of ideas but based only on the properties of weakly compact subsets of a Banach space. This point of view has been used by this author to give a new proof of a classical result of Dunford, Pettis and Phillips in [5]. Some of our observations lead to a more compact proof of the Orlicz-Pettis theorem, than the one given in [7, Corollary 2].

Our notation and terminology is standard and can be found in [2], [3] and

[4]. All the spaces are considered over the real scalar field.

2. Main result

THEOREM 1. *Let K be a compact space and $f \in WC(K,X)$. There exists a net $\{f_\alpha\} \subset C(K,X)$ such that $f_\alpha \to f$ pointwise in the norm topology.*

Proof. Let $f \in WC(K,X)$. Define $T : X^* \to C(K)$ by $T(x^*) = x^* \circ f$.

It is easy to see that T is a bounded linear operator and $T^* : C(K)^* \to X^{**}$. Also if K is embedded via the canonical Dirac map as a subset of $C(K)^*$, then

$$T^*|K = f.$$

Since the unit ball of $C(K)^*$ is the w^*-closed convex hull of $K \cup -K$ and since f is weakly continuous, it is easy to see that T^* actually takes values in X. Since any $C(K)$ space and its dual have the metric approximation property, by a standard application of the Principle of local reflexivity, it follows that there exists a net $\{P_\alpha\}$ of operators of finite rank in $C(K)$ such that $P_\alpha^* \to I$, in the strong operator topology (see [4] and [8]).

Let $f_\alpha = T^* \circ P_\alpha^*/K$. Since P_α's are of finite rank, $f_\alpha \in C(K,X)$. Clearly $f_\alpha \to f$ pointwise. ∎

Remark 1. When K is metrizable, one can choose in the above proof a sequence of finite rank contractions, $\{P_n\}$, such that $P_n \to I$ in the strong operator topology. However the adjoint operators will in general do not converge to I in the strong operator topology. In the proof of the following corollary, we over come this difficulty by using a theorem of Mazur.

COROLLARY. *Assume further that K is a metric space. Then there exists a sequence $\{f_n\}$ such that $f_n \to f$ pointwise in the norm topology.*

Proof. Since $f(K)$ is a metrizable weakly compact subset of X, we can assume w.l.o.g that K is a metrizable weakly compact subset of the unit ball of X and f is the identity mapping i.

Let $\{x_k\}$ be a norm dense sequence in K (since K is weakly separable, it is norm separable). We shall exhibit a sequence $\{f_n\} \subset C(K,X)$ such that $\|f_n(x) - f_n(y)\| \leq \|x - y\|$ for all $x, y \in K$ and $f_n(x_k) \to x_k$ in the norm, for each k. Once the existence of such a sequence is proved, the following argument shows that $f_n(x) \to x$ in the norm for every $x \in K$. To see this, let

$x \in K, \epsilon > 0$. Choose k such that $\|x - x_k\| < \epsilon/3$ and a N such that for all $n \geq N$, $\|f_n(x_k) - x_k\| < \epsilon/3$. Now for $n \geq N$,

$$\|f_n(x) - x\| \leq \|f_n(x) - f_n(x_k)\| + \|f_n(x_k) - x_k\| + \|x_k - x\| \leq \epsilon.$$

As in the theorem above, let $T : X^* \to C(K)$ be the restriction map. Clearly $\|T\| \leq 1$. Since K is metrizable, we can choose a sequence of finite rank operators $P_n^* : C(K)^* \to C(K)^*$ such that $\|P_n^*\| = 1$ and $P_n^* \to I$ pointwise in the w^*-topology (such a sequence will converge to I in the strong operator topology only when K is a dispersed space).

Now $g_n = P_n^* \circ T^*/K$ is such that

1. $g_n \in C(K, X)$,
2. $\|g_n(x) - g_n(y)\| \leq \|x - y\|, \quad \forall x, y \in K$ and
3. $g_n \to i$ pointwise weakly on K.

Since $g_n(x_1) \to x_1$ weakly, by Mazur's theorem (see [3]), we can get a subsequence of the sequence of rational convex combinations of all the g_n's, say $\{g_n^1\}$ in $C(K, X)$ such that $g_n^1(x_1) \to x_1$ in the norm. Note that $g_n^1(x_2) \to x_2$ weakly. Repeating this procedure we choose a sequence $\{g_n^2\}$ in $C(K, X)$ such that $g_n^2(x_j) \to x_j$ in the norm for $j = 1, 2$. Thus at the kth step we have the sequence $\{g_n^k\}$. Note that we still have

$$\|g_n^k(x) - g_n^k(y)\| \leq \|x - y\|, \quad \forall x, y \in K.$$

Let n_1 be the smallest integer such that $\|g_{n_1}^1(x_1) - x_1\| < 1$. Put $f_1 = g_{n_1}^1$. Let n_2 be the smallest integer such that $\|g_{n_2}^2(x_j) - x_j\| < 1/2$, for $j = 1, 2$. Put $f_2 = g_{n_2}^2$. In general let $f_m = g_{n_m}^m$ where n_m is the smallest integer such that

$$\|g_{n_m}^m(x_j) - x_j\| < \frac{1}{2^{m-1}}, \quad 1 \leq j \leq m.$$

Clearly the sequence $\{f_m\} \subset C(K, X)$ is such that $f_m(x_k) \to x_k$ in the norm for each k. And for each m

$$\|f_m(x) - f_m(y)\| \leq \|x - y\|, \quad \forall x, y \in K.$$

This completes the proof. ∎

Remark 2. Suppose X is such that there exists a sequence $\{T_n\}$ of compact operators on X with $T_n \to I$ pointwise, then for any $f \in WC(K, X), f_n = T_n \circ f$ is clearly in $C(K, X)$ and $f_n \to f$ pointwise. This clearly is the case for example when X has a Schauder basis. This idea leads to a 'shorter' proof of the Orlicz-Pettis theorem (Corollary 2.1 in [7]).

THEOREM. (ORLICZ-PETTIS) *Let X be a Banach space. Every weakly subseries convergent series in X is subseries convergent in the norm.*

Proof. It is enough to prove the Theorem when X is a separable Banach space. Since any such X is isometric to a subspace of $C[0,1]$, it is clearly enough to prove the result in $C[0,1]$. Now if we define $F : 2^w \to C[0,1]$ as in the proof of Corollary 2.1 in [7], from our remark above, since $C[0,1]$ has a Schauder basis and F is weakly continuous, it easily follows that F is of Baire class 1 for the norm topology. Therefore F has a point of continuity. The rest of the arguments are identical to the ones given in [7]. ∎

We are now ready to prove the characterization mentioned in the introduction. The proof uses several standard ideas from Chapter 5 of [2]. Recall that a topological space satisfies the countable chain condition (C.C.C), if each disjoint family of open sets is countable.

THEOREM 2. *Let K be any compact set. The following assertions are equivalent:*

1. *K satisfies the countable chain condition (C.C.C).*
2. *For any Banach space X, any $f \in WC(K,X)$ is of Baire class 1 for the norm topology.*
3. *Every weakly compact subset of $C(K)$ is norm separable.*

Proof. $1 \Rightarrow 2$. Let $f \in WC(K,X)$. $f(K)$ is now a weakly compact subset of X. Clearly $f(K)$ satisfies the C.C.C, hence it follows from a well known result of Rosenthal (see [6]) that $f(K)$ is separable. Hence $f(K)$ is metrizable. It now follows from the arguments indicated before that f is of Baire class 1 for the norm topology.

$2 \Rightarrow 3$. Let M be a weakly compact subset of $C(K)$. Let A be the closed subalgebra (over real scalars) generated by M and the constant function 1. Since the product of two weakly compact subsets of $C(K)$ is again weakly compact, we get that A is a weakly compactly generated space. Also there exists a compact set K' and a continuous onto map $f : K \to K'$ such that A is isometric to $C(K')$. Since $C(K')$ now is weakly compactly generated, we get that K' is Eberlian compact. So we assume w.l.o.g that K' is a weakly compact subset of a Banach space. By our assumption it now follows that f is of Baire class 1. Consequently K' is separable and hence metrizable. Therefore A and hence M is separable.

$3 \Rightarrow 1$. Suppose K fails the C.C.C. There exists an uncountable set Γ and a family $\{A_\alpha\}_{\alpha\in\Gamma}$ of pairwise disjoint open subsets of K. Chose $t_\alpha \in A_\alpha$ and continuous functions $0 \le f_\alpha \le 1$, such that $f_\alpha(t_\alpha) = 1$, $f_\alpha(A_\alpha^c) = 0$. Now the map $T : c_0(\Gamma) \to C(K)$ defined by $T(e_\alpha) = f_\alpha$ is an isometry. This contradicts the hypothesis. ∎

Remark 3. Let $\mathcal{F}(X, C(K))$ denote the space of weakly compact operators. Using the canonical identification of this space with $WC(K, X^*)$ one can reformulate the above result in terms of approximation by sequences of finite rank operators. Similarly for any positive, finite measure μ on a measure space $(\Omega, \mathcal{A})$, the identification of $\mathcal{F}(L^1(\mu), X)$ with $WC(K, X)$,where K is the stone space of $L^\infty(\mu)$ (see [5]) leads to approximation by sequences of finite rank operators of elements in $\mathcal{F}(L^1(\mu), X)$.

We conclude by giving a proof of Proposition 4 of [1] based on the ideas of this note and apply it to give a proof of the Krein-Šmulian theorem.

PROPOSITION. *Let K be a compact Hausdorff space and μ a finite, regular, positive, Borel measure on K. Then any $f \in WC(K, X)$ is μ-Bochner integrable.*

Proof. Let E be the closed support of μ. Clearly $f/E \in WC(E, X)$ and E satisfies the C.C.C. It follows from Theorem 2 that there exists $f_n \in C(E, X), f_n \to f$ pointwise on E. Hence f is μ-Bochner integrable. ∎

Remark 4. As an application of the above Proposition, one can prove the Krein-Šmulian theorem using Bochner integrals (like the way it was done in [3]). Our approach, briefly indicated below for the sake of completeness, is to proceed as suggested in Exercise 3 on page 29 of [3]. We note that (i) of that exercise involves the Eberlein-Šmulian theorem.

THEOREM. (Krein-Šmulian) *The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

Proof. Let K be a weakly compact subset of X. The Bochner integrability of the identity mapping w.r.t every regular Borel measure on K is guaranteed by the above Proposition. Since the set of probability measures is a *weak** compact convex set in $C(K)^*$, the conclusion follows from the observation that the map induced by the Bochner integral on this set is a continuous map whose range contains the closed convex hull of K. ∎

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