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Equations of Lax Type with a Triple Bracket *

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1. Introduction

Equations with several brackets arose originally in the works of Brockett [2], [3] (for ordinary differential equations) and Felipe [5] (for partial differential equations) of double bracket equations of Lax type.

The purpose of this note is to study triple bracket equations of the form

$$\frac{\partial L}{\partial t} = [L,[L,[L,P]]]. \tag{1}$$

We deal with some algebraic properties of these equations, in particular we show that, as in the classical case, they are related to the presence of an infinite sequence of first integrals. Also we exhibit some new integrable systems.

2. Notations and preliminary results

A pseudodifferential operator is a formal series of the form

$$R = \sum_{-\infty}^{n} r_i(x)\partial^i, \qquad n \in \mathbb{N}. \tag{2}$$

As usual, ∂ denotes $\frac{\partial}{\partial x}$. The coefficients $r_i(x)$ are supposed to lie in some differential algebra over $\mathbb{C}$, of smooth functions of x, for example $\mathbb{C}[[x]]$. To

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multiply two such operators, we need to know how to move ∂^{-1} across a function $r(x)$:

$$\partial^{-1} r \cdot = \sum_{j=0}^{\infty} (-1)^j r^{(j)} \partial^{-(j+1)}$$

where $r^{(j)} = \frac{\partial^j r}{\partial x^j}$. It is easy to check that this makes the set of all pseudodifferential operators an associative algebra, which we denote by Psd.

If $R = \sum r_i \partial^i$ is a pseudodifferential operator, we shall write R_+ for the differential operator part, $R_+ = \sum_{i \geq 0} r_i \partial^i$ and $R_- = \sum_{i<0} r_i \partial^i$.

Thus $R = R_+ + R_-$. An element $R \in$ Psd has order n if

$$R = \sum_{-\infty}^{n} r_i \partial^i$$

and $r_n \neq 0$, we denote $\operatorname{ord}(R) = n$. If $\operatorname{ord}(R) = n$ and $r_n = 1$, R is called a monic operator. Let $E^{(n)}$ be the set of all elements in Psd of order at most n. Then we have formally a direct sum decomposition

$$\mathrm{Psd} = D \oplus E^{(-1)}$$

where D is the set of all differential operator ($R = R_+$ if $R \in D$). Psd is a Lie algebra with the bracket defined as usual, $[R_1, R_2] = R_1 R_2 - R_2 R_1$.

We call a first order monic operator in Psd of the form

$$L = \partial + u_{-1} \partial^{-1} + u_{-2} \partial^{-2} + \cdots$$

a Lax operator ($u_{-n} = u_{-n}(x,t)$).

We recall that, if $R = \sum r_i \partial^i \in$ Psd , then $\frac{\partial R}{\partial t} = \sum \frac{\partial r_i}{\partial t} \partial^i$.

Let L be a Lax operator. A triple bracket equation formed from L is an equation of the form

$$\frac{\partial L}{\partial t} = [L, [L, [L, P]]] \tag{3}$$

where $P \in D$. The possible operators P are determined by the requirement that $[L, [L, [L, P]]] \in E^{(-1)}$. The pair (P, L), where $P \in D$ and L is a Lax operator, is called admissible if the triple bracket $[L, [L, [L, P]]]$ belongs to $E^{(-1)}$.

Let L be a Lax operator. Then, as in the classical case, for every $n \in \mathbb{N}$, $((L^n)_+, L)$ is an admissible pair. In fact $((L^n)_+, L)$is a Brockettt pair (see [5]), i.e. $[L, [L, (L^n)_+]]$ is in $E^{(-1)}$. Hence $[L, [L, [L, (L^n)_+]]] \in E^{(-1)}$. This shows that $((L^n)_+, L)$ is an admissible pair.

LEMMA 1. *Let L be a Lax operator and (P, L) an admissible pair such that $[L, P] \in E^{(-1)}$. Then P is a linear combination of $(L^n)'_+s$ with coefficients in $\mathbb{C}$.*

Proof. This lemma follows directly from the lemma 3.1 of [7]. ∎

It should be noted that the condition $[L, P] \in E^{(-1)}$ for $P \in D$ implies that the top order coefficient of P is a constant. On the other hand, if (P, L) is an admissible pair, then the top order coefficient of P is a polynomial of the form $(ax^2 + bx + c)$, where a, b and c are complex numbers.

LEMMA 2. *Let L be a Lax operator, and (P, L) be an admissible pair such that the top order coefficient $(ax^2 + bx + c)$ of P is not constant. Then, if $\operatorname{ord}(P) = n$, P can be written in the following form*

$$P = (ax^2 + bx + c)(L^n)_+ + \sum_{k=0}^{n-1} r_k (L^k)_+ \tag{4}$$

where $r_k \in \mathbb{C}\,[[x]]$ for every k.

Proof. Suppose the lemma is true for all admissible $P \in D$ of order less than n, such that the top order coefficient is not constant. Let us take an admissible pair (P, L), such that, $\operatorname{ord}(P) = n$ and with the same property. Let $(ax^2 + bx + c)$ be the top order coefficient of P. We define

$$L^n_{(a,b,c)} = (ax^2 + bx + c)(L^n)_+ + \sum_{k=0}^{n-1} q_k (L^k)_+$$

where the coefficients q_k are formal power series that are obtained by imposing the condition: $((P - L^n_{(a,b,c)}), L)$ is an admissible pair. To eliminate the coefficient of ∂^{n-1} in $[L, [L, [L, (P - L^n_{(a,b,c)})]]]$ we can take $q_{n-1} = p_{n-1} + (dx^2 + ex + f)$, where p_{n-1} is the coefficient of ∂^{n-1} in P (because $[\partial, [\partial, [\partial, (dx^2 + ex + f)\partial^{n-1}]]] = 0$). Notice that we may assume that $d \neq 0$ or $e \neq 0$. Now, the equation equivalent to the annihilation of ∂^k for $k = 0, 1, \cdots, n-2$ in $[L, [L, [L, (P - L^n_{(a,b,c)})]]]$, only contains $q_k, q_{k+1}, \cdots, q_{n-1}$ and their derivatives, and it is of the form $q_k''' = Q_k(q_{k+1}, \cdots, q_{n-1})$, where Q_k is a differential polynomial in $q_{k+1}, \cdots, q_{n-1}$ with coefficients in $\mathbb{C}[[x]]$. This fact allows us to calculate $q_{n-2}, q_{n-3}, \cdots, q_0$ recursively.

Since, $\operatorname{ord}(P - L^n_{(a,b,c)}) < n$ and the top order coefficient of $P - L^n_{(a,b,c)}$ is not constant, we have

$$P - L^n_{(a,b,c)} = (dx^2 + ex + f)(L^{n-1})_+ + \sum_{k=0}^{n-2} t_k (L^k)_+$$

where $t_k \in \mathbb{C}\,[[x]]$. Therefore P can be represented as a $\mathbb{C}[[x]]$-linear combination of $(L^k)_+$'s. ∎

Let L be a Lax operator, the following system of infinitely many triple bracket equations is called the triple bracket hierarchy

$$\frac{\partial L}{\partial t_n} = [L, [L, [L, (L^n)_+]]] \tag{5}$$

We assume that the coeficients of L in (5) are functions dependent on some additionals variables $t_1, t_2, t_3, \cdots, t_n, \cdots$. We remark that, $L = \partial$ is a trivial solution of (5).

In many cases it is convenient to represent the Lax operator L in a formal dressing form, $L = \phi\partial\phi^{-1}$, where ϕ is determined up to multiplication on the right by a series in ∂^{-1} with constant coefficients starting with 1. In terms of ϕ, the equations of the triple bracket hierarchy are

$$\frac{\partial \phi}{\partial t_n} = [L, [L, (L^n)_-]]\phi \tag{6}$$

In fact, let ϕ be a solution of (6), where $L = \phi\partial\phi^{-1}$. Then L is a Lax operator that satisfies (5). To see this, notice that from (6) we immediately obtain the equation

$$\frac{\partial \phi^{-1}}{\partial t_n} = -\phi^{-1}[L, [L, (L^n)_-]] \tag{7}$$

Now,

$$\frac{\partial L}{\partial t_n} = \frac{\partial \phi}{\partial t_n}\partial\phi^{-1} + \phi\partial\frac{\partial \phi^{-1}}{\partial t_n} \tag{8}$$

If we replace $\frac{\partial \phi}{\partial t_n}$ and $\frac{\partial \phi^{-1}}{\partial t_n}$ in (8) by the right parts of (6) and (7), respectively, we have

$$\begin{aligned}\frac{\partial L}{\partial t_n} &= [L, [L, (L^n)_-]]\phi\partial\phi^{-1} - \phi\partial\phi^{-1}[L, [L, (L^n)_-]] \\ &= -[L, [L, [L, (L^n)_-]]] \\ &= [L, [L, [L, (L^n)_+]]].\end{aligned}$$

3. Invariant polynomial functionals

In this section we will work with pseudodifferential operators (1) whose coefficients $r_i(x)$ are periodic real valued C^∞ functions on $\mathbb{R}$ of period 1. A conservation law is an identity

$$\frac{\partial H}{\partial t} = \frac{\partial J}{\partial x}$$

that follows formally from (3). The conserved density H and flux J are differential polinomials in $u_{-1}, u_{-2}, u_{-3}, \cdots$ and its x-derivatives $u_{-n}^{(k)}$. An invariant polynomial functional for (3) is a functional of the form

$$F(H) = \int_0^1 H\,dx$$

where H is a conserved density. Notice that if F is an invariant polynomial functional, then $\frac{\partial F}{\partial t} = 0$.

For $R \in \mathrm{Psd}$ we define the residue

$$\operatorname{res} R = r_{-1}$$

Next, we shall also use the Adler functional

$$Tr\,R = \int_0^1 \operatorname{res} R\,dx$$

This functional has the following property, $Tr\,[R_1, R_2] = 0$, for every $R_1, R_2 \in \mathrm{Psd}$ (see the proof of theorem 4).

LEMMA 3. *For any $k \geq 2$, by virtue of the equations* (5),

$$\frac{\partial L^k}{\partial t_n} = [L^k, [L, [L, (L^n)_+]]]$$

holds.

Proof. Let $k = 2$, then

$$\begin{aligned}
\frac{\partial L^2}{\partial t_n} &= \frac{\partial L}{\partial t_n} L + L \frac{\partial L}{\partial t_n} \\
&= [L, [L, [L, (L^n)_+]]]L + L[L, [L, [L, (L^n)_+]]] \\
&= (L[L, [L, (L^n)_+]] - [L, [L, (L^n)_+]]L)L \\
&\quad + L(L[L, [L, (L^n)_+]] - [L, [L, (L^n)_+]]L) \\
&= L^2[L, [L, (L^n)_+]] - [L, [L, (L^n)_+]]L^2 \\
&= [L^2, [L, [L, (L^n)_+]]].
\end{aligned}$$

Now, suppose the lemma is true for $k = r$, i.e. $\frac{\partial L^r}{\partial t_n} = [L^r, [L, [L, (L^n)_+]]]$. We have

$$\begin{aligned}
\frac{\partial L^{r+1}}{\partial t_n} &= \frac{\partial L^r}{\partial t_n} L + L^r \frac{\partial L}{\partial t_n} \\
&= [L^r, [L, [L, (L^n)_+]]]L + L^r[L, [L, [L, (L^n)_+]]] \\
&= (L^r[L, [L, (L^n)_+]] - [L, [L, (L^n)_+]]L^r)L \\
&\quad + L^r(L[L, [L, (L^n)_+]] - [L, [L, (L^n)_+]]L) \\
&= L^{r+1}[L, [L, (L^n)_+]] - [L, [L, (L^n)_+]]L^{r+1} \\
&= [L^{r+1}, [L, [L, (L^n)_+]]].
\end{aligned}$$

∎

THEOREM 4. *The polynomial functionals*

$$F_k = Tr\, L^k = \int_0^1 \operatorname{res} L^k \, dx \qquad k = 1, 2, 3, \cdots$$

are invariant polynomial functionals.

Proof. It is well known that $\operatorname{res}[R_1, R_2] = \partial h$, where h is a differential polynomial in the coefficients of R_1 and R_2, here $R_1, R_2 \in$ Psd. Hence

$$\begin{aligned}
\frac{\partial}{\partial t_n}(\operatorname{res} L^k) = \operatorname{res} \frac{\partial L^k}{\partial t_n} &= \operatorname{res} [L^k, [L, [L, (L^n)_+]]] \\
&= \frac{\partial J_k}{\partial x}
\end{aligned}$$

where J_k are differential polynomials in u_{-n}, $n = 1, 2, 3, \cdots$ and it is x-derivates $u_{-n}^{(i)}$, $i, n = 1, 2, 3, \cdots$. ∎

4. ZAKHAROV-SHABAT TYPE EQUATIONS AND SOME NEW INTEGRABLE SYSTEMS

The purpose of this section is to indicate how the system (5) generates new integrable equations. As we will see, they have the form of zero-curvature type equations.

THEOREM 5. *Let $B_s = (L^s)_+$, $s \geq 1$. Equations* (5) *imply*

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} = [L, [L, [B_n, B_m]]]_+. \tag{9}$$

Remark. Every equation given by (9) is equivalent to a closed system of partial differential equations.

Proof. It is easy to see that if $s \geq 1$, then

$$[L^s, [L, [L, (L^n)_+]]] = [L, [L, [L^s, (L^n)_+]]]. \tag{10}$$

From (10), and according to lemma 3

$$\begin{aligned}\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} &= [L, [L, [L^n, (L^m)_+]]]_+ - [L, [L, [L^m, (L^n)_+]]]_+ \\ &= [L, [L, [L^n, B_m] - [L^m, B_n]]]_+ \\ &= [L, [L, [B_n, B_m]]]_+ - [L, [L, [(L^n)_-, (L^m)_-]]]_+ \\ &= [L, [L, [B_n, B_m]]]_+.\end{aligned}$$

∎

Next we consider the case $n = 3$ and $m = 2$. It is well known that

$$B_2 = \partial^2 + 2u_{-1}$$

and

$$B_3 = \partial^3 + 3u_{-1}\partial + 3(u_{-2} + u'_{-1}),$$

where the prime denotes derivative with respect to x. On the other hand

$$[B_3, B_2] = -3(2u'_{-2} + u''_{-1})\partial + 6u_{-1}u'_{-1} - 3u''_{-2} - u'''_{-1}. \tag{11}$$

Then by virtue of (11) we can see that for $n = 3$ and $m = 2$, equation (9) is equivalent to the equations

$$\frac{\partial u_{-1}}{\partial t_2} = -(2u'''_{-2} + u''''_{-1}) \tag{12}$$

and

$$3\frac{\partial}{\partial t_2}(u_{-2} + u'_{-1}) = 2\frac{\partial u_{-1}}{\partial t_3} + 6(u_{-1}u'_{-1})'' - 3u''''_{-2} - u'''''_{-1}. \tag{13}$$

From (12) and (13) we get

$$\frac{\partial^2 u_{-1}}{\partial t_2^2} = -\Big(\frac{4}{3}\frac{\partial u_{-1}}{\partial t_3} + 4(u_{-1}u'_{-1})'' + \frac{1}{3}u''''_{-1}\Big)'''.$$

It is clear that this equation can be written in the form

$$\frac{\partial^2 u_{-1}}{\partial t_2^2} = \Big(\frac{4}{3}\frac{\partial u_{-1}}{\partial t_3} - 4(u_{-1}u'_{-1})'' - \frac{1}{3}u''''_{-1}\Big)'''. \tag{14}$$

The equation (14) might be called the $(KP)^3$-equation.

If u_{-1} does not depend on t_2, then (14) reduces to

$$\frac{4}{3}\frac{\partial u_{-1}}{\partial t_3} = 4(u_{-1}u'_{-1})'' + \frac{1}{3}u''''_{-1} + h(x), \tag{15}$$

where $h(x)$ is a polynomial of degree 2 in x. In the case $h(x) = 0$, this equation might be called the $(KdV)^3$-equation.

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