EXTRACTA MATHEMATICAE Vol. **14**, Núm. 3, 355 – 371 (1999)

Valdivia Compacta and Subspaces of $C(K)$ Spaces *

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(Research paper presented by J.M.F. Castillo)

AMS *Subject Class.* (1991): 46B26, 46E15, 54C05 *Received June* 30, 1999

1. Introduction

Corson and Valdivia compact spaces play an important role in functional analysis. The spaces of continuous functions on such compact spaces, as well as Banach spaces with dual unit ball in some of these classes were studied for example in [2], [18], [19], [20], [5], [7], [11], [12].

In [10] it was proved that a compact space, whose every continuous image is Valdivia, is already Corson. First example of a non-Valdivia continuous image of a Valdivia compact space was given in [21]. In view of these results it is natural to ask whether similar results hold in the framework of Banach spaces. Namely, *is the dual unit ball of a Banach space Corson provided the dual unit ball of every subspace is Valdivia?* This question was posed to the author first by M. Fabian and V. Zizler. An easy example of a Banach space X and its subspace Y such that B_{X^*} is Valdivia and B_{Y^*} is not Valdivia, is given in [9]. In the present paper we prove a partial positive answer to the above question, namely we prove that the dual unit ball of X is Corson whenever X is of the form $C(K)$ where K is a continuous image of a Valdivia compact space, and the dual unit ball of every subspace of X is Valdivia. A related result, showing that the dual unit ball of a Banach space is Corson provided the dual unit ball of every equivalent norm is Valdivia, is given in [12].

Let us start with basic definitions.

*The research was supported by Research grants GAUK 1/1998, GAUK 160/1999 and GAČR 201/97/0216.

Definition 1. Let Γ be a set.

1. For $x \in \mathbb{R}^\Gamma$ we denote $\text{supp} x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}$.
2. We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \text{supp} x \text{ is countable}\}$.

Definition 2. Let K be a compact Hausdorff space.

1. K is called a *Corson compact* space if K is homeomorphic to a subset of $\Sigma(\Gamma)$ for some Γ.
2. K is called a *Valdivia compact* space if K is homeomorphic to a subset K' of $\mathbb{R}^\Gamma$ for some Γ such that $K' \cap \Sigma(\Gamma)$ is dense in K'.

It turned out to be useful to introduce the following auxiliary notion.

Definition 3. Let K be a compact Hausdorff space and $A \subset K$ be arbitrary. We say that A is a *Σ-subset* of K if there is a homeomorphic injection φ of K into $\mathbb{R}^\Gamma$ for some Γ such that $\varphi(A) = \varphi(K) \cap \Sigma(\Gamma)$.

In this setting a compact K is Valdivia if it has a dense Σ-subset.

We will need also the following notion of property (M), which was used in [2] to characterize those Corson compact spaces K such that the dual unit ball $B_{C(K)^*}$ is Corson as well.

Definition 4. A compact Hausdorff space is said to have *property (M)* if every Radon probability measure on K has separable support.

It turned out ([19], [20] and [7]) that there is a closed connection between Valdivia compacta and projectional resolutions of the identity and Markuševič bases. Let us now recall the definitions of these notions.

Definition 5. Let X be a Banach space of the density $\kappa > \aleph_0$. The *projectional resolutions of the identity (PRI)* on X is an indexed family $(P_\alpha \mid \omega \leq \alpha \leq \kappa)$ of projections on X with the following properties.

(i) $P_\omega = 0$, $P_\kappa = \text{Id}_X$;

(ii) $\|P_\alpha\| = 1$ for $\omega < \alpha \leq \kappa$;

(iii) $\text{dens} P_\alpha X \leq \text{card} \alpha$ for $\omega < \alpha \leq \kappa$;

(iv) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\omega \leq \alpha \leq \beta \leq \kappa$;

(v) $P_\alpha X = \overline{\bigcup_{\beta<\alpha} P_\beta X}$ if $\alpha \leq \kappa$ is limit.

DEFINITION 6. Let X be a Banach space.

1. A *Markuševič basis* of X is an indexed family $(x_a, f_a)_{a\in A} \subset X \times X^*$ such that the following conditions are fulfilled.

   (a) $f_a(x_b) = 0$ if $a \neq b$, $f_a(x_a) = 1$ for $a, b \in A$;
   (b) $\overline{\operatorname{span}\{x_a \mid a \in A\}} = X$;
   (c) For every $x \in X$, $x \neq 0$ there is $a \in A$ with $f_a(x) \neq 0$.

2. A Markuševič basis $(x_a, f_a)_{a\in A}$ is *countably 1-norming* if for every $x \in X$ we have $\|x\| = \sup\{f(x) \mid f \in M, \|f\| \leq 1\}$, where $M = \{f \in X^* \mid \{a \in A \mid f(x_a) \neq 0\}$ is countable$\}$.

2. MAIN RESULTS

To formulate our results in a simple general way we introduce the following class of compact spaces.

DEFINITION 7. A compact Hausdorff space is said to belong to the *class* $\mathcal{G}\Omega$ if for every nonempty open subset $U \subset K$ the following condition is satisfied.

> Either U contains at least one G_δ point of K, or the one-point compactification of U contains a homeomorphic copy of the ordinal segment $[0, \omega_1]$.

It is clear from the definition that the class $\mathcal{G}\Omega$ contains all compact Hausdorff spaces with a dense set of G_δ points. It follows from Proposition 4 below that every continuous image of a Valdivia compact space belongs to this class as well. Example 1 shows that this class does not contain all compact spaces. Now we are ready to formulate main theorems.

The first one characterizes Corson compact spaces with property (M) using properties of all subspaces of the space of continuous functions.

THEOREM 1. *Let K be a compact Hausdorff space from the class $\mathcal{G}\Omega$. Then the following assertions are equivalent.*

1. *K is a Corson compact space with the property (M).*
2. *For every subspace $Y \subset C(K)$ the dual unit ball (B_{Y^*}, w^*) has a dense convex symmetric Σ-subset.*

3. *For every subspace* $Y \subset C(K)$ *the dual unit ball* (B_{Y^*}, w^*) *is a Valdivia compact.*
4. *Every subspace of* $C(K)$ *has a countably 1-norming Markuševič basis.*

If the topology of K *has a basis with cardinality* $\aleph_1$ *then these conditions are also equivalent to the following one.*

5. *Every subspace of* $C(K)$ *has a projectional resolution of the identity.*

In particular, the assumptions of this theorem are satisfied if K *is a continuous image of a Valdivia compact space.*

The second theorem gives a characterization of Corson compact spaces (regardless of the property (M)) using properties of those subspaces of the space of continuous functions which are also of the form $C(L)$. This theorem can be viewed as a strengthening of [10, Theorem 3.1].

THEOREM 2. *Let* K *be a compact Hausdorff space from the class* $\mathcal{G}\Omega$. *Then the following assertions are equivalent.*

1. K *is a Corson compact space.*
2. *For every continuous image* L *of* K *the dual unit ball* $(B_{C(L)^*}, w^*)$ *has a dense convex symmetric* Σ*-subset.*
3. *For every continuous image* L *of* K *the dual unit ball* $(B_{C(L)^*}, w^*)$ *is a Valdivia compact.*
4. *For every continuous image* L *of* K *the space* $C(L)$ *has a countably 1-norming Markuševič basis.*

If the topology of K *has a basis with cardinality* $\aleph_1$ *then these conditions are also equivalent to the following one.*

5. *For every continuous image* L *of* K *the space* $C(L)$ *has a projectional resolution of the identity.*

In particular, the assumptions of this theorem are satisfied if K *is a continuous image of a Valdivia compact space.*

The following example shows both that the class $\mathcal{G}\Omega$ does not contain all compact spaces, and that the assumption on K in Theorems 1 and 2 is not the best possible.

EXAMPLE 1. Let $K = \beta\mathbb{N} \setminus \mathbb{N}$ be the remainder of $\mathbb{N}$ in its Čech-Stone compactification. Then the following holds.

(a) K does not belong to $\mathcal{G}\Omega$.

(b) The dual unit ball $(B_{C(K)^*}, w^*)$ is not a Valdivia compact.

3. Auxiliary results

We start with the following lemma which sums up basic properties of Σ-subsets.

LEMMA 1. [10, Proposition 2.2] *Let K be a compact Hausdorff space and $A \subset K$ be a dense Σ-subset of K. Then the following assertions hold.*

1. *A is countably closed in K, i.e. $\overline{C} \subset A$ for every $C \subset A$ countable.*
2. *A is a Fréchet-Urysohn space, i.e. whenever $x \in A$, $C \subset A$ are such that $x \in \overline{C}$, then there are $x_n \in C$ with $x_n \to x$.*
3. *If $G \subset K$ is a G_δ set, then $G \cap A$ is dense in G.*

Let us remark that the point (1) of the previous lemma is obvious and the point (2) immediately follows from [15, Theorem 2.1]. In the point (3) it would be enough to suppose that A is a dense countably compact subset of K (cf. [10, Lemma 2.3]). The following lemma slightly strengthens [11, Theorem 4.10(3) $\Rightarrow$ (1)].

LEMMA 2. *Let K be a compact Hausdorff space such that the dual unit ball $(B_{C(K)^*}, w^*)$ is a Valdivia compact. If we denote by M the set of all G_δ points of K, then $\overline{M}$ is a Valdivia compact as well.*

Proof. We use the same idea as in the proof of [11, Theorem 4.10]. Let A be a dense Σ-subset of $B_{C(K)^*}$. If $m \in M$ is a G_δ point of K, then there is $f \in C(K)$ such that $f(m) = 1$ and $f(k) \in [0, 1)$ if $k \neq m$. Then we have

$$\{\delta_m\} = \{\mu \in B_{C(K)^*} \mid \langle \mu, f \rangle = 1\},$$

where δ_m is the Dirac measure supported by the point m. It follows that δ_m is a weak* G_δ point of $B_{C(K)^*}$, hence $\delta_m \in A$ by Lemma 1(3). If we identify k with δ_k for every $k \in K$, we have $M \subset A$, so $\overline{M} \cap A$ is dense in $\overline{M}$. Therefore $\overline{M}$ is a Valdivia compact. ∎

The following lemma generalizes the result of [17] that every Corson compact has a dense set of G_δ points.

LEMMA 3. *Let K be a compact Hausdorff space such that there is a residual Σ-subset A of K. Then K has a dense set of G_δ points.*

Proof. At first we prove the following statement.

Every nonempty Corson compact space has at least one G_δ point.

This follows from the result of [17] but we give an easy proof for the sake of completeness. Let $H \subset \Sigma(\Gamma)$ be compact. Let us introduce on H the following order.

$$x \leq y \Leftrightarrow (\forall \gamma \in \Gamma)(x(\gamma) \neq 0 \Rightarrow x(\gamma) = y(\gamma)).$$

This is a partial order and it is clear from compactness of H that any subset of H totally ordered by this relation has an upper bound. So by Zorn's lemma, there is a maximal element x_m of H. It is clear that

$$\{x_m\} = \{y \in H \mid y(\gamma) = x_m(\gamma) \text{ for all } \gamma \in \operatorname{supp} x_m\},$$

which is a G_δ set as $\operatorname{supp} x_m$ is countable. This completes the proof of $(*)$.

Now, let A be a residual Σ-subset of K and $U \subset K$ a nonempty open set. It follows easily from the regularity of K that there is a nonempty closed G_δ set $H \subset A \cap U$. As H is a Corson compact, by $(*)$ it has a G_δ point, which is also a G_δ point of K contained in U. This completes the proof. ∎

Now we are ready to prove the following proposition.

PROPOSITION 1. *Let K be a compact Hausdorff space, M be the set of all G_δ points of K. If $\overline{M}$ is not Corson, then there are $a, b \in K$ such that that $B_{C(L)^*}$ is not Valdivia compact where L is the quotient space made from K by identifying a and b.*

Proof. If $\overline{M}$ is not Valdivia, it is enough to choose $a = b$, due to Lemma 2. Next suppose that $\overline{M}$ is Valdivia. Let A be a dense Σ-subset of $\overline{M}$. Choose $a \in A$ an non-isolated point and $b \in \overline{M} \setminus A$. Let L be the quotient space made from K by identifying a and b, with Q being the canonical quotient mapping. Then $Q(\overline{M})$ is not Valdivia by [10, Proposition 3.2]. Further, $M \setminus \{a\}$ is dense in $\overline{M}$ and it is clear that $Q(m)$ is a G_δ point of L for every $m \in M \setminus \{a\}$. It follows that $Q(\overline{M})$ is the closure of the set of all G_δ points of L. Therefore, by Lemma 2, the dual unit ball $B_{C(L)^*}$ is not Valdivia. ∎

To deal with compact spaces without G_δ points we need some finer properties of Σ-subsets. We start by the following lemma which follows easily from [4, Theorem 1] and [8, Theorem 2] and was proved in [10, Proposition 2.5].

LEMMA 4. *Let K be a compact Hausdorff space and $A \subset K$ be a dense subset of K. Then A is a Σ-subset of K, if and only if A is homeomorphic to a countably compact subset of some $\Sigma(\Gamma)$ and $K = \beta A$ (i.e., each bounded continuous real function on A can be continuously extended to K).*

As a consequence we get the following lemma which refines Lemma 1(3).

LEMMA 5. *Let K be a compact Hausdorff space and $G = \bigcap_{n\in\mathbb{N}} \overline{U}_n$ where each U_n is an open subset of K. If A is a dense Σ-subset of K, then $G \cap A$ is dense in G.*

Proof. Let A be a fixed dense Σ-subset of K.

Step 1. If U is open in K, then $\overline{U} \cap A$ is dense in $\overline{U}$.

This follows simply by density of A.

Step 2. If U and V are disjoint open subsets of K, then $\overline{U} \cap \overline{V} \cap A$ is dense in $\overline{U} \cap \overline{V}$.

This was proved in [10, Proposition 2.6], we give the proof for the sake of completeness. Without loss of generality we can suppose that $K = \overline{U} \cup \overline{V}$. Let $x \in \overline{U} \cap \overline{V}$ and $W \subset K$ be an open neighborhood of x such that $\overline{W} \cap \overline{U} \cap \overline{V} \cap A = \emptyset$. Then $A \cap \overline{W}$ is a dense Σ-subset of $\overline{W}$. The sets $A \cap \overline{W} \cap \overline{U}$ and $A \cap \overline{W} \cap \overline{V}$ are disjoint relatively clopen subsets covering $A \cap \overline{W}$ (as $\overline{W} \cap \overline{U} \cap \overline{V} \cap A = \emptyset$). Let f be the characteristic function of $\overline{W} \cap \overline{U} \cap A$. This is a bounded continuous function on $A \cap \overline{W}$. By Lemma 4 this function can be continuously extended on $\overline{W}$. But the point x belongs both to $\overline{\overline{W} \cap \overline{U} \cap A}$ and to $\overline{\overline{W} \cap \overline{V} \cap A}$, which is impossible. This completes the proof.

Step 3. If $U_1, \dots, U_n$ are pairwise disjoint open subsets of K, then $G \cap A$ is dense in G, where $G = \overline{U}_1 \cap \dots \cap \overline{U}_n$.

We will prove it by induction. By Steps 1 and 2 the assertion holds for $n = 1, 2$. Suppose we have proved it up to n and that $U_1, \dots, U_{n+1}$ are pairwise disjoint open sets. Put $H_1 = \overline{U}_1 \cap \dots \cap \overline{U}_n$, $H_2 = \overline{U}_{n+1}$ and $H = H_1 \cup H_2$. By induction hypothesis we get that $H_1 \cap A$ is dense in H_1, by Step 1 we have $H_2 \cap A$ dense in H_2. It follows that $H \cap A$ is a dense Σ-subset of H. Now we are going to prove that $H_1 \cap H_2 \cap A$ is dense in $H_1 \cap H_2$.

Put $V = \operatorname{int}_{H_1}(H_1 \cap H_2)$. Clearly $\overline{V} \cap A$ is dense in $\overline{V}$ (Step 1). Further, $H_1 \setminus H_2$ and $H_2 \setminus H_1$ are two disjoint open subsets of H (the first one is equal to $H \setminus H_2$, the second one to $H \setminus H_1$), hence by Step 2 the intersection $H' \cap A$ is dense in $H' = \overline{H_1 \setminus H_2} \cap \overline{H_2 \setminus H_1}$. To finish Step 3 it is enough to show that $H_1 \cap H_2 = H' \cup \overline{V}$.

The inclusion "$\supset$" is obvious, let us prove the inverse one. Let $x \in H_1 \cap H_2$. If $x \in V$, then clearly $x \in H' \cup \overline{V}$. Suppose that $x \in (H_1 \cap H_2) \setminus V$. Observe that $H_1 \cap H_2$ is nowhere dense in H_2, so $\overline{H_2 \setminus H_1} = H_2$ and thus $x \in \overline{H_2 \setminus H_1}$. Further, $(H_1 \cap H_2) \setminus V$ is nowhere dense in H_1, so

$$H_1 = \overline{H_1 \setminus ((H_1 \cap H_2) \setminus V)} = \overline{(H_1 \setminus H_2) \cup V} = \overline{H_1 \setminus H_2} \cup \overline{V}.$$

Therefore either $x \in \overline{V}$ or $x \in \overline{H_1 \setminus H_2} \cap \overline{H_2 \setminus H_1} = H'$, which completes the proof of Step 3.

Step 4. If $U_1, \dots, U_n$ are open subsets of K, then $G \cap A$ is dense in G where $G = \overline{U}_1 \cap \dots \cap \overline{U}_n$.

For $J \subset \{1, \dots, n\}$ we put

$$W_J = \operatorname{int}\left(\bigcap_{j \in J} \overline{U}_j \setminus \bigcup_{j \notin J} U_j \right),$$

where we adopt the convention that the union of the empty collection is the empty set and the intersection of the empty collection is the whole space K.

It is clear that the sets W_J are open and pairwise disjoint. In view of Step 3 it is enough to show that the set G is the union of some sets of the form $\overline{W}_{J_1} \cap \dots \cap \overline{W}_{J_k}$.

First let us show that

$$(*) \qquad \overline{U}_k = \bigcup_{k \in J} \overline{W}_J, \qquad k = 1, \dots, n.$$

The inclusion "$\supset$" follows from the definition of W_J. To prove the inverse one let us remark that

$$\overline{U}_k = \bigcup_{k \in J} \left(\bigcap_{j \in J} \overline{U}_j \setminus \bigcup_{j \notin J} U_j \right).$$

In this way we have $\overline{U}_k$ covered by finitely many closed sets, hence the union of their interiors is dense in $\overline{U}_k$, which yields $(*)$. In particular, it follows from $(*)$ that the closures of all W_J cover K.

Choose $x \in G$ arbitrary. Put $\mathcal{J} = \{J \subset \{1, \dots, n\} \mid x \in \overline{W}_J\}$. By the above we have $\mathcal{J} \neq \emptyset$. We claim that

$$x \in \bigcap_{J \in \mathcal{J}} \overline{W}_J \subset G.$$

The first relation is satisfied due to the choice of $\mathcal{J}$. Let us prove the second one. By the definition of W_J we have

$$\overline{W}_J \subset \bigcap_{j\in J} \overline{U}_j,$$

hence

$$\bigcap_{J\in\mathcal{J}} \overline{W}_J \subset \bigcap_{j\in\bigcup\mathcal{J}} \overline{U}_j,$$

so it suffices to prove that $\bigcup\mathcal{J} = \{1,\dots,n\}$. Suppose this is not the case, let $k \notin \bigcup\mathcal{J}$. Then $x \notin \bigcup_{k\in J} \overline{W}_J$. By (*) we get $x \notin \overline{U}_k$ which contradicts the fact $x \in G$. This completes the proof of Step 4.

Step 5. Let U_n be a sequence of open subsets of K. Then $G \cap A$ is dense in A where $G = \bigcap_{n\in\mathbb{N}} \overline{U}_n$.

Let $x \in G$ and W be an open neighborhood of x. By regularity of K there is an open set V with $x \in V \subset \overline{V} \subset W$. By Step 4 the set $\overline{V} \cap \bigcap_{j=1}^{n} \overline{U}_n \cap A$ is nonempty for every $n \in \mathbb{N}$. As A is countably compact (Lemma 1(1)), it is clear that $\overline{V} \cap G \cap A \neq \emptyset$ which completes the proof. ∎

PROPOSITION 2. *Let K be a compact Hausdorff space such that there are two disjoint closed nowhere dense, mutually homeomorphic subsets $M, N \subset K$ such that N has a dense set of (relatively) G_δ points and is not a Valdivia compact. Then there is L, an at most two-to-one continuous image of K, such that $B_{C(L)^*}$ is not Valdivia.*

Proof. Let $h : M \to N$ be a homeomorphism and put $L = K \setminus M$ with the quotient topology defined by the mapping

$$\varphi(x) = \begin{cases} x & x \in K \setminus M, \\ h(x) & x \in M. \end{cases}$$

There are disjoint open sets $U', V' \subset K$ such that $U' \supset M$, $V' \supset N$ and $\overline{U'} \cap \overline{V'} = \emptyset$. Now it follows from the definition of quotient topology that the sets $U = \varphi(U') \setminus N$ and $V = \varphi(V') \setminus N$ are open disjoint in L, and moreover, $\overline{U} \cap \overline{V} = N$.

Suppose that $B_{C(L)^*}$ is Valdivia. By Lemma 1(3) the space of Radon probabilities $P(L)$ is Valdivia as well. Let us note, that $P(\overline{W})$ is of the form $\bigcap_{n\in\mathbb{N}} \overline{G}_n$ with G_n open in $P(L)$ whenever $W \subset L$ is open. Indeed,

$$
\begin{aligned}
P(\overline{W}) = \overline{P(W)} &= \overline{\bigcap_{n\in\mathbb{N}} \{\mu \in P(L) \mid \mu(W) > 1 - \tfrac{1}{n}\}} \\
&\subset \bigcap_{n\in\mathbb{N}} \overline{\{\mu \in P(L) \mid \mu(W) > 1 - \tfrac{1}{n}\}} \\
&\subset \bigcap_{n\in\mathbb{N}} \{\mu \in P(L) \mid \mu(\overline{W}) \geq 1 - \tfrac{1}{n}\} = P(\overline{W})
\end{aligned}
$$

We used the standard identification $P(B) = \{\mu \in P(K) \mid \mu(B) = 1\}$ for any Borel set $B \subset K$, and the well known facts that $\{\mu \in P(L) \mid \mu(W) > c\}$ is open in $P(L)$ for W open in L and $\{\mu \in P(L) \mid \mu(F) \geq c\}$ is closed in $P(L)$ for F closed in L (see e.g [14]). Now clearly $P(\overline{U}) \cap P(\overline{V}) = P(N)$. So it follows by Lemma 5 that $P(N)$ is Valdivia as well. As N has a dense set of G_δ points, by Lemma 2 the space N is Valdivia, which is a contradiction. ∎

COROLLARY 1. *Let K be a compact Hausdorff space which contains four pairwise disjoint nowhere dense homeomorphic copies of the ordinal segment $[0, \omega_1]$. Then there is L, an at most four-to-one continuous image of K, such that $B_{C(L)^*}$ is not Valdivia.*

Proof. Let $M_1, \dots, M_4$ be the copies of $[0, \omega_1]$ with $m_1, \dots, m_4$ being the respective points ω_1. Let K' be the quotient space made from K by identifying m_1 with m_2 and m_3 with m_4. Denote by Q the quotient mapping. Then $Q(M_1 \cup M_2)$ and $Q(M_3 \cup M_4)$ are pairwise disjoint closed nowhere dense mutually homeomorphic sets. Further, each of them has a dense set of G_δ (even isolated) points, and by [10, Example 3.4] they are not Valdivia. It remains to use Proposition 2. ∎

Further we will need the following lemma which is a Banach space counterpart of [10, Lemma 2.8]. We will need it only for a hyperplane, but we prefer to formulate it in this more general setting.

LEMMA 6. *Let X be a Banach space and Y its complemented subspace such that the quotient X/Y is separable. Let i denote the canonical injection of Y into X and i^* its adjoint mapping. If K is a weak* compact set in X^* and A is a Σ-subset of $i^*(K)$, then $(i^*)^{-1}(A) \cap K$ is a Σ-subset of K.*

Proof. By our assumptions there is a closed separable subspace $Z \subset X$ such that $X = Y \oplus Z$. Denote by j the canonical injection of Z into X and by j^* the adjoint mapping.

Let $K \subset X^*$ be weak* compact and $A \subset i^*(K)$ a Σ-subset. Then there is a homeomorphic injection $h_0 : i^*(K) \to \mathbb{R}^\Gamma$ with $h_0(A) = h_0(i^*(K)) \cap \Sigma(\Gamma)$.

Further, $j^*(K)$ is a metrizable compact (as Z is separable), therefore there is a homeomorphic injection $h_1 : j^*(K) \to \mathbb{R}^\mathbb{N}$.

We are ready to define $h : K \to \mathbb{R}^{\Gamma\cup\mathbb{N}}$ by the formula

$$h(x)(\gamma) = h_0(i^*(x))(\gamma), \qquad \gamma \in \Gamma;$$
$$h(x)(n) = h_1(j^*(x))(n), \qquad n \in \mathbb{N}.$$

It is clear that h is continuous. Further, h is one-to-one. Indeed, if $h(x) = h(y)$, then $i^*(x) = i^*(y)$ and $j^*(x) = j^*(y)$ (as h_0 and h_1 are one-to-one), which means $x \restriction Y = y \restriction Y$ and $x \restriction Z = y \restriction Z$, so $x = y$. Finally, it is obvious that $h(x) \in \Sigma(\Gamma \cup \mathbb{N})$ if and only if $i^*(x) \in A$. ∎

LEMMA 7. *Let K be a compact Hausdorff space, $k \in K$ a non-isolated point and μ be a continuous Radon probability on K (i.e. all singletons have zero μ-measure). Put $Y = \{f \in C(K) \mid f(k) = \langle \mu, f\rangle\}$. Let i be the canonical embedding of Y into $C(K)$ and i^* be its adjoint mapping. Then $i^*(G)$ has nonempty interior in $i^*(P(K))$ whenever $G \subset P(K)$ is a nonempty open set.*

Proof. Denote $H = \operatorname{supp}\mu$. Let $G \subset P(K)$ be a nonempty open set. By [14] there are $U_1, \dots, U_n \subset K$ pairwise disjoint nonempty open sets and numbers $c_1, \dots, c_n > 0$ with $c_1 + \cdots + c_n < 1$ such that

$$G_1 = \{\nu \in P(K) \mid \nu(U_j) > c_j,\ j = 1, \dots, n\} \subset G,$$

and this G_1 is a nonempty open set. We can chose nonempty open sets $V_j \subset U_j$ such that $\overline{V}_j \subset U_j$, $k \notin \overline{V}_j$, and $\mu(U_j \setminus \overline{V}_j) > 0$ whenever $\mu(U_j) > 0$. This is possible by regularity of K using the fact that k is not an isolated point, and by regularity of μ using the fact that μ is continuous. Then the set

$$G_2 = \{\nu \in P(K) \mid \nu(V_j) > c_j,\ j = 1, \dots, n\}$$

is a nonempty open set contained in G_1. Further, it is easy to construct open sets $W_1, \dots, W_n \subset K$ such that

(i) the sets $\overline{V}_1, \dots, \overline{V}_n, \overline{W}_1, \dots, \overline{W}_n$ are pairwise disjoint;

(ii) $k \notin \overline{W}_j$, $j = 1, \dots, n$;

(iii) $W_j \cap H \neq \emptyset$, $j = 1, \dots, n$.

Now, for every $j = 1, \dots, n$ we choose $f_j, g_j \in C(K)$ such that

(a) $0 \leq f_j \leq 1$, $0 \leq g_j \leq 1$;

(b) $\mathrm{supp} f_j \subset V_j$, $\mathrm{supp} g_j \subset W_j$;
(c) $\|f_j\| = 1$;
(d) $\langle \mu, f_j \rangle = \langle \mu, g_j \rangle$.

If $V_j \cap H = \emptyset$, put $g_j = 0$ and choose f_j satisfying (a)–(c) by complete regularity of K. Then (d) is fulfilled automatically.

If $V_j \cap H \neq \emptyset$, choose first g'_j such that $0 \leq g'_j \leq 1$, $\mathrm{supp} g'_j \subset W_j$ and $\langle \mu, g'_j \rangle > 0$. This is possible due to (iii). Choose $L_j \subset V_j \cap H$ compact such that $0 < \mu(L_j) < \langle \mu, g'_j \rangle$. By regularity of μ and normality of K, there is $M_j \subset K$ open such that $L_j \subset M_j \subset \overline{M}_j \subset V_j$ and $\mu(M_j) < \langle \mu, g'_j \rangle$. By Tietze theorem there is $f_j \in C(K)$, $0 \leq f_j \leq 1$ such that $f_j \restriction L_j = 1$ and $f_j \restriction (K \setminus M_j) = 0$. Then f_j satisfies conditions (a)–(c) and $0 < \langle \mu, f_j \rangle < \langle \mu, g'_j \rangle$. Put $g_j = \frac{\langle \mu, f_j \rangle}{\langle \mu, g'_j \rangle} \cdot g'_j$. Then clearly f_j and g_j satisfy all conditions (a)–(d).

Put

$$G_3 = \{\nu \in P(K) \mid \langle \nu, f_j - g_j \rangle > c_j, j = 1, \dots, n\}.$$

Then G_3 is clearly open in $P(K)$. Moreover, $G_3 \subset G_2$, as for $\nu \in G_3$ we have clearly

$$\nu(V_j) \geq \langle \nu, f_j \rangle > \langle \nu, g_j \rangle + c_j \geq c_j.$$

Further, remark that $G_3 \neq \emptyset$. Indeed, put $\Delta = 1 - (c_1 + \cdots + c_n)$ and choose $x_j \in V_j$ such that $f_j(x_j) = 1$. Then the measure $\nu = \sum_{j=1}^{n} (c_j + \frac{\Delta}{n}) \delta_{x_j}$ belongs to G_3.

Finally, as $f_j - g_j \in Y$ for every j, we get that

$$i^*(G_3) = \{\xi \in i^*(P(K)) \mid \langle \xi, f_j - g_j \rangle > c_j, j = 1, \dots, n\}$$

which is a nonempty open set contained in $i^*(G)$. This completes the proof. ∎

Proposition 3. *Let K be a Corson compact space without property (M). Then there is a hyperplane $Y \subset C(K)$ such that B_{Y^*} is not Valdivia.*

Proof. Let $P(K)$ denote the space of Radon probabilities on K endowed with the weak* topology inherited from $C(K)^*$. By [11, Theorem 3.2] $P(K)$ has a dense convex Σ-subset A. This is the unique Σ-subset, due to [10, Proposition 2.4], as $P(K)$ has a dense set of G_δ points by Lemma 3 and [11, Lemma 4.11]. By [2, Theorem 3.5] $P(K)$ is not Corson, so there is $\mu_0 \in P(K) \setminus A$. This μ_0, as each element of $P(K)$, is a convex combination of a countably supported measure μ_d and a continuous measure μ. It is easy to check that $\mu_d \in A$ (cf. [2, Theorem 3.5] or [11, Theorem 3.2]), so $\mu \notin A$ as A is convex. Hence we have a continuous measure $\mu \in P(K) \setminus A$. Let H denote the support of μ. Further, choose $k \in H$ an arbitrary point. We put

$$Y = \{f \in C(K) \mid f(k) = \langle \mu, f \rangle\}.$$

This is clearly a hyperplane. We are going to prove that B_{Y^*} is not Valdivia. For contradiction suppose it is Valdivia.

Denote by i the inclusion of Y into $C(K)$. Remark, that $i^*(P(K)) = \{\xi \in B_{Y^*} \mid \langle \xi, 1 \rangle = 1\}$. Indeed, the inclusion "$\subset$" is trivial and the inverse one follows for example from Hahn-Banach theorem together with the well-known fact that $P(K) = \{\nu \in B_{C(K)^*} \mid \langle \nu, 1 \rangle = 1\}$. So $i^*(P(K))$ is weak* G_δ weak* closed subset of B_{Y^*}, hence it is a Valdivia compact as well by Lemma 1(3). Let B be a dense Σ-subset of $i^*(P(K))$. By Lemma 6 the set $C = (i^*)^{-1}(B) \cap P(K)$ is a Σ-subset of $P(K)$. It follows from Lemma 7 that C is dense, hence $C = A$ (as A is the unique dense Σ-subset), so $\delta_k \in C$ and $\mu \notin C$, which is a contradiction as $i^*(\delta_k) = i^*(\mu)$. ∎

PROPOSITION 4. *Let K be a compact Hausdorff space which is a continuous image of a Valdivia compact. Then every nonempty open subset of K contains either a G_δ point or a homeomorphic copy of $[0, \omega_1]$. In particular, K belongs to the class $\mathcal{G}\Omega$.*

Proof. There is a Valdivia compact space L and a continuous onto mapping $f : L \to K$. Let $U \subset K$ be a nonempty open set. By regularity of K there is a nonempty open set V such that $\overline{V} \subset U$. Let $H = \overline{f^{-1}(V)}$. Then H is the closure of an open subset of the Valdivia compact L, hence H is Valdivia as well (Lemma 5). It is clear that $f(H) = \overline{V}$, so $\overline{V}$ is a continuous image of a Valdivia compact space. If $\overline{V}$ is Corson, then the set V contains a G_δ point by Lemma 3. If $\overline{V}$ is not Corson, then it contains a homeomorphic copy of $[0, \omega_1]$ by [13, Theorem 1]. This completes the proof. ∎

4. Proofs of the main results

Proof of Theorem 2. (1) $\Rightarrow$ (2) Let K be a Corson compact and L be a continuous image of K. Then L is Corson by [1, Corollary IV.3.15]. In particular, L is Valdivia, so $(B_{C(L)^*}, w^*)$ has a dense convex symmetric Σ-subset by [11, Theorem 3.2].

(2) $\Rightarrow$ (3) This is trivial.

(1) $\Rightarrow$ (4) & (5) This follows e.g. from [20, Corollary 2.2] using the fact that every continuous image of a Corson compact space is again Corson [1, Corollary IV.3.15].

(4) $\Rightarrow$ (3) This is proved in [9, Lemma 3].

(5) $\Rightarrow$ (3) If K has a basis of cardinality $\aleph_1$, then it is well-known and easy to see that $C(K)$ has density $\aleph_1$. So we can apply [7, Lemma 2].

(3) $\Rightarrow$ (1) Let K be a non-Corson compact from the space $\mathcal{G}\Omega$. Let M denote the set of all G_δ points of K. If $\overline{M}$ is not Corson, then we use Proposition 1 to get a continuous image L of K with non-Valdivia $B_{C(L)^*}$.

If $\overline{M}$ is Corson, then $K \setminus \overline{M} \neq \emptyset$. In fact, this set is uncountable as it has no isolated points. Therefore, there are $V_1, \dots, V_4$ nonempty open sets with pairwise disjoint closures such that $\overline{V}_i \cap \overline{M} = \emptyset$ for every i. Put

$$L_0 = K \setminus \bigcup_{i=1}^{4} \partial V_i \cup \{\partial V_i \mid i = 1, \dots, 4\}$$

and equip L_0 with the quotient topology induced by the mapping $q : K \to L_0$ defined by the formula

$$q(k) = \begin{cases} k & k \in K \setminus \bigcup_{i=1}^{4} \partial V_i, \\ \partial V_i & k \in \partial V_i, i = 1, \dots, 4. \end{cases}$$

It is clear that L_0 is a compact Hausdorff space, and it follows from the definition of the class $\mathcal{G}\Omega$ that L_0 contains four pairwise disjoint nowhere dense homeomorphic copies of $[0, \omega_1]$. To finish it is enough to use Corollary to Proposition 2. ∎

Proof of Theorem 1. (1) $\Rightarrow$ (2) & (4) & (5) Let K be a Corson compact with property (M). Then $B_{C(K)^*}$ is Corson by [2, Theorem 3.5]. If Y is a subspace of $C(K)$, then B_{Y^*} is a continuous image of $B_{C(K)^*}$, hence it is Corson by [1, Corollary IV.3.15]. Thus (2) is satisfied. The validity of (4) and (5) follows from [20, Corollary 2.2].

(2) $\Rightarrow$ (3) This is trivial.

(4) $\Rightarrow$ (3) This is proved in [9, Lemma 3].

(5) $\Rightarrow$ (3) If K has a basis of cardinality $\aleph_1$, this follows from [7] similarly as in the proof of Theorem 2.

(3) $\Rightarrow$ (1) Suppose that (3) holds. By Theorem 2 we get that K is Corson. If it had not the property (M), we finish by using Proposition 3. ∎

Proof of Example 1. Let $K = \beta\mathbb{N} \setminus \mathbb{N}$. It follows from [6, p.132, Theorem 3.5.4] that K contains no nontrivial convergent sequence. Hence it is clear that K contains no homeomorphic copy of $[0,\omega_1]$. Further let us remark that K has no G_δ point. Indeed, if g is a G_δ point of K, then it is easy to see that g has a countable neighborhood basis. Then, as g cannot be an isolated point of K, one can easily construct a one-to-one sequence converging to g. This proves that K does not belong to the class $\mathcal{G}\Omega$.

It remains to show that $B_{C(K)^*}$ is not a Valdivia compact. Suppose it is not the case. Then the space $C\left(B_{C(K)^*}, w^*\right)$ of continuous functions on $B_{C(K)^*}$ has an equivalent locally uniformly rotund norm by [19, Corollary]. And hence $C(K)$, as a subspace of this space, has an equivalent locally uniformly rotund norm as well, which contradicts [3]. ∎

5. Final remarks and open problems

We would like to sum up several natural open questions in this area.

Question 1. Let X be a Banach space such that B_{Y^*} is Valdivia for every subspace $Y \subset X$. Is then B_{X^*} Corson?

This is a general question which motivated our investigation in the present paper. We used substantially the structure of $C(K)$ spaces, and it seems not to be clear how to transfer these results to general Banach spaces. For example, it seems to be unknown whether the dual unit ball of every subspace of $\ell_1(\Gamma)$ is Valdivia. But we do not know the answer even for $X = C(K)$ with no additional assumptions on K. So it is natural to ask the following question.

Question 2. Let K be a compact Hausdorff space such that $B_{C(K)^*}$ is Valdivia. Is then K Valdivia as well?

If the answer on Question 2 was positive, then Theorem 2 would follow directly from [10, Theorem 3.1], and the assumption than K is in the class $\mathcal{G}\Omega$ could be dropped. Let us remark that the answer is yes, if K has a dense set of G_δ points (Lemma 2, [11, Theorem 4.10]). Further, the converse holds true by the proof of [16, Corollary 5] (see also [11, Theorem 3.2]).

Another related question concerns the class $\mathcal{G}\Omega$. We could drop the assumption on K from both theorems if the answer to the following question would be positive.

Question 3. Let K be a compact Hausdorff space such that $B_{C(K)^*}$ is Valdivia. Does K belong to the class $\mathcal{G}\Omega$? In particular, does K contain a homeomorphic copy of $[0, \omega_1]$ provided K has at most one G_δ point and $C(K)$ has an equivalent locally uniformly rotund norm?

Acknowledgements

Thanks are due to M. Fabian and V. Zizler for helpful discussions and comments.

References

[1] Archangel'skij, A.V., "Topological Spaces of Functions", Moscow State Univ. (in Russian), 1989, Kluver Acad. Publ. Dordrecht–Boston–London 1992 (English).

[2] Argyros, S., Mercourakis, S., Negrepontis, S., Functional analytic properties of Corson compact spaces, *Studia Math.,* **89** (1988), 197–229.

[3] Bourgain, J., ℓ_∞/c_0 has no equivalent strictly convex norm, *Proc. Amer. Math. Soc.,* **78** (1985), 225–226.

[4] Corson, H.H., Normality in subsets of product spaces, *Amer. J. Math.,* **81** (1959), 785–796.

[5] Deville, R., Godefroy, G., Some applications of projective resolutions of identity, *Proc. London Math. Soc.*, **67** (1) (1993), 183–199.

[6] Engelking, R., "Outline of General Topology", North-Holland, 1968.

[7] Fabian, M., Godefroy, G., Zizler, V., A note on Asplund generated Banach spaces, *Bull. Acad. Polon. Sci.,* **47** (3) (1999), 221–230.

[8] Glicksberg, I., Stone-Čech compactifications of products, *Trans. Amer. Math. Soc.,* **90** (1959), 369–382.

[9] Kalenda, O., An example concerning Valdivia compact spaces, *Serdica Math. J.,* **25** (1999), 131–140.

[10] Kalenda, O., Continuous images and other topological properties of Valdivia compacta, *Fund. Math.,* **162** (1999), 181–192.

[11] Kalenda, O., A characterization of Valdivia compact spaces, *Collectanea Math.,* to appear.

[12] Kalenda, O., Valdivia compacta and equivalent norms, *Studia Math.,* to appear.

[13] Kalenda, O., Embedding of the ordinal segment $[0, \omega_1]$ into continuous images of Valdivia compacta, *Comment. Math. Univ. Carolinae,* to appear.

[14] Koumoullis, G., Baire Category in Spaces of Measures, *Advances in Math.,* **124** (1996), 1–24.

[15] Noble, N., The continuity of functions on Cartesian products, *Trans. Amer. Math. Soc.,* **149** (1970), 187–198.

[16] Orihuela, J., On weakly Lindelöf Banach spaces, in "Progress in Funct. Anal.", eds: K.D. Bierstedt, J. Bonet, J. Horváth, M. Maestre, Elsevier Sci. Publ. B.V., 1992, 279–291.

[17] Shapirovskii, B.E., Special types of embeddings in Tychonoff cubes. Subspaces of Σ-products and cardinal invariants, in "Collection: Topology", Vol II, (Colloq. Math. Soc. Janos Bolyai 23), North-Holland, 1980, 1055–1086.

[18] Valdivia, M., Resolutions of the identity in certain Banach spaces, *Collectanea Math.,* **39** (2) (1988), 127 – 140.
[19] Valdivia, M., Projective resolutions of the identity in $C(K)$ spaces, *Archiv der Math.,* **54** (1990), 493 – 498.
[20] Valdivia, M., Simultaneous resolutions of the identity operator in normed spaces, *Collectanea Math.,* **42** (3) (1991), 265 – 285.
[21] Valdivia, M., On certain compact topological spaces, *Revista Matemática de la Universidad Complutense de Madrid,* **10** (1) (1997), 81 – 84.