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Mathematical Foundations of Geometric Quantization

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Contents

1. INTRODUCTION

There are two kinds of theories for describing the dynamical behaviour of a physical system: *Classical* and *Quantum* theories. The quantum description is obtained from the classical one following an appropriate procedure, which is called *quantization* of the system.

In Theoretical Physics, there are different ways to quantize a classical theory; such as, *canonical quantization*, *Feynmann-path integral quantization* *Weyl-Wigner quantization*, *Moyal quantization*, and other methods derived from these ones. In many cases, the first of them is a direct and easy way of quantization.

Canonical quantization is based in the so-called *Dirac's rules for quantization*. It is applied to "simple" systems: finite number of degrees of freedom and "flat" classical phase spaces (an open set of $\mathbb{R}^{2n}$). The lines of the method are mainly the following [1], [22], [24], [29]:

- *Classical description* (starting data). The system is described by the *Hamiltonian* or *canonical formalism*: its classical phase space is locally coordinated by a set of *canonical coordinates* (q^j, p_j), the *position* and *momentum* coordinates. Classical observables are real functions $f(q^j, p_j)$. Eventually, a Lie group G of symmetries acts on the system.
- *Quantum description*. The quantum phase space is a complex Hilbert space $\mathcal{H}$. Quantum observables are self-adjoint operators acting on $\mathcal{H}$,

$\mathcal{O}(\mathcal{H})$ [1]. The symmetries of the system are realized by a group of unitary operators $U_G(\mathcal{H})$.

- *Quantization method.* As a Hilbert space we take the space of square integrable complex functions of the configuration space; that is, functions depending only on the position coordinates, $\psi(q^j)$. The quantum operator associated with $f(q^j, p_j)$ is obtained by replacing p_j by $-i\hbar\frac{\partial}{\partial q^j}$, and hence we have the correspondence $f(q^j, p_j) \mapsto O_f(q^j, -i\hbar\frac{\partial}{\partial q^j})$. In this way, the classical commutation rules between the canonical coordinates are assured to have a quantum counterpart: the commutation rules between the quantum operators of position and momentum (which are related to the "uncertainty principle" of Quantum Mechanics).

Nevertheless, canonical quantization involves several problems. The principal ones are the following:

- As we have said, it can be applied to finite dimensional systems with "flat" classical phase spaces. Some difficulties arise when this is not the case.
- The method exhibits a strong coordinate dependence: it needs the existence of global canonical coordinates and depends on their choice, that is, it is not invariant under canonical transformations. In addition, the result of quantization depends on the order on which q^j and p_j appear in the expression of the classical observables.
- The procedure is easy for simple systems, but serious difficulties arise when we deal with constrained systems or systems with internal degrees of freedom.
- There are several ways to obtain the quantization of a system: the so-called *Schrödinger representation*, *Bargmann-Fock representation*, etc. Canonical quantization does not provide a unified frame for all of them.

In order to solve these and other questions, a new theory, called *geometric quantization*, was developed in the 70's. Its main goal is to set a relation between classical and quantum mechanics from a geometrical point of view, taking as a model the canonical quantization method. In this sense, it is a theory which removes some ambiguities involved in the canonical quantization procedure. Thus, for instance:

[1] The Hilbert space is complex in order to take into account the interference phenomena of wave functions representing the quantum states. The operators are self-adjoint in order to assure their eigenvalues are real.

- It gives a unified frame for the various kinds of representations.
- It generalizes the quantization procedure for classical phase spaces which are not necessarily "flat" (and, even, without being a cotangent bundle).
- Since it is a geometrical theory, it is a coordinate-free quantization procedure.
- It clarifies the analogies between the mathematical structures involved in classical and quantum theories.

On the other hand, a relevant feature of geometric quantization is its close relationship with the theory of irreducible unitary representations of Lie groups [10], [47], [51]. This relation can be understood in the following way: in the geometrical description of a regular system the classical phase space is, in a lot of cases, a symplectic manifold (M, Ω). The classical observables are the real smooth functions $\Omega^0(M)$. Suppose G is a Lie group of symmetries with a strongly symplectic action on (M, Ω). Geometric quantization tries to establish a correspondence between the categories $(M, \Omega^0(M))$ and $(\mathcal{H}, \mathcal{O}(\mathcal{H}))$ and such that the group of symmetries G is realized as a group of unitary operators $U_G(\mathcal{H})$ [32]. The situation is summarized in the following diagram:

$$\begin{array}{ccc} G & \longrightarrow & (M, \Omega) \\ {\scriptstyle \text{irr. unit. rep.}} \big\downarrow & & \big\downarrow {\scriptstyle \text{geom. quant.}} \\ U_G(\mathcal{H}) & \longrightarrow & \mathcal{H} \end{array}$$

Then, the way of constructing irreducible unitary representations of G (by the orbit method) is related to the way of constructing the Hilbert space $\mathcal{H}$ (made of quantum states) from (M, Ω).

The first works on geometric quantization are due to J.M. Souriau [75], B. Kostant [51] and I.E. Segal [70], although many of their ideas were based on previous works by A.A. Kirillov [47], [48]. Nowadays, their results constitute what is known as *prequantization procedure*.

Nevertheless, the so-obtained quantum theory is unsatisfactory from the point of view of the irreducibility of the quantum phase space, so that another basic structure in the geometric quantization programme has to be introduced: the so-called *polarization*, (this concept was due to B. Kostant and J.M. Souriau in the real case, and to L. Auslander and B. Kostant in the complex one). Concerning to this question, the relation between quantizations of the same system arising from different choices of polarizations was also studied: it is performed by means of the so-called *Blattner-Kostant-Sternberg kernels* (see [14], [15], [16], [40]).

Once the polarization condition is imposed, in a lot of cases, other structures have to be added, because the inner product between *polarized quantum states* is not well-defined in general. The key is to introduce the *bundles of densities and half-forms* [90] in order to define the inner product between quantum states. Finally, in many cases the so-called *metaplectic correction* must be done for obtaining the correct energy levels of the quantum theory [14], [40], [52].

Although the geometric quantization programme was initially developed for quantizing regular systems; that is, symplectic manifolds, it was applied soon to *presymplectic manifolds*, as an attempt for giving a geometric framework for the canonical quantization rules which P.A.M. Dirac applied to *singular systems* (see, for instance, [8], [18], [19], [33], [36], [57], [61], [74], [78]). Nevertheless, the method shows important limitations; mainly, the noncommutativity of the procedures of constraining and quantizing. In order to overcome these problems new geometrical structures were introduced, which led to the so-called *BRST quantization* (from *Becchi-Rouet-Stora-Tuytin*) [4], [5], [27], [43], [53], [58], [79], [80].

In addition, geometric quantization has been extended in order to be applied to *Poisson manifolds* [84], [85]. The origin of this question is that, in the most general cases, the phase space of classical dynamical systems are not symplectic manifolds merely, but Poisson manifolds. As a generalization of these ideas, quantization of *Jacobi manifolds* has been also considered recently [21], [56]. The interest of this topic is that, from the mathematical point of view, Jacobi manifolds are the natural generalization of Poisson manifolds (in particular, of symplectic, cosymplectic and Lie-Poisson manifolds); and their physical interest lies on their relation with the *Batalin-Vilkovisky algebras*.

As a final remark, it is interesting to point out that geometric quantization is a theory developed essentially for the quantization of finite dimensional systems. Few things are known about the geometric quantization of field theories, which is a topic under research.

The aim of this paper is to give a mathematical detailed description of geometric quantization. In particular, our study concerns just to the "standard theory" (up to the metaplectic correction); that is, we only consider geometric quantization of symplectic manifolds.

We pay special attention to several questions, namely:

- The analysis of the mathematical aspects related to the structures involved in the geometric quantization theory, such as complex line bundles, hermitian connections, real and complex polarizations, metalinear bundles and bundles of densities and half-forms.

- The justification of all the steps followed in the geometric quantization programme, from the standpoint definition to the structures which are sucesivelly introduced.

Next, we give some indications on the organization of the paper.

In Section 2, we begin with a discussion on the ideas and postulates on which the canonical quantization is based and which justify the steps of the geometric quantization programme. Section 3 contains a careful and detailed exposition of the mathematical concepts needed in the first stage of prequantization. Next we begin the geometric quantization programme, properly said. So, Section 4 is devoted to explain its first steps which lead to the so-called prequantization procedure. Once the problems arising above have been discussed, a new structure for geometric quantization is introduced and justified in Section 5: the concept of *polarization*, its properties as well as their application to quantization. Section 6 is devoted to introduce new mathematical structures: the *metalinear structure* and the *bundle of densities* and *half-forms*. These structures are then used in order to complete the quantization programme. The final section is devoted to discuss some problems concerning to the geometric quantization of constrained systems.

All manifolds are assumed to be finite dimensional, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood. As far as possible, we follow the notation of references [1] and [2]. In particular, if (M, Ω) is a symplectic manifold ($\dim M = 2n$), the Hamiltonian vector fields are defined as $i(X_f)\Omega = df$. Then, we have

$$\{f, g\} = \Omega(X_f, X_g) = -\,i(X_f)\,i(X_g)\Omega = -X_f(g) = X_g(f)$$

and $X_{\{f,g\}} = [X_g, X_f]$. In a chart of canonical coordinates the expression of the symplectic form is $\Omega = dq^j \wedge dp_j$, and

$$\{f, g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}.$$

2. Preliminary statements

Since our goal is to construct a geometrical theory of quantization based on the canonical quantization programme, we will take as the standpoint model the geometrical framework which describes the *canonical formalism* of the classical physical systems [1], [6]. Then, a set of postulates is stated in order to construct the corresponding quantum description.

There are several ways of choosing a set of axioms or postulates for Quantum Mechanics (see, for example, [22], [24], [29], [46], [59], [62], [69], [87]). In general, these postulates can be arranged into three groups: those which we can call the *"kinematical" postulates*, the *"dynamical" postulate* and the *"statistical" postulates*. Nevertheless, in this paper we are only interested in the "kinematical" and, eventually, in the "dynamical" aspects of quantum theory and, therefore, we omit (if possible) any reference to "statistical" considerations.

Thus, this section is devoted to state and comment those postulates of Quantum Mechanics in which the standpoint definition of the geometric quantization programme is based. Many details of this presentation can be found also in [49] and [81].

2.1. The postulates of Quantum Mechanics

2.1.1. On the space of states.

In the canonical formalism of Classical Mechanics, the phase space of a physical system is assumed to be a manifold $\mathcal{M}$ which is endowed with a symplectic structure Ω or, more generically, with a Poisson structure $\{\ ,\ \}$. Every point of this manifold represents a *(classical) physical state* of the system. Since, as we will see in Section 4, the geometric quantization program deals with symplectic manifolds, if $\mathcal{M}$ is Poissonian, remembering that a Poisson manifold is the union of symplectic manifolds, (its *symplectic leaves* [55]), with this programme we quantize every symplectic leaf, which is only a partial representation of the phase space [2]. This is an essential fact because, given a symplectic manifold (M, Ω), there is a natural way to define a Hilbert space associated with M and a set of self-adjoint operators acting on it and satisfying the suitable conditions which we will discuss afterwards.

In contrast to this fact, in Quantum Mechanics the framework for the description of a physical system is a separable complex Hilbert space. In general, there are different ways of working, mainly: the *Hilbert space formulation* and the *projective Hilbert space formulation* [3]. Next we are going to compare both

[2] See the comments in the Introduction about geometric quantization of Poisson manifolds.

[3] There is also a third possibility: the so-called *unit sphere formulation*. Nevertheless we

of them.

1. *The Hilbert space formulation:*

   The initial framework is a Hilbert space $\mathcal{H}$. At first, it seems reasonable to identify each element $|\psi\rangle \in \mathcal{H}$ as a quantum state, but as it is well known, the dynamical equations in Quantum Mechanics are linear in the sense that the set of solutions is a linear subspace. Hence we cannot identify each element $|\psi\rangle \in \mathcal{H}$ as a quantum state since, for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, there is no way to choose between the solutions $|\psi\rangle$ and $\lambda|\psi\rangle$, so they must represent the same quantum state [4]. Then, the true quantum states are really rays in that Hilbert space. Hence, in this formulation there is a redundancy which has to be taken into account when we define the quantum states.

2. *The projective Hilbert space formulation*:

   If we want to eliminate this redundancy in the definition of quantum states, the only way is passing to the *projective Hilbert space* $P\mathcal{H}$ which is made of the complex lines in $\mathcal{H}$:

   $$|\psi\rangle_{\mathbb{C}} := \{\lambda|\psi\rangle : \ \lambda \in \mathbb{C}, \ |\psi\rangle \in \mathcal{H}\}.$$

   We have the following projection

   $$\begin{array}{ccc} \mathcal{H} - \{0\} & \xrightarrow{\pi} & P\mathcal{H} \\ |\psi\rangle & \mapsto & |\psi\rangle_{\mathbb{C}} \end{array}$$

   Hence, in this picture, a quantum state is given by an unique element of $P\mathcal{H}$. It must be remarked that, like in the classical situation, $P\mathcal{H}$ is a differentiable manifold but, in contrast, it is infinite-dimensional.

We can summarize this discussion in the following postulate:

POSTULATE 1. In the framework of Quantum Mechanics, a physical system is described by a *separable (complex) Hilbert space* $\mathcal{H}$. Every state of this system at time t is represented by a *ray* $|\psi(t)\rangle_{\mathbb{C}}$ belonging to the Hilbert space. Any element $|\psi(t)\rangle$ (different from zero) of this ray is called a vector state.

do not consider it here, since it is not relevant for our presentation of geometric quantization. You can see a detailed exposition of it, as well as its use in an alternative presentation of geometric quantization, in [76] and [81].

[4] When $|\lambda| = 1$, λ will be called a *phase factor*.

Remark. There is a one-to-one correspondence between the states in this postulate and the so-called *pure states* in several axiomatic formulations of Quantum Mechanics [59], [62], [69], [87], [89]. This correspondence is established in the following way: the pure states are the projection operators over the one-dimensional subspaces of $\mathcal{H}$, meanwhile the so-called *mixed states* are convex combinations of projection operators (which are not necessarily projection operators).

2.1.2. ON THE OBSERVABLES. In the classical picture, a *physical observable* (that is, a measurable quantity) is a real smooth function $f \in \Omega^0(M)$ and the result of a measure of a classical observable is the value taken by the representative function on a point (state) of the classical phase space. In contrast:

POSTULATE 2. In the framework of Quantum Mechanics, every observable of a physical system is represented by a *self-adjoint linear operator* which acts on its associated Hilbert space [5].

The result of a measure of a quantum observable is an eigenvalue of the corresponding operator.

2.1.3. ON THE DYNAMICS. The above two postulates establish the "kinematical" framework for the description of Quantum Mechanics. The following step lies in stating the dynamical equations. In the canonical formulation of the classical theory, the usual way is to give a function $H \in \Omega^0(M)$ containing the dynamical information of the system (the *Hamiltonian function*) and hence to take the *Hamilton equations* as the equations of motion. Then, the dynamical evolution of an observable f is given by

$$\frac{df}{dt} = X_H(f) = \{f, H\}. \tag{1}$$

In Quantum Mechanics we have two possible options:

POSTULATE 3. In the framework of Quantum Mechanics, the dynamics of the system is defined by a quantum observable O_H called the *Hamiltonian operator* of the system. Then:

[5] Technical difficulties concerning the domains of unbounded self-adjoint operators can be ignored hereafter since they are not important for the purposes of geometric quantization.

1. *Heisenberg picture*: The dynamical evolution of the system is carried out by the quantum observables and, in the interval of time between two consecutive measures, the evolution of every observable $O_f(t)$ is given by the *Heisenberg equation* [6]

$$i\hbar\frac{d}{dt}O_f(t) = [O_f(t), O_H(t)].$$

In this picture the states of the system are constant in time.

2. *Schrödinger picture*: The dynamical evolution of the system is carried out by the states and, in the interval of time between two consecutive measures, on each ray $|\psi(t)\rangle_{\mathbb{C}}$, there is some representative vector state $|\psi(t)\rangle$ such that the evolution of the system is given by the *Schrödinger equation*

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = O_H|\psi(t)\rangle.$$

In this picture the observables of the system are represented by operators which are constant in time.

Really, geometric quantization concerns only to the "kinematical" aspects of the quantum theory. Nevertheless, several attempts have been made trying to set up the dynamical postulate in geometrical terms, although we do not treat this subject on our exposition (see, for instance [9], [44], [45], [76] for more information on this topic).

There are other postulates which are related with the *probability interpretation* of Quantum Mechanics. As we have said earlier, we do not consider them in this study (for further information, see the references given at the beginning).

2.2. THE GEOMETRIC QUANTIZATION PROGRAMME. Given a symplectic manifold (M, Ω) (the phase space of a classical system), the aim of the quantization programme is to construct a Hilbert space $\mathcal{H}$ (the space of states of a quantum system), and associate a self-adjoint operator O_f to every smooth function f in a Poisson subalgebra of $\Omega^0(M)$ [7]. In addition, as the set of self-adjoint operators $\mathcal{O}(\mathcal{H})$ is also a Lie algebra with the bracket operation, it

[6] Observe the analogy with the classical equation (1).

[7] The apparent modesty of this purpose is due to the fact that it is impossible to represent the full Poisson algebra $\Omega^0(M)$ in the conditions that we will specify soon [86].

seems reasonable to demand this correspondence between classical and quantum observables to be a Lie algebra homomorphism.

Next, we are going to justify another condition to be satisfied by this representation.

2.2.1. On the irreducibility of the space of states. First of all we need to set the concept of irreducibility, both in the classical and the quantum picture.

Definition 1. Let (M, Ω) be a symplectic manifold. A set of smooth functions $\{f_j\} \subset \Omega^0(M)$ is said to be a *complete set of classical observables* if every other function $g \in \Omega^0(M)$ such that $\{f_j, g\} = 0$, for all f_j, is constant.

Observe that this imply that the functions $\{f_j\}$ separate points in M. Moreover, for every $m \in M$, there exists an open set U and a subset $\{f_i'\}$ of $\{f_j\}$ which is a local system of coordinates on U. Then, let $X_{f_i'}$ be the Hamiltonian vector fields associated with these functions. This set is a local basis for the vector fields in M (notice that, in general, the subset $\{f_i'\}$ is not a global system of coordinates on M, since this would imply that M is parallelizable. As a consequence, this notion is used only locally by means of canonical systems of coordinates). Therefore, if $\{\varphi_{it}\}$ are local one-parameter groups of $\{X_{f_i'}\}$ defined in an open set $V \subset M$, and S is a submanifold of M with $\dim S < \dim M$, then $S \cap V$ is not invariant by the action of $\{\varphi_{it}\}$, that is, M is irreducible under the action of this set of local groups of diffeomorphisms.

The quantum analogy of this concept can be established as follows:

Definition 2. Let $\mathcal{H}$ be a Hilbert space. A set of self-adjoint operators $\{O_j\}$ (acting in $\mathcal{H}$) is said to be a *complete set of operators* if every other operator O which commutes with all of them is a multiple of the identity.

If $\mathcal{H}$ is considered as the quantum representation of a physical system, then this set is called a *complete set of quantum observables.*

Notice that the operators $\{O_j\}$ and O involved in this statement do not come necessarily from any set of classical observables.

As above, this concept can be related to the irreducibility of $\mathcal{H}$ under the action of this set as follows:

Proposition 1. *If a set of self-adjoint operators $\{O_j\}$ on $\mathcal{H}$ is a complete set of operators then $\mathcal{H}$ is irreducible under the action of $\{O_j\}$ (that is, every* *closed subspace of $\mathcal{H}$ which is invariant under the action of this set is either equal to $\{0\}$ or $\mathcal{H}$).*

Proof. Let $\{O_j\}$ be a complete set and $F \subset \mathcal{H}$ a closed subspace invariant under the action of this set. Let Π_F be the projection operator over F. Then Π_F commutes with all the elements of the complete set. In fact, if O is a self-adjoint operator on $\mathcal{H}$ which leaves F invariant (and therefore $F^\perp$ is also invariant), and $\psi \in \mathcal{H}$, $\psi = \psi_1 + \psi_2$, with $\psi_1 \in F$ and $\psi_2 \in F^\perp$, we have:

$$\begin{aligned}[O, \Pi_F]\psi &= O(\Pi_F(\psi_1)) + O(\Pi_F(\psi_2)) - \Pi_F(O(\psi_1)) - \Pi_F(O(\psi_2)) \\ &= O(\psi_1) - O(\psi_1) = 0,\end{aligned}$$

since $\Pi_F(\psi_1) = \psi_1$ and $\Pi_F(\psi_2) = 0$. Then $\Pi_F = \lambda Id_{\mathcal{H}}$, so $F = \{0\}$ if $\lambda = 0$, or $F = \mathcal{H}$ if $\lambda \neq 0$. ∎

It is interesting to point out that if the operators O_j and O are continuous, then the converse also holds (see, for instance, [23] and [54]).

Taking into account the above discussion, we will demand:

IRREDUCIBILITY POSTULATE (FIRST VERSION). If $\{f_j\}$ is a complete set of classical observables of a physical system then, in the framework of Quantum Mechanics, their associated quantum operators make up a complete set of quantum observables (which implies that the Hilbert space $\mathcal{H}$ is irreducible under the action of the set $\{O_{f_j}\}$).

Remark. Let $M = \mathbb{R}^{2n}$ be the classical phase space, and (q^j, p_j) the position and momentum canonical coordinates. Then the uniparametric groups associated with the Hamiltonian fields X_{q^j} and X_{p_j}, are the groups of translations in position and momentum which act irreducibly in the classical phase space, that is, there are no proper submanifolds of M invariant by these actions. In the old canonical quantization scheme, $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$ (the space of q^j-dependent square integrable functions), and the self-adjoint operators corresponding to the complete set (q^j, p_j) are

$$O_{q^j} = q^j, \quad O_{p_j} = -i\hbar \frac{\partial}{\partial q^j}.$$

This set of operators is a complete set of quantum observables, and this is the translation to the quantum picture of the irreducibility of the phase space. The above operators satisfy the following commutation rules:

$$[O_{q^k}, O_{q^j}] = 0, \quad [O_{p_k}, O_{p_j}] = 0, \quad [O_{q^k}, O_{p_j}] = i\hbar\delta_{kj} Id$$ [8].

[8]This last equality is related to the known *uncertainty principle* of Quantum Mechanics.

2.2.2. CLASSICAL AND QUANTUM SYMMETRIES. A relevant concept in Physics is the notion of *symmetry* of a system. Next we are going to discuss this subject, both from the classical and the quantum point of view. As we will see at the end, we can relate this discussion to the irreducibility postulate.

Let (M,Ω) be a symplectic manifold. A *symmetry* of the system described by (M,Ω) is an element g of a Lie group G which acts symplectically on (M,Ω). So, every symmetry is represented by a symplectomorphism $\phi_g\colon M\to M$ (that is, such that $\phi_g^*\Omega=\Omega$). Let $Sp(M,\Omega)$ be the group of these symplectomorphisms. A group of symmetries of (M,Ω) is then represented by a subgroup of $Sp(M,\Omega)$.

Remember that, if $f\in\Omega^0(M)$, then every local uniparametric group $\{\varphi_t\}$ associated with the Hamiltonian vector field X_f is a group of local symmetries of (M,Ω).

The quantum counterpart of this concept is the following:

DEFINITION 3. (Wigner) A *symmetry* of the quantum description of a physical system is a map $P\mathcal{H}\to P\mathcal{H}$ such that:

i) It is bijective.

ii) It preserves the map

$$\begin{array}{ccc} P\mathcal{H}\times P\mathcal{H} & \longrightarrow & \mathbb{R}^+\cup\{0\} \\ (\pi|\psi\rangle,\pi|\psi'\rangle) & \mapsto & \frac{|\langle\psi|\psi'\rangle|^2}{\|\psi\|^2\|\psi'\|^2} \end{array}$$

which means that the"transition probabilities" are conserved.

Remarks:

- Incidentally, we can point out that this definition means that a symmetry of the quantum description is an isometry in $P\mathcal{H}$ for the *Fubini-Studdi metric* [91].
- The space $P\mathcal{H}$ is naturally endowed with a strongly symplectic form. It can be proved that a bijection $g\colon P\mathcal{H}\to P\mathcal{H}$ conserving this symplectic structure and the natural complex structure of $P\mathcal{H}$ is a quantum symmetry (if $\mathcal{H}$ is finite-dimensional, then the converse is also true: every quantum symmetry preserves the natural symplectic form and the complex structure of $P\mathcal{H}$) (see [82] and the references quoted therein).

These comments reveal the analogy between the classical and quantum concepts of symmetry.

A way of realizing quantum symmetries is to projectivize unitary or antiunitary operators on $\mathcal{H}$. The following proposition proves that this is actually the only possible way.

PROPOSITION 2. (Wigner) *Let g be a symmetry of the quantum description of a physical system. Then:*

i) *There exists either a unitary or alternativelly an antiunitary operator U_g on $\mathcal{H}$ such that U_g induces g, that is, for every $|\psi\rangle \in \mathcal{H} - \{0\}$, we have $\pi(U_g|\psi\rangle) = g(\pi|\psi\rangle)$.*
ii) *If U_g and U'_g are unitary or antiunitary operators on $\mathcal{H}$ which induce g, then U_g and U'_g differ on a phase factor.*

COROLLARY 1. *If g_1, g_2 are quantum symmetries and U_{g_1}, U_{g_2} are unitary operators on $\mathcal{H}$ inducing those symmetries, then $U_{g_1 g_2} = \alpha(g_1, g_2) U_{g_1} U_{g_2}$; where $\alpha(g_1, g_2) \in U(1)$.*

Let G be a connected group of quantum symmetries. As a consequence of the previous results, if $U(\mathcal{H})$ denotes the set of unitary operators on $\mathcal{H}$, there exists a subgroup $G' \subset U(\mathcal{H})$ such that the following sequence is exact

$$1 \to U(1) \to G' \to G \to 1,$$

that is, G' is a *central extension* of G by $U(1)$.

In other words, if $U(\mathcal{H})$ induces the quantum symmetries and $PU(\mathcal{H}) := U(\mathcal{H})/U(1)$ is the group of projectivized unitary operators on $\mathcal{H}$, then, it is isomorphic to the group of quantum symmetries, and every subgroup of quantum symmetries is isomorphic to a quotient of a subgroup of $U(\mathcal{H})$ by $U(1)$.

Summarizing, the situation is the following: Let G be a Lie group, then it is a group of symmetries of the physical system, both for the classical and the quantum descriptions, if we have the following representations of G:

- As classical symmetries, by a representation as symplectomorphisms acting on (M, Ω) (the phase space of the system in the classical picture).
- As quantum symmetries, by a representation as unitary transformations acting on $\mathcal{H}$ (the space supporting the quantum states of the system in the quantum picture).

Then, we can state the following version of the irreducibility postulate:

IRREDUCIBILITY POSTULATE (SECOND VERSION). Suppose G is a group of symmetries of a physical system both for the classical and the quantum descriptions. If G acts transitively on (M, Ω) (by means of the corresponding group of symplectomorphisms), then the Hilbert space $\mathcal{H}$ is an irreducible representation space for a $U(1)$-central extension of the corresponding group of unitary transformations.

Suppose G is a Lie group which acts on (M, Ω) and the action is *strongly symplectic*, that is the fundamental vector fields associated to the Lie algebra of G by this action are global Hamiltonian vector fields $\{X_{f_i}\} \subset \mathcal{X}_h(M)$. In this case, the connection between the first and second version of the irreducibility postulate can be established as follows: if the action is transitive, then the Hamiltonian functions $\{f_i\}$ make up a complete set of classical observables for (M, Ω) [9]. Conversely, if $\{f_i\}$ is a complete set of classical observables, then these functions can be thought as the generators of a group G of infinitesimal symplectomorphisms whose action on (M, Ω) is strongly symplectic and transitive [49].

Now, a remaining question is the following: *can every classical symmetry of a physical system be translated into a quantum symmetry?* This question can be reformulated and generalized in a more precise way. In fact, since every classical symmetry must be a symplectomorphism of the classical phase space (M, Ω), the maximal set of classical symmetries is the group of all symplectomorphisms $Sp(M, \Omega)$. In an analogous way, since every quantum symmetry must be a projective unitary transformation of $P\mathcal{H}$, the maximal set of them is the group of projective unitary transformations $PU(\mathcal{H})$. Then, the question can be generalized in the following terms: *if (M, Ω) represents the classical phase space of a physical system and $\mathcal{H}$ is the Hilbert space for the quantum description, are the groups $Sp(M, \Omega)$ and $PU(\mathcal{H})$ isomorphic?*

As we will remark at the end of the following section, in general the answer is negative [1], [32], [39], [86]: on the one hand, there is no way to associate an element of $PU(\mathcal{H})$ to every element of $Sp(M, \Omega)$. Physically this means that, in some cases, if G is a group of symmetries of a classical system, in the quantization procedure some symmetries are preserved but other ones are broken. These situations are called *quantum anomalies* in the physical

[9]If the action is transitive and symplectic but not strongly symplectic (or Hamiltonian), then the fundamental vector fields are locally Hamiltonian and the corresponding locally Hamiltonian functions make up a local complete set of classical observables.

literature (see, for instance, [31] for a more detailed explanation on this topic). On the other hand, neither every quantum symmetry comes necessarily from a classical one. In physical terms this is related to the fact that it may have unitary operators which have not classical counterpart.

Finally, a relevant fact is that, as it is clear from the discussion made in this section, geometric quantization of a classical system is closely related to the study of irreducible representations of a Lie group [10]. Pioneering works on these topics are [47] (for nilpotent groups), [10], [13] (for solvable Lie groups) and [14], [16], [68] (for semisimple groups). (For a summarized guideline of some of these methods see, for instance, [49] and [90]).

2.3. THE STANDPOINT DEFINITION OF THE GEOMETRIC QUANTIZATION PROGRAMME. Taking into account the previous postulates and the discussion made in the above paragraphs, we can say that the objective of the geometric quantization programme is to try of finding a correspondence between the set of pairs *(Symplectic manifolds* (M,Ω), *smooth real functions* $C^\infty(M)$*)* and *(Complex Hilbert spaces* $\mathcal{H}$, *self-adjoint operators* $\mathcal{O}(\mathcal{H})$*)*; or, in a more general way, a functor between the categories *(Symplectic manifolds* (M,Ω), *symplectomorphisms* $Sp(M,\Omega)$*)* and *(Complex Hilbert spaces* $\mathcal{H}$, *unitary operators* $U(\mathcal{H})$*)*. This functorial relation must satisfy certain properties.

Hence, we can establish the following standpoint definition:

DEFINITION 4. A *full quantization* of the classical system (M,Ω) is a pair $(\mathcal{H}_Q, O)$ where:

a) $\mathcal{H}_Q$ is a separable complex Hilbert space. The elements $|\psi\rangle \in \mathcal{H}_Q$ are the *quantum wave functions* and the elements $|\psi\rangle_{\mathbb{C}} \in P\mathcal{H}_Q$ are the *quantum states* of the system. $\mathcal{H}_Q$ is called the *intrinsic Hilbert space* and $P\mathcal{H}_Q$ is the *space of quantum states* of the system.

b) O is a one to one map, taking classical observables (i.e., real functions $f \in \Omega^0(M)$) to self adjoint operators O_f on $\mathcal{H}_Q$, such that

   i) $O_{f+g} = O_f + O_g$.
   ii) $O_{\lambda f} = \lambda O_f, \ \forall \lambda \in \mathbb{C}$.
   iii) $O_1 = Id_{\mathcal{H}_Q}$.
   iv) $[O_f, O_g] = i\hbar O_{\{f,g\}}$.

[10] And, really, first developments of geometric quantization arose from works on the second problem.

v) If $\{f_j\}$ is a complete set of classical observables of (M, Ω), then $\mathcal{H}_Q$ has to be irreducible under the action of the set $\{O_{f_j}\}$. Alternatively, suppose G is a group of symmetries of a physical system both for the classical and the quantum descriptions. If G acts transitively on (M, Ω), then $\mathcal{H}_Q$ provides an irreducible representation space for a $U(1)$-central extension of the corresponding group of unitary transformations.

The set of these operators is denoted $\mathcal{O}(\mathcal{H}_Q)$ and its elements are called *quantum observables* or *quantum operators.*

The justification of this definition lies in the discussion held in the previous sections. Thus, part (a) of definition arises as a consequence of Postulate 1. On the other hand, part (b) is the translation of Postulate 2 and, in relation to conditions listed there, we point out that:

- Conditions (i) and (ii) establish the *linearity* of the map O which, although it has not, in general, a physical interpretation, it is a desirable property from the mathematical point of view.
- Condition (iii) gives account of the fact that, if the result of a measurement has to be equal to 1 in every state of the classical description of the system, then we want the same result in the quantum description; that is the only expected value has to be 1, so the corresponding operator must be the identity.
- Condition (iv) imposes that, moreover, the map O is a Lie algebra morphism (up to a factor).
- Finally, condition (v) is the irreducibility Postulate.

Hence, the quantization programme consists of constructing a Hilbert space $\mathcal{H}_Q$ on which the Lie algebra of classical observables could be represented irreducibly by self-adjoint operators on $\mathcal{H}_Q$ satisfying conditions in part (b) of Definition 4.

It is important to point out that, as it is proved in [1], [32], [39] and [86], it is not possible to find a full quantization for every classical system, even in the case of $M = \mathbb{R}^{2n}$. (In this last case, it is not possible to quantize all the classical observables of the system. Then, the usual way is to quantize only a subset of all the classical observables which is called a *Hilbert subalgebra.* We will treat this feature afterwards).

3. Hermitian line bundles

Before starting the explanation of the geometric quantization programme, several geometric tools (which are basic in this task) have to be known. This section deals with the study and development of all these concepts. General references are [17], [30], [51] and [92].

3.1. Complex line bundles

Definition 5. Let $\pi : L \to M$ be a projection of manifolds. (L, π, M) is said to be a *complex line bundle* if:

i) For every $m \in M$, the fiber $L_m = \pi^{-1}(m)$ is a one dimensional complex vector space.
ii) There exists an open covering $\{U_l\}$ of M and sections $s_l : U_l \to L$ such that the maps
$$\begin{array}{rccc} \eta_l : & \mathbb{C} \times U_l & \longrightarrow & \pi^{-1}(U_l) \\ & (z, m) & \mapsto & z s_l(m) \end{array}$$
are diffeomorphisms.

Observe that the family $\{(U_l, \eta_l)\}$ is a bundle trivialization and that $s_l(m) \neq 0$ for all $m \in U_l$.

Taking into account that

$$\mathbb{C} \times U_{lj} \xrightarrow{\eta_j} \pi^{-1}(U_{lj}) \xrightarrow{\eta_l^{-1}} \mathbb{C} \times U_{lj},$$

(where $U_{lj} = U_l \cap U_j$), we have that the *transition functions* Ψ_{lj} are given by $\Psi_{lj} = \eta_l^{-1} \circ \eta_j$, that is,

$$\Psi_{lj}(z, m) = \eta_l^{-1}(\eta_j(z, m)) = \eta_l^{-1}(z s_j(m)) = \left(\frac{z s_j(m)}{s_l(m)}, m \right).$$

Observe that $\frac{s_j(m)}{s_l(m)}$ is well defined. We can also write $\Psi_{lj}(z, m) = (z c_{lj}(m), m)$ where

$$\begin{array}{rccl} c_{lj} : & U_{lj} & \hookrightarrow & \mathbb{C}^* \\ & m & \mapsto & \frac{s_j(m)}{s_l(m)} = (\pi_1 \circ \Psi_{lj})(1, m) \end{array}$$

satisfying that $c_{lj} c_{jk} = c_{lk}$, in U_{ljk}, $c_{ii} = 1$, and $c_{lj} = c_{ji}^{-1}$. These functions c_{lj} are also called "transition functions" and the above conditions *cocycle conditions*.

DEFINITION 6. Two complex line bundles (L, π, M) and (L', π', M) are said to be *equivalent* if there exists a fiber diffeomorphism $\phi : L \to L'$ such that the restrictions to the fibers $\phi_m : L_m \to L'_m$ are $\mathbb{C}$-linear and the induced map on M is the identity.

In this case, we can construct trivializations with the same transition functions, since they can be carried from one to the other by means of the diffeomorphism. We denote by $L(M)$ the set of equivalence classes of complex line bundles.

In the set $L(M)$ we consider the tensor product (over $\mathbb{C}$, that is, $(L \otimes L')_x := L_x \otimes_{\mathbb{C}} L'_x$); then we have a group structure in which the unit is the trivial bundle and the inverse L^{-1} is the dual L^*. The transition functions of $L \otimes L'$ are the product of those of L and L'. This is called the *Picard's group* of M.

Let $\sigma : M \to L$ be a section. We are going to analyze the relation between the restrictions of σ to two open sets of a trivialization. Let $\{U_l, \eta_l\}$ be a trivialization with transition functions c_{lj} and let σ_l be the restriction of σ to the open set U_l. Considering the following diagram

$$\begin{array}{ccc} \mathbb{C} \times U_l & \xrightarrow{\eta_l} & \pi^{-1}(U_l) \\ & & \uparrow{\scriptstyle \sigma_l} \\ & & U_l \end{array}$$

the section σ defines local functions (with values in $\mathbb{C}$) as $f_l := \pi_1 \circ \eta_l^{-1} \circ \sigma_l$; and in the same way for U_j. Now, in U_{lj} we have:

$$\begin{aligned} f_l(m) &= (\pi_1 \circ \eta_l^{-1})(\sigma(m)) = \pi_1\Big((\eta_l^{-1} \circ \eta_j \circ \eta_j^{-1})(\sigma(m))\Big) \\ &= \pi_1\Big(\Psi_{lj}\Big(\eta_j^{-1}(\sigma(m))\Big)\Big) = \pi_1\Big(\Psi_{lj}(f_j(m), m)\Big) \\ &= \pi_1(c_{lj}(m) f_j(m), m) = c_{lj}(m) f_j(m), \end{aligned}$$

that is, $f_l = c_{lj} f_j$. Therefore, a section of L induces, on each open set U_l, a function (with values on $\mathbb{C}$) and the relation between the functions defined in two intersecting trivializing open sets is obtained taking the product by the transition functions.

3.2. CHERN CLASSES. Let (L, π, M) be a complex line bundle and $\{U_l, s_l\}$ a trivialization with transition functions c_{lj}. Let $\mathcal{F}$ be the sheaf of germs of complex smooth functions in the manifold M, and $\mathcal{F}^*$ the set of the nowhere-vanishing ones, considered as a sheaf of groups with the product. The transition functions define a Čech 1-cocycle in M, $c : U_{lj} \mapsto c_{lj}$, with values in the sheaf $\mathcal{F}^*$, and then it determines an element in the cohomology group $\check{H}^1(M, \mathcal{F}^*)$.

PROPOSITION 3. *The above Čech cohomology class does not depend on the trivialization used but only on the class of L in $L(M)$. Moreover, the assignment $L(M) \longrightarrow \check{H}^1(M, \mathcal{F}^*)$ is a group isomorphism.*

Proof. The independence of the trivialization is a consequence of the fact that the union of two trivializations is a trivialization whose associated covering is a refinement of those of the initial trivializations.

The assignment is a group morphism because the transition functions of the tensor product are the ordinary product of the transition functions of the factors. ∎

Consider now the exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{\epsilon} \mathcal{F} \xrightarrow{e} \mathcal{F}^* \to 0,$$

where ϵ is the natural injection and $e(f) := e^{2\pi i f}$. The corresponding cohomology sequence is

$$\check{H}^1(M, \mathbb{Z}) \to \check{H}^1(M, \mathcal{F}) \to \check{H}^1(M, \mathcal{F}^*) \to \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathcal{F}),$$

but $\mathcal{F}$ is a fine sheaf (with partitions of the unity), therefore the cohomology groups with values in $\mathcal{F}$ and degree greater than zero are null and then $\check{H}^1(M, \mathcal{F}^*) \cong \check{H}^2(M, \mathbb{Z})$.

THEOREM 1. *The group $L(M)$ is canonically isomorphic to $\check{H}^2(M, \mathbb{Z})$.*

Proof. It is a consequence of the last proposition and the above arguments. ∎

DEFINITION 7. If $[L]$ is an element of $L(M)$, then the element $K([L])$ of $\check{H}^2(M, \mathbb{Z})$ corresponding to $[L]$, will be called the *first Chern class* of L.

3.3. CONNECTIONS AND CURVATURE. Let $\Gamma(L)$ be the $\mathcal{F}$-module of sections of (L, π, M) and $\mathcal{X}^{\mathbb{C}}(M)$ the module of complex vector fields on M $(\mathcal{X}^{\mathbb{C}}(M) = \mathcal{X}(M) \otimes \mathbb{C})$.

DEFINITION 8. A *connection* in the complex line bundle (L, π, M) is a $\mathbb{C}$-linear map

$$\begin{array}{rccc} \nabla : & \mathcal{X}^{\mathbb{C}}(M) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(\Gamma(L), \Gamma(L)) = \Gamma(L) \otimes_{\mathbb{C}} \Gamma(L)^* \\ & X & \mapsto & \nabla_X \end{array}$$

satisfying that

1) $\nabla_{fX} = f\nabla_X$,
2) $\nabla_X(fs) = (Xf)s + f\nabla_X s$,

that is, an element of $(\mathcal{F}^1 \otimes_{\mathcal{F}^0} \Gamma(L)) \otimes_{\mathbb{C}} \Gamma(L)^*$ satisfying condition (2), where $\mathcal{F}^p$ denotes the module of complex p-forms on M.

Equivalently, a connection is a map

$$\begin{array}{rccc} \nabla : & \Gamma(L) & \longrightarrow & \mathcal{F}^1 \otimes_{\mathcal{F}^0} \Gamma(L) \\ & s & \mapsto & \nabla s \end{array}$$

such that

i) It is $\mathbb{C}$-linear,
ii) $\nabla(fs) = df \otimes s + f\nabla s$.

(That is, $\nabla \in (\mathcal{F}^1 \otimes_{\mathcal{F}^0} \Gamma(L)) \otimes_{\mathbb{C}} \Gamma(L)^*$ and, moreover, it satisfies condition (ii), from which, taking $\nabla_X s := i(X)\nabla s$, the equivalence between both definitions is immediate).

$\nabla_X s$ is called the *covariant derivative* of the section s with respect to X and the connection ∇.

PROPOSITION 4. *$(\nabla_X s)(m)$ depends only on X_m and on the germ of s in m.*

Proof. If s has null germ in m, then it vanishes in a neighbourhood U of m. Let $f \in \mathcal{F}^0$ be null in an open set $V \subset U$ and taking the value equal to 1 in the complementary of an open set W contained in U, such that $V \subset W \subset U$. It is clear that $s = sf$ and we have that

$$(\nabla s)(m) = (\nabla(fs))(m) = (df)(m) \otimes s(m) + f(m)(\nabla s) = 0,$$

therefore two sections with the same germ in m have the same covariant derivative in m. On the other hand, let U and V be open neighbourhoods of

m in M with $U \subset V$ and $f \in \mathcal{F}^0$ such that $f|_U = 1$, $f|_{M-V} = 0$. If s is a section it is clear that s and fs have the same germ in m and then we have

$$(\nabla_X s)(m) = (\nabla_X(fs))(m) = X_m(f)s(m) + f(m)\, i(X_m)(\nabla s) = i(X_m)(\nabla s),$$

and then it only depends on X_m. ∎

In order to find the local expression of a connection, let $\{U_l, s_l\}$ be a trivialization of the bundle. Every section in U_l has the form $s = fs_l$; hence, according to the property (ii), in order to calculate ∇s, it suffices to calculate ∇s_l. Writing $\nabla s_l = 2\pi i \omega^l \otimes s_l$,[11] (where ω^l is an element of $\mathcal{F}^1$ in U_l), in U_{lj} we have that $s_j = c_{lj} s_l$, therefore

$$\nabla s_j = \nabla(c_{lj} s_l) = dc_{lj} \otimes s_l + c_{lj} \nabla s_l,$$

that is

$$2\pi i \omega^j \otimes s_j = 2\pi i c_{lj} \omega^j \otimes s_l = dc_{lj} \otimes s_l + c_{lj} 2\pi i \omega^l \otimes s_l,$$

therefore

$$2\pi i c_{lj} \omega^j = dc_{lj} + c_{lj} 2\pi i \omega^l,$$

and hence, in U_{lj},

$$\omega^j = \frac{1}{2\pi i} \frac{dc_{lj}}{c_{lj}} + \omega^l.$$

Observe now that we can write the following relation for the covariant derivative of a section:

$$\nabla_X s = \nabla_X(fs_l) = (X(f) + 2\pi i \langle X, \omega^l \rangle f) s_l. \tag{2}$$

The family $\{U_l, \omega^l\}$ is called *connection* 1-*form*, but this is not a global form in M. Nevertheless, in U_{lj} the equality $d\omega^j|_{U_{lj}} = d\omega^l|_{U_{lj}}$ holds, hence there exists a global complex 2-form $\bar{\Omega}$ on M such that $\bar{\Omega}|_{U_l} = d\omega^l$.

DEFINITION 9. The 2-form $\bar{\Omega}$ so defined is called the *curvature form* of the connection ∇.

The relation with the "classical" curvature is as follows:

[11] Be careful: the summation convention does not apply in this expression and those ones related with it.

PROPOSITION 5. *If* $X_1, X_2 \in \mathcal{X}^{\mathbb{C}}(M)$ *and* $s \in \Gamma(L)$, *then*

$$2\pi i\bar{\Omega}(X_1, X_2)s = (\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})s.$$

Proof. Taking a trivialization $\{U_l, s_l\}$, in the open set U_l we have

$$\bar{\Omega}(X_1, X_2) = d\omega^l(X_1, X_2) = X_1\omega^l(X_2) - X_2\omega^l(X_1) - \omega^l([X_1, X_2]),$$

but

$$\nabla_{X_1}\nabla_{X_2}s_l = \nabla_{X_1}(2\pi i\omega^l(X_2)s_l) = 2\pi iX_1(\omega^l(X_2))s_l + (2\pi i)^2\omega^l(X_2)\omega^l(X_1)s_l,$$
$$\nabla_{X_2}\nabla_{X_1}s_l = \nabla_{X_2}(2\pi i\omega^l(X_1)s_l) = 2\pi iX_2(\omega^l(X_1))s_l + (2\pi i)^2\omega^l(X_1)\omega^l(X_2)s_l,$$
$$\nabla_{[X_1,X_2]}s_l = 2\pi i\omega^l([X_1, X_2])s_l,$$

and thus the result follows immediately. ∎

Another way of defining the curvature is the following: we can extend the action of ∇ to the p-forms on M with values in $\Gamma(L)$ in the following way

$$\mathcal{F}^p \otimes_{\mathcal{F}^0} \Gamma(L) \xrightarrow{\nabla^p} \mathcal{F}^{p+1} \otimes_{\mathcal{F}^0} \Gamma(L)$$

by means of the expression

$$\nabla^p(\alpha \otimes s) = d\alpha \otimes s + (-1)^p\alpha \wedge \nabla s.$$

Denoting $\varpi := \nabla^1 \circ \nabla$, we have that

$$\begin{aligned}\varpi(fs) &= (\nabla^1 \circ \nabla)(fs) = \nabla^1(df \otimes s + f\nabla s)\\ &= d(df) \otimes s - df \wedge \nabla s + df \wedge \nabla s + f(\nabla^1 \circ \nabla)s = f\varpi(s),\end{aligned}$$

therefore ϖ is $\mathcal{F}^0$-linear, that is,

$$\varpi \in (\mathcal{F}^2 \otimes_{\mathcal{F}^0} \Gamma(L)) \otimes_{\mathcal{F}^0} \Gamma(L)^* \cong \mathcal{F}^2 \otimes_{\mathcal{F}^0} (\Gamma(L) \otimes_{\mathcal{F}^0} \Gamma(L)^*).$$

Let $\bar{\Omega}$ be the 2-form obtained from ϖ by means of the natural contraction of the last two factors of ϖ. The so-obtained form is a complex 2-form on M. In order to see that it coincides with the curvature form, it suffices to calculate it in an open set U_l of a trivialization $\{U_l, s_l\}$:

$$\begin{aligned}(\nabla^1 \circ \nabla)s_l &= \nabla^1(\nabla s_l) = \nabla^1(\omega^l \otimes s_l) = d\omega^l \otimes s_l + \omega^l \wedge \nabla s_l\\ &= d\omega^l \otimes s_l + \omega^l \wedge \omega^l \otimes s_l = d\omega^l \otimes s_l,\end{aligned}$$

therefore $\varpi|_{U_l} = d\omega^l \otimes s_l \otimes s_l^*$, where s_l^* is the dual of s_l. Hence, $\bar{\Omega} = d\omega^l$, as we wanted to prove. (We have not considered the factor $2\pi i$ which depends on the definition of the connection form).

3.4. Hermitian structures

DEFINITION 10. Let (L, π, M) be a complex line bundle endowed with a connection ∇. A *hermitian structure* on (L, π, M) is a correspondence such that assigns a *hermitian metric* h_m on L_m, for every $m \in M$, in a differentiable way; that is, if s, s' are differentiable sections, then the function $h(s, s')$ is differentiable. This is equivalent to say that h is an element of $\Gamma(L^* \otimes \bar{L}^*)$, (where $\bar{L}$ is the conjugate bundle of L), that is, h satisfies condition $h(s, s') = \overline{h(s', s)}$.

Given a hermitian structure h on (L, π, M), ∇ is said to be a *hermitian connection* with respect to h if

$$X(h(s, s')) = h(\nabla_X s, s') + h(s, \nabla_X s'), \ \forall X \in \mathcal{X}(M), \ \forall s, s' \in \Gamma(L).$$

This is equivalent to say that $\nabla h = 0$, when the connection is extended to $\Gamma(L^* \otimes \bar{L}^*)$ in the usual way.

PROPOSITION 6. *If ∇ is a hermitian connection with respect to h, then the curvature $\bar{\Omega}$ of ∇ is a real form* [12].

Proof. We are going to see it for trivializing neighbourhoods. So, let $\{U_l, s_l\}$ be a trivialization, $\{\omega^l\}$ the connection forms of ∇ on $\{U_l\}$ and X a real vector field; we have

$$\begin{aligned} X(h(s_l, s_l)) &= h(\nabla_X s_l, s_l) + h(s_l, \nabla_X s_l) \\ &= 2\pi i h(\omega^l(X) s_l, s_l) - 2\pi i h(s_l, \omega^l(X) s_l) \\ &= 2\pi i(\omega^l(X) - \overline{\omega^l(X)}) h(s_l, s_l), \end{aligned}$$

and therefore

$$\omega^l(X) - \overline{\omega^l(X)} = \frac{1}{2\pi i} \frac{X(h(s_l, s_l))}{h(s_l, s_l)}.$$

That is

$$\omega^l - \overline{\omega^l} = \frac{1}{2\pi i} \frac{dh(s_l, s_l)}{h(s_l, s_l)},$$

hence $d\omega^l - d\overline{\omega^l} = 0$, that is, $d\omega^l$ is real, for every l and thus the curvature $\bar{\Omega}$ is real. ∎

[12] A vector field X is *real* if $X(f) \in \Omega^0(M)$, $\forall f \in \Omega^0(M)$. A 1-form α is real if $\alpha(X)$ is real, for every real vector field. And so on.

Remarks. Let (L, π, M) be a complex line bundle with hermitian metric h and hermitian connection ∇.

1. Let $\{U_j, s_j\}$ be a trivialization of L with $h(s_j, s_j)(m) = 1$, for every $m \in U_j$. Then if ω^j is the connection form of ∇ in U_j, from the above equality we have that $\omega^j - \overline{\omega^j} = 0$, so they are real forms.

   These special trivializations can be obtained from any one by dividing every s_j by its module.

2. If $\{U_j, s_j\}$ is one of such trivializations, then $|c_{lj}(m)| = 1$, for every $m \in U_{lj}$, since

$$1 = h(s_l, s_l) = h(s_j, s_j) = c_{lj}\bar{c}_{lj}h(s_j, s_j).$$

   Then $c_{lj} = e^{if_{lj}}$ with $f_{lj} \colon U_{lj} \to \mathbb{R}$ a differentiable function. So we have that every hermitian line bundle admits a trivialization with transition functions taking values in S^1, the group of isometries of h.

3.5. EXISTENCE OF HERMITIAN CONNECTIONS. Let M be a differentiable manifold and $\bar{\Omega}$ a real closed two form on M.

THEOREM 2. *The necessary and sufficient condition for $\bar{\Omega}$ to be the curvature 2-form of a hermitian connection ∇ on a complex line bundle (L, π, M) endowed with a hermitian metric is that the cohomology class $[\bar{\Omega}] \in H^2(M, \mathbb{R})$ is integer, that is, the Čech cohomology class canonically associated with $[\bar{\Omega}]$ belongs to the image of the morphism $\varepsilon^2 \colon \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathbb{R})$ induced by the inclusion $\varepsilon \colon \mathbb{Z} \hookrightarrow \mathbb{R}$.*

Moreover, in this case, $\bar{\Omega}$ is a representative of the image by ε^2 of the Chern class of the bundle (L, π, M).

Before starting the proof, we remind how to construct the canonical isomorphism between the de Rham cohomology and the Čech cohomology of degree two; and what means that $[\bar{\Omega}]$ is integer.

Let $\bar{\Omega} \in \Omega^2(M)$ be a real 2-form with $d\bar{\Omega} = 0$ and $[\bar{\Omega}] \in H^2(M, \mathbb{R})$ its cohomology class. We can associate an element of $\check{H}^2(M, \mathbb{R})$ to $[\bar{\Omega}]$. Let $\{U_l\}$ be a contractible covering of M [13]. Since U_l are contractible, we have that

[13] What we understand by "contractible covering" is that all the open sets of the covering and all their intersections are contractible. In order to prove its existence, it suffices to endow M with a Riemannian metric and to take a covering by geodesically convex open sets.

$\bar{\Omega}|_{U_l} = d\omega^l$, where $\omega^l \in \Omega^1(U_l)$. But U_{lj} is also contractible and $d\omega^l|_{U_{lj}} = d\omega^j|_{U_{lj}}$, hence $(\omega^j - \omega^l)|_{U_{lj}} = df^{lj}$, where $f^{lj} \in \Omega^0(U_{lj})$. In U_{ljk} we have that

$$(df^{lj} + df^{jk} - df^{lk})|_{U_{ljk}} = 0,$$

therefore $f^{lj} + f^{jk} - f^{lk} = \alpha^{ljk}$ is constant in U_{ljk}.

Let a be the map defined by $(U_l, U_j, U_k) \mapsto \alpha^{ljk}$. We have that a is a Čech cochain associated with the cohomology class $[\bar{\Omega}]$ and then we can construct a map

$$\begin{array}{ccc} H^2(M,\mathbb{R}) & \longrightarrow & \check{H}^2(M,\mathbb{R}) \\ [\bar{\Omega}] & \mapsto & [a] \end{array}$$

since $da = 0$, as

$$\begin{aligned} da(U_i, U_j, U_k, U_l) = a(U_j, U_k, U_l) &- a(U_i, U_k, U_l) \\ &+ a(U_i, U_j, U_l) - a(U_i, U_j, U_k) = 0. \end{aligned}$$

It is evident that $[a]$ does not depend on $\bar{\Omega}$ but only on its cohomology class, since, taking into account that, if $\Omega' = \bar{\Omega} + d\eta$, then $\omega'^l = \omega^l + \eta$ in U_l. Therefore $(\omega'^j - \omega'^l)|_{U_{lj}} = (\omega^j - \omega^l)|_{U_{lj}} = df^{lj}$ and hence both $\bar{\Omega}$ and $\bar{\Omega}'$ have the same associated cochain.

On the other hand, the natural injection $\varepsilon : \mathbb{Z} \hookrightarrow \mathbb{R}$ induces a morphism

$$\begin{array}{cccc} \varepsilon^2 : & \check{H}^2(M,\mathbb{Z}) & \longrightarrow & \check{H}^2(M,\mathbb{R}) \\ & [a] & \mapsto & [\varepsilon(a)] \end{array}$$

therefore we obtain that $[\bar{\Omega}]$ is in the image of ε^2 if, and only if, there exists a contractible covering $\{U_l\}$ of M, a family of 1-forms $\{\omega^l\}$ and a family of functions $\{f^{lj}\}$ such that

$$\bar{\Omega}|_{U_l} = d\omega^l, \quad (\omega^j - \omega^l)|_{U_{lj}} = df^{lj}, \quad (f^{lj} + f^{jk} - f^{lk})|_{U_{ljk}} \in \mathbb{Z}.$$

Now we prove the theorem.

Proof. ($\Longrightarrow$) Let $\bar{\Omega}$ be the curvature form of a connection ∇ on a complex line bundle (L, π, M) with hermitian metric h. Let $\{U_l, s_l\}$ be a trivialization of the bundle with $h(s_l, s_l) = 1$. We know that the forms ω^l are real, and since

$$(\omega^j - \omega^l)|_{U_{lj}} = \frac{1}{2\pi i}\frac{dc_{lj}}{c_{lj}} = df^{lj},$$

then the functions f^{lj} can be chosen real also. But we have $2\pi i f^{lj} = \log c_{lj}$; and since $c_{lj}c_{jk} = c_{lk}$ we obtain that $\log c_{lj} + \log c_{jk} - \log c_{lk}$ is an integer multiple of $2\pi i$, hence $f^{lj} + f^{jk} - f^{lk}$ is an integer, so, as we have seen above, $[\bar{\Omega}]$ is integer.

($\Longleftarrow$) Suppose $\bar{\Omega}$ is a closed 2-form in M such that $[\bar{\Omega}]$ is an integer class. Let $\{U_l\}$ be a contractible covering of M and $\{\omega^l\}$ and $\{f^{lj}\}$ as above. Now we have that $f^{lj}+f^{jk}-f^{lk} \in \mathbb{Z}$. Put $c_{lj} = e^{2\pi i f^{lj}}$ in U_{lj}. Taking into account that $c_{lj}c_{jk} = c_{lk}$, we deduce that a complex line bundle (L, π, M) with transition functions c_{lj} can be constructed. The sections s_l can be taken equal to 1 on each U_l. Once the bundle is constructed, we can take the connection ∇ which is determined by the connection forms ω^l, and whose curvature form is obviously $\bar{\Omega}$. The hermitian structure is given by

$$h_m(e, e') = z\bar{z}',$$

where $e = (m, z)$, $e' = (m, z')$ in any open set of the trivialization containing m. We are going to see that this is a hermitian connection with respect to this metric. Consider $X \in \mathcal{X}(M)$ and $s, s' \in \Gamma(L)$. If $m \in M$ and U_l is a trivializing open set with $m \in U_l$ and $s|_{U_l} = f_l s_l$, $s'|_{U_l} = f_l' s_l$, then we have

$$\begin{aligned}\Big(X\big(h(s,s')\big)\Big)(m) &= \Big(X\big(h(s,s')\big)\Big)\Big|_{U_l}(m) = \Big(X\big(h(f_l s_l, f_l' s_l)\big)\Big)\Big|_{U_l}(m)\\ &= (X(f_l\bar{f}_l'))|_{U_l}(m) = (Xf_l)(m)\bar{f}_l'(m) + f_l(m)(X\bar{f}_l')(m).\end{aligned}$$

On the other hand,

$$\begin{aligned}h(\nabla_X s, s')|_{U_l}(m) &= h((Xf_l)s_l + 2\pi i f_l\omega^l(X)s_l, f_l's_l)\big|_{U_l}(m)\\ &= ((Xf_l)\bar{f}_l' + 2\pi i f_l\omega^l(X)\bar{f}_l')(m),\\ h(s, \nabla_X s')|_{U_l}(m) &= \Big(h(f_l s_l, (Xf_l')s_l + 2\pi i f_l'\omega^l(X)s_l)\Big)(m)\\ &= (f_l(X\bar{f}_l') - f_l 2\pi i \bar{f}_l'\omega^l(X))(m),\end{aligned}$$

because $\omega^l(X)$ is real. Therefore

$$X\big(h(s,s')\big) = h(\nabla_X s, s') + h(s, \nabla_X s').$$

∎

3.6. Classification of hermitian connections

3.6.1. Affine structure on the set of connections. In the following (M, Ω) will be a symplectic manifold and (L, π, M) a complex line bundle endowed with an hermitian metric h.

Denote by $\mathrm{Con}(L)$ the set of connections on the complex line bundle L. If $\nabla, \nabla' \in \mathrm{Con}(L)$, we have that, for every $s \in \Gamma(L)$ and $f \in \mathcal{F}^0$,

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s.$$

Therefore

$$\nabla - \nabla' \in \mathcal{F}^1 \otimes_{\mathcal{F}^0} \Gamma(L) \otimes_{\mathcal{F}^0} \Gamma(L)^* \simeq \Gamma(L \times L^*) \otimes_{\mathcal{F}^0} \mathcal{F}^1 \simeq \Gamma(\mathbb{C} \times M) \otimes_{\mathcal{F}^0} \mathcal{F}^1 \simeq \mathcal{F}^1.$$

On the other hand, if $\eta \in \mathcal{F}^1$ and ∇ is a connection, so is $\nabla + \eta$. So we have:

PROPOSITION 7. *The $\mathcal{F}^0$-module of the one-forms $\mathcal{F}^1$ acts freely and transitively on the set* $\mathrm{Con}(L)$.

Remarks.

- An equivalent way of stating the above proposition is the following: $\mathrm{Con}(L)$ has an affine structure modeled on the vector space $\mathcal{F}^1(M)$.
- Observe that if $\{U_j, s_j\}$ is a trivialization of the fiber bundle and $\{\omega^j\}$ are the connection forms of ∇, then $\{\omega^j + \eta\}$ are the connection forms of $\nabla + \eta$.

Denote by $\mathrm{Con}(L, h)$ the set of hermitian connections with respect to h. Then, if $\nabla \in \mathrm{Con}(L, h)$ and $\eta \in \mathcal{F}^1$, the connection $\nabla + \eta$ is hermitian with respect to h if, and only if, η is real. In order to see this, it suffices to remind that, in a trivializing system $\{U_j, s_j\}$ such that $\{U_j\}$ is a contractible covering, the connection forms are real for the hermitian connections. Then we have proved that:

PROPOSITION 8. *The Ω^0-module Ω^1 acts freely and transitively on the set* $\mathrm{Con}(L, h)$.

Now, let $\mathrm{Con}(L, h, \bar{\Omega})$ be the set of hermitian connections with curvature $\bar{\Omega}$. If ∇ and $\nabla' = \nabla + \eta$ are two of these connections, taking into account the relation between the corresponding connection forms and their curvature, we deduce that $d\eta = 0$. Thus:

PROPOSITION 9. *The group $Z^1(M)$ of the differential 1-cocycles of M acts freely and transitively on the set* $\mathrm{Con}(L, h, \bar{\Omega})$.

3.6.2. Equivalence of line bundles with connection. Let (L, π, M) and (L', π', M) be complex line bundles and $\phi\colon L \to L'$ a complex line bundle diffeomorphism inducing the identity on M. If $s\colon M \to L$ is a section of L, then $s' = \phi \circ s$ is a section of L'. The map $\Gamma(\phi)\colon \Gamma(L) \to \Gamma(L')$ given by $\Gamma(\phi)(s) := \phi \circ s$ is an isomorphism of $\mathcal{F}^0$-modules, because it is a diffeomorphism and is ϕ is $\mathbb{C}$-linear on the fibers.

In order to calculate $\Gamma(\phi)$ locally, let $\{U_j, s_j\}$ and $\{U_j, s'_j\}$ be trivializing systems of L and L'. Then $\phi \circ s_j = \varphi_j s'_j$, where $\varphi_j\colon U_j \to \mathbb{C}^*$ is a differentiable function with values in $\mathbb{C}^*$, because $\phi_m\colon L_m \to L'_m$ is a $\mathbb{C}$-isomorphism for every $m \in M$. But $s_j = c_{kj} s_k$ and $s'_j = c'_{kj} s'_k$, where c_{kj} and c'_{kj} are the transition functions of L and L' respectively. Then, on U_{jk} we have:

$$\phi \circ s_j = \varphi_j s'_j = \varphi_j c'_{kj} s'_k,$$
$$\phi \circ s_j = \phi \circ (c_{kj} s_k) = c_{kj} \phi \circ s_k = c_{kj} \varphi_k s'_k,$$

therefore $\varphi_j c'_{kj} = c_{kj} \varphi_k$, that is $\varphi_j = c_{kj} \varphi_k (c'_{kj})^{-1}$. Hence, $\Gamma(\phi)$ is represented by the family $\{U_j, \varphi_j\}$ with $\varphi_j\colon U_j \to \mathbb{C}^*$ satisfying $\varphi_j = c_{kj} \varphi_k (c'_{kj})^{-1}$.

Let ∇ be a connection in L. We can construct a connection in L' induced by ∇ and ϕ.

Definition 11. The *connection induced* by ∇ and ϕ in L' is the unique connection ∇' in L' such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(L) & \xrightarrow{\Gamma(\phi)} & \Gamma(L) \\ \nabla \big\downarrow & & \big\downarrow \nabla' \\ \mathcal{F}^1 \otimes_{\mathcal{F}^0} \Gamma(L) & \xrightarrow{id \otimes \Gamma(\phi)} & \mathcal{F}^1 \otimes_{\mathcal{F}^0} \Gamma(L) \end{array}$$

Proposition 10. *If ∇' is the connection induced by ∇ and the diffeomorphism ϕ, then both connections have the same curvature.*

Proof. Let $\{U_j, s_j\}$ and $\{U_j, s'_j\}$ be trivializing systems of L and L'. We have

$$\nabla s_j = 2\pi i \omega^j \otimes s_j, \quad \nabla' s'_j = 2\pi i \omega'^j \otimes s'_j,$$

where $\{\omega^j\}$ and $\{\omega'^j\}$ are the connection forms of ∇ and ∇' for the open covering $\{U_j\}$. According to the above notations, $\phi \circ s_j = \varphi_j s'_j$, then:

$$(id \otimes \Gamma(\phi))(\nabla s_j) = (id \otimes \Gamma(\phi))(2\pi i \omega^j \otimes s_j) = 2\pi i \omega^j \otimes \varphi_j s'_j,$$
$$\nabla'(\Gamma(\phi) s_j) = \nabla'(\varphi_j s_j) = d\varphi_j \otimes s'_j + 2\pi i \varphi_j \omega'^j \otimes s'_j.$$

But $(id \otimes \Gamma(\phi)) \circ \nabla = \nabla' \circ \Gamma(\phi)$, then

$$\omega^j = \frac{1}{2\pi i} \frac{d\varphi_j}{\varphi_j} + \omega'^j,$$

hence $d\omega^j = d\omega'^j$; and the result follows. ∎

Remarks.

1. Observe that if ∇' is induced by ∇ and ϕ, we have proved that their connection forms are related by

$$\omega^j = \frac{1}{2\pi i} \frac{d\varphi_j}{\varphi_j} + \omega'^j, \tag{3}$$

where $\varphi_j : U_j \to \mathbb{C}^*$ is defined by $\phi \circ s_j = \varphi_j s'_j$.

2. Let $\{U_j, s_j\}$ be a trivializing system of L. Then $\{U_j, s'_j = \phi \circ s_j\}$ is a trivializing system of L', because ϕ is a diffeomorphism. In these trivializing systems the functions $\varphi_j : U_j \to \mathbb{C}^*$ are identically equal to one, then $\omega^j = \omega'^j$.

3. Let L and L' be complex line bundles on M and $\{U_j, s_j\}$, $\{U_j, s'_j\}$ trivializing systems. Suppose $\varphi_j : U_j \to \mathbb{C}^*$ is a family of functions related by $\varphi_j = c_{kj}\varphi_k(c'_{kj})^{-1}$ in U_{jk}. Then there exists a unique diffeomorphism $\phi : L \to L'$ which induces the family $\{\varphi_j\}$. In fact, if $l \in L$ and $\pi(l) \in U_j$, put

$$\phi(l) = \frac{l}{s_j(\pi(l))} \varphi_j(\pi(l)) s'_j(\pi(l)).$$

Observe that $\phi(l)$ is well defined, since if $\pi(l) \in U_{jk}$, then the expression of $\phi(l)$ does not depend on the chosen index and the calculus can be made using the trivializing systems. Hence we have proved the following:

PROPOSITION 11. *The necessary and sufficient condition for a connection ∇' in L' to be induced by a connection ∇ in L and a diffeomorphism from L to L', is that there exist trivializations $\{U_j, s_j\}$ and $\{U_j, s'_j\}$ on L and L' and a family of functions $\varphi_j : U_j \to \mathbb{C}^*$ satisfying $\varphi_j = c_{kj}\varphi_k(c'_{kj})^{-1}$.*

4. Now suppose that $L = L'$, $\phi : L \to L$ is a diffeomorphism and $\{U_j, s_j\}$ a trivializing system of L. In this case we have that $\phi \circ s_j = \varphi_j s'_j$, but

$\varphi_j = c_{kj}\varphi_k(c_{kj})^{-1} = \varphi_k$ in U_{jk}. Therefore there exists a global function $\varphi\colon M \to \mathbb{C}^*$ such that $\phi \circ s = \varphi s$, for every $s \in \Gamma(L)$.

If ∇ is a connection in L and ∇' is induced by ∇ and ϕ, then the connection forms are related by (3). Now, repeating the arguments of the comment 3 above, we have that:

- Given $\varphi\colon M \to \mathbb{C}^*$, there exists a unique diffeomorphism $\phi\colon L \to L$ such that $\Gamma(\phi)s = \varphi s$, for $s \in \Gamma(L)$. In this case, if $l \in L$, we have that $\phi(l) = \varphi(\pi(l))l$.
- The necessary and sufficient condition for the connection ∇' to be induced by ∇ in L is that there exists one function $\varphi\colon M \to \mathbb{C}^*$ such that, if $\{U_j, s_j\}$ is a trivializing system of L, their connection forms are related by (3).

Taking into account the above results and comments, the equivalence we will use is the following:

DEFINITION 12. Two complex line bundles with connection (L, ∇) and (L', ∇') on M are said to be *equivalent* if there exists a diffeomorphism $\phi\colon L \to L'$ such that ∇' is the connection induced by ∇ and ϕ.

Remark. Observe that, if $(L, \nabla) \simeq (L', \nabla')$ by the diffeomorphism ϕ, taking trivializing systems of L and L', $\{U_j, s_j\}$ and $\{U_j, s'_j = \phi \circ s_j\}$, according to comment 2 above, we have

$$\begin{aligned} \nabla s_j &= 2\pi i \omega^j \otimes s_j, \\ \nabla'(\phi \circ s_j) &= \nabla' \varphi_j s'_j = 2\pi i \omega^j \otimes s'_j. \end{aligned}$$

That is, the local connection forms associated with ∇ and ∇' are the same with respect to the given trivializing systems.

3.6.3. THE CASE OF HERMITIAN LINE BUNDLES. In the same way as in Definition 12, we state:

DEFINITION 13. Two complex line bundles with hermitian connection (L, h, ∇) and (L', h', ∇') on M are said to be *equivalent* if there exists a diffeomorphism $\phi\colon L \to L'$ such that $h = \phi^* h'$ and ∇' is the connection induced by ∇ and ϕ.

We are interested in studying the set of equivalence classes of complex line bundles with hermitian connection. Previously we need the following:

LEMMA 1. a) *Let h_1, h_2 be two hermitian metrics in $\mathbb{C}$. There exists a $\mathbb{C}$-linear isomorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ such that, if $z_1, z_2 \in \mathbb{C}$, we have that $h_1(z_1, z_2) = h_2(\phi(z_1), \phi(z_2))$; that is $h_1 = \phi^* h_2$. Moreover ϕ is real.*

b) *Let (L, π, M) be a complex line bundle and h_1, h_2 two hermitian metrics in L. There exists a diffeomorphism $\phi\colon L \to L$ such that, if $l_1, l_2 \in L$, then $h_1(l_1, l_2) = h_2(\phi(l_1), \phi(l_2))$; that is $h_1 = \phi^* h_2$. In addition, ϕ is generated by a real function $\varphi\colon M \to \mathbb{R}^*$.*

c) *Let L and L' be complex line bundles over M with the same Chern class and h and h' hermitian metrics on L and L'. There exists a diffeomorphism $\phi\colon L \to L'$ such that, if $l_1, l_2 \in L$, then $h_1(l_1, l_2) = h_2(\phi(l_1), \phi(l_2))$; that is $h_1 = \phi^* h_2$.*

Proof. a) Let $z \in \mathbb{C}$ with $h_1(z, z) = 1$ and let $\lambda = h_2(z, z)$. Consider the isomorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ given by $\phi(w) = \lambda^{1/2} w$. Then ϕ satisfies the required conditions.

b) If $m \in M$, then h_{1m} and h_{2m} are hermitian metrics on L_m. Consider the function $\varphi\colon M \to \mathbb{R}^+$ such that $h_{1m}(l_1, l_2) = h_{2m}(\varphi(m) l_1, \varphi(m) l_2)$: for every l_1 and l_2 in L_m. The function φ exists by the item (a), and it is differentiable because h_1 and h_2 are differentiable. The diffeomorphism $\phi\colon L \to L$ given by $\phi(l) = \varphi(\pi(l)) l$ satisfies the required condition.

c) Let $\psi\colon L \to L'$ be a diffeomorphism. Its existence is assured because L and L' have the same Chern class. Let $\hat{h}$ be the metric on L' induced by ψ and h, that is, $h = \psi^* \hat{h}$. Let $\eta\colon L' \to L'$ be the diffeomorphism mapping $\hat{h}$ into h', that is, $\hat{h} = \eta^* h'$. Then $\phi = \eta \circ \psi\colon L \to L'$ is a diffeomorphism and $\phi^* h' = h$. ∎

The equivalence of complex line bundles with hermitian connection is given by the following:

PROPOSITION 12. *Let (L, h) and (L', h') be complex line bundles with hermitian metric and having the same Chern class. Let $\phi\colon L \to L'$ be the diffeomorphism satisfying $h = \phi^* h'$. If ∇ is an hermitian connection in L with respect to h, then ∇', the connection induced by ϕ on L', is hermitian with respect to h'.*

Proof. We have that $\nabla' \circ \Gamma(\phi) = (id \otimes \Gamma(\phi)) \circ \nabla$, that is, if $X \in \mathcal{X}(M)$ and $\sigma \in \Gamma(L)$, then $\nabla'_X(\phi \circ \sigma) = (\phi \circ \nabla_X)\sigma$. Now, take $s, s' \in \Gamma(L')$, then there exist $\sigma, \sigma' \in \Gamma(L)$ with $s = \phi \circ \sigma$, $s' = \phi \circ \sigma'$, and, if $X \in \mathcal{X}(M)$, we have

$$\begin{aligned} X(h'(s,s')) &= X(h(\phi^{-1} \circ s, \phi^{-1} \circ s')) = X(h(\sigma, \sigma')) \\ &= h(\nabla_X \sigma, \sigma') + h(\sigma, \nabla_X \sigma') \\ &= h'(\phi \circ \nabla_X \sigma, \phi \circ \sigma') + h'(\phi \circ \sigma, \phi \circ \nabla_X \sigma') \\ &= h'(\nabla'_X s, s') + h'(s, \nabla'_X s'). \end{aligned}$$

∎

Remarks.

1. The last proposition proves that it is irrelevant to take different hermitian metrics on a complex line bundle or different complex line bundles with the same Chern class. That is, if we fix (L, h), then in the quotient set defined by the equivalence relation introduced in Definition 13, every class has a representative of the form (L, h, ∇).
2. According to this, from now on we will take a fixed complex line bundle L with one fixed hermitian metric h on L. Then, all the results obtained in the following will refer to complex line bundles having the same Chern class as L.

Now, the result we are interested in is:

PROPOSITION 13. *Let ∇_1 and ∇_2 be equivalent connections on the complex line bundle (L, π, M) with hermitian metric h. Suppose that ∇_1 is hermitian. Then ∇_2 is hermitian if, and only if, the function $\varphi: M \to \mathbb{C}^*$ relating the connection forms of ∇_1 and ∇_2 has constant modulus.*

Proof. Let $\{U_j, s_j\}$ be a trivializing system with $h(s_j, s_j) = 1$. We have that $\omega_1^j = \frac{1}{2\pi i}\frac{d\varphi}{\varphi} + \omega_2^j$. As every hermitian connection has real connection forms, then $\frac{1}{2\pi i}\frac{d\varphi}{\varphi}$ must be a real form. Then the statement is a consequence of the following lemma whose proof is immediate. ∎

LEMMA 2. *Let $\varphi: M \to \mathbb{C}^*$ be a differentiable function. The necessary and sufficient condition for $\frac{d\varphi}{\varphi}$ to be imaginary is that φ has constant modulus.*

Consider now the group $B = \{\varphi\colon M \to \mathbb{C}^* : |\varphi| = \text{const.}\}$ with the product operation, and the morphism

$$\begin{array}{rccc} \eta\colon & B & \longrightarrow & Z^1(M) \\ & \varphi & \mapsto & \frac{1}{2\pi i}\frac{d\varphi}{\varphi} \end{array}$$

$\ker\eta$ is made of the constant functions. Let C be the image of η. According to the above discussion, we have proved that:

PROPOSITION 14. *The group* $\frac{Z^1(M)}{C}$ *acts freely and transitively on the set* $\mathrm{Con}(L, h, \bar{\Omega})/\sim$; *where* $\sim$ *is the equivalence of hermitian connections.*

3.6.4. CALCULATION OF $Z^1(M)/C$ AND $C/B^1(M)$.

LEMMA 3. $B^1(M) \subset C \subset Z^1(M)$.

Proof. Consider $f \in \Omega^0(M)$. Then $df \in B^1(M)$. Let $\varphi\colon M \to \mathbb{C}^*$ be the map defined by $\varphi := e^{2\pi i f}$. So we have that $\frac{1}{2\pi i}\frac{d\varphi}{\varphi} = df$; therefore the result follows. (Observe that, in general, $B^1(M) = C$ does not hold, as we will see later.) ∎

Therefore we have the following exact sequence of groups:

$$0 \to \frac{C}{B^1(M)} \to \frac{Z^1(M)}{B^1(M)} \to \frac{Z^1(M)}{C} \to 0,$$

and from here

$$\frac{Z^1(M)}{C} \simeq \frac{Z^1(M)/B^1(M)}{C/B^1(M)} = \frac{H^1(M,\mathbb{R})}{C/B^1(M)}.$$

Now, we are going to study the group $\frac{C}{B^1(M)}$ in order to characterize the last quotient. Consider the natural injection $\varepsilon\colon \mathbb{Z} \to \mathbb{R}$, which is understood as a morphism between constant sheaves over M. We have:

PROPOSITION 15. *The morphism* $\varepsilon^1\colon \check{H}^1(M,\mathbb{Z}) \to \check{H}^1(M,\mathbb{R})$ *(induced by* ε*) is injective.*

Proof. Consider the exact sequence of constant sheaves over M:

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{R} \xrightarrow{e} S^1 \to 0,$$

with $e(\alpha) = e^{2\pi i\alpha}$, whose associated exact cohomology sequence is

$$0 \to \check{H}^0(M,\mathbb{Z}) \xrightarrow{\varepsilon^0} \check{H}^0(M,\mathbb{R}) \xrightarrow{e^0} \check{H}^0(M,S^1) \xrightarrow{\partial^0} \check{H}^1(M,\mathbb{Z}) \xrightarrow{\varepsilon^1} \check{H}^1(M,\mathbb{R}),$$

that is

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{R} \xrightarrow{e} S^1 \xrightarrow{\partial^0} \check{H}^1(M,\mathbb{Z}) \xrightarrow{\varepsilon^1} \check{H}^1(M,\mathbb{R}) \dots ,$$

hence $\ker \partial^0 = S^1$, and then $\operatorname{Im} \partial^0 = \ker \varepsilon^1 = 0$. ∎

On the other hand, we have a canonical isomorphism between $H^1(M,\mathbb{R})$ and $\check{H}^1(M,\mathbb{R})$ which is constructed in the following way: consider $[\eta] \in H^1(M,\mathbb{R})$ and $\eta \in [\eta]$. If $\{U_j\}$ is a contractible covering of M, then there exist $f_j \colon U_j \to \mathbb{R}$ such that $df_j = \eta\,|_{U_j}$. In U_{jk} we have that $d(f_j - f_k) = 0$, then $f_j - f_k\,|_{U_{jk}} \in \mathbb{R}$, that is, it is constant. In this way we have the assignment $[\eta] \mapsto \{U_{jk} \mapsto f_j - f_k\}$, which is an isomorphism. Let $\alpha \colon \check{H}^1(M,\mathbb{R}) \to H^1(M,\mathbb{R})$ be the inverse isomorphism. Consider now the sequence of maps:

$$\check{H}^1(M,\mathbb{Z}) \xrightarrow{\varepsilon^1} \check{H}^1(M,\mathbb{R}) \xrightarrow{\alpha} H^1(M,\mathbb{R}).$$

Let $H^1(M,\mathbb{Z})$ be the image of $\check{H}^1(M,\mathbb{Z})$ in $H^1(M,\mathbb{R})$ by $\alpha \circ \varepsilon^1$, this subgroup is characterized in the following way: $[\eta] \in H^1(M,\mathbb{Z})$ if, and only if, there exists a contractible covering $\{U_j\}$ of M and functions $f_j \colon U_j \to \mathbb{R}$ such that $df_j = \eta\,|_{U_j}$ and $f_j - f_k\,|_{U_{jk}} \in \mathbb{Z}$; for every representative $\eta \in [\eta]$. Taking this into account we can characterize the group $\frac{C}{B^1(M)}$ as follows:

PROPOSITION 16. *$\frac{C}{B^1(M)}$ is isomorphic to $H^1(M,\mathbb{Z})$.*

Proof. According to the definition of C, we have a natural injection

$$\frac{C}{B^1(M)} \to \frac{Z^1(M)}{B^1(M)} = H^1(M,\mathbb{R}).$$

Consider $\varphi \colon M \to \mathbb{C}^*$ with $|\varphi| = r$. Its associate element in C is $\frac{1}{2\pi i}\frac{d\varphi}{\varphi}$. We denote by $\left[\frac{1}{2\pi i}\frac{d\varphi}{\varphi}\right]$ its class in $\frac{C}{B^1(M)}$ and maintain this notation when it is considered in $H^1(M,\mathbb{R})$. We are going to see that $\left[\frac{1}{2\pi i}\frac{d\varphi}{\varphi}\right] \in H^1(M,\mathbb{Z})$. The image of φ is in the circumference with radius equal to r in the complex plane. Consider the following open sets in this image

$$U_1 = \{z \in \mathbb{C} : |z| = r, z \neq r\}, \quad U_2 = \{z \in \mathbb{C} : |z| = r, z \neq -r\}.$$

Consider $U_j' = \varphi^{-1}(U_j)$ and $\varphi_j = \varphi \mid_{U_j'}$ $(j = 1, 2)$. Taking determinations of the logarithm in U_1 and U_2, we can construct differential functions $f_j \colon U_j \to \mathbb{R}$ such that $\varphi_j = re^{2\pi i f_j}$. We have that $df_j = \frac{1}{2\pi i}\frac{d\varphi_j}{\varphi_j}$, and if $m \in U_1' \cap U_2'$, then $re^{2\pi i f_1(m)} = re^{2\pi i f_2(m)}$, hence $f_1 - f_2 \in \mathbb{Z}$. Therefore $[\frac{1}{2\pi i}\frac{d\varphi}{\varphi}] \in H^1(M, \mathbb{Z})$, as we wanted.

On the other hand, if $[\eta] \in H^1(M, \mathbb{Z})$ and $\{U_j\}$ is a contractible open covering of M, let $f_j \colon U_j \to \mathbb{R}$ be maps such that $\eta \mid_{U_j} = df_j$ and $f_j - f_k \mid_{U_{jk}} \in \mathbb{Z}$. Let $\varphi_j \colon U_j \to \mathbb{C}^*$ the map defined by $\varphi_j(m) := e^{2\pi i f_j(m)}$. If $m \in U_{jk}$ we have

$$\frac{\varphi_j(m)}{\varphi_k(m)} = e^{2\pi i (f_j(m) - f_k(m)} = 1,$$

therefore $\varphi_j \mid_{U_{jk}} = \varphi_k \mid_{U_{jk}}$, hence there exists $\varphi \colon M \to \mathbb{C}^*$ such that $\varphi \mid_{U_j} = \varphi_j$, for every j. Moreover $|\varphi| = 1$, then $\frac{1}{2\pi i}\frac{d\varphi}{\varphi} \in C$ and $\left[\frac{1}{2\pi i}\frac{d\varphi}{\varphi}\right] \in \frac{C}{B^1(M)}$. The image of $\left[\frac{1}{2\pi i}\frac{d\varphi}{\varphi}\right]$ in $H^1(M; \mathbb{R})$ is $[\eta]$, since

$$\left(\eta - \frac{1}{2\pi i}\frac{d\varphi}{\varphi}\right) \mid_{U_j} = df_j - df_j = 0,$$

and the assertion holds. ▮

In addition, taking into account the canonical isomorphisms $\check{H}^1(M, \mathbb{R}) \simeq H^1(M, \mathbb{R})$, $\check{H}^1(M, \mathbb{Z}) \simeq H^1(M, \mathbb{Z})$, we have proved that:

THEOREM 3. *The group* $\frac{\check{H}^1(M,\mathbb{R})}{\check{H}^1(M,\mathbb{Z})}$ *acts freely and transitively on the set of hermitian connections in* (L, π, M) *related to a given metric* h *and with curvature* Ω, *module the equivalence of connections.*

And taking into account the influence of the hermitian metric and the diffeomorphisms of L, we have proved also the following:

THEOREM 4. *Let* M *be a differentiable manifold and* $\bar{\Omega}$ *a real closed 2-form in* M. *The group* $\frac{\check{H}^1(M,\mathbb{R})}{\check{H}^1(M,\mathbb{Z})}$ *acts freely and transitively on the set of equivalence classes of complex line bundles with hermitian connection* (L, ∇) *in* M *which have the same Chern class* $c(L)$ *(that is, which are diffeomorphic to* L*) and the same curvature* $\bar{\Omega}$.

COROLLARY 2. *If* M *is simply connected, then there exists only one equivalence class of complex line bundles with hermitian connection* (L, ∇) *with the same Chern class and the same curvature.*

3.6.5. Influence of the ker of the morphism $\varepsilon^2\colon \check{H}^2(M,\mathbb{Z}) \to \check{H}^2(M,\mathbb{R})$. Given a manifold M and a real closed 2-form $\bar{\Omega}$ on it, if L is a hermitian complex line bundle, we have studied the equivalence classes of hermitian connections in L with curvature $\bar{\Omega}$. In this case $\bar{\Omega}$ is a representative of the image of the Chern class of L by the morphism ε^2. As $L(M) \simeq \check{H}^2(M,\mathbb{Z})$, and ε^2 is not injective, then the following problem arises: there are non-diffeomorphic line bundles whose Chern classes in $\check{H}^2(M,\mathbb{Z})$ have the same image under ε^2. Now we are going to study the effects of this problem on the classification of complex line bundles with connections (L,∇) on M.

Consider the morphism $\varepsilon^2\colon \check{H}^2(M,\mathbb{Z}) \to \check{H}^2(M,\mathbb{R})$ induced by $\varepsilon\colon \mathbb{Z} \to \mathbb{R}$; and denote $\bar{G} = \ker \varepsilon^2$. Given $[\bar{\Omega}] \in H^2(M,\mathbb{R}) \simeq \check{H}^2(M,\mathbb{R})$, let V be the antiimage of $[\bar{\Omega}]$ by ε^2. The group $\bar{G}$ acts freely an transitively in V in the natural way. If $\sigma \in V$, let P_σ be the set of equivalence classes of complex line bundles over M with hermitian connection (L,∇) whose Chern class is σ and with curvature $\bar{\Omega}$. We have that $\varepsilon^2(\sigma) = [\bar{\Omega}]$. If $\sigma, \sigma' \in V$ with $\sigma \neq \sigma'$, then $P_\sigma \cap P_{\sigma'} = \emptyset$. Then we have a natural projection $\pi\colon P = \bigcup_{\sigma\in V} P_\sigma \to V$. Moreover, as we have seen above, the action of the group $G = \frac{\check{H}^1(M,\mathbb{R})}{\check{H}^1(M,\mathbb{Z})}$ in P preserves its fibers and is free and transitive on these fibers. Hence (P,π,V) is a G-principal bundle of sets.

The already known exact sequence of constant sheaves over M

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{R} \xrightarrow{e} S^1 \to 0$$

gives the cohomology sequence

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{R} \xrightarrow{e^0} S^1 \xrightarrow{\partial^0} \check{H}^1(M,\mathbb{Z}) \xrightarrow{\varepsilon^1} \check{H}^1(M,\mathbb{R}) \xrightarrow{e^1} \check{H}^1(M,S^1) \xrightarrow{\partial^1} \dots$$

and taking into account that $\operatorname{Im}\partial^0 = \ker \varepsilon^1 = 0$ we have

$$0 \to \check{H}^1(M,\mathbb{Z}) \xrightarrow{\varepsilon^1} \check{H}^1(M,\mathbb{R}) \xrightarrow{e^1} \check{H}^1(M,S^1) \xrightarrow{\partial^1} \ker\varepsilon^2 \to 0,$$

therefore, as ε^1 is injective, we have

$$0 \to \frac{\check{H}^1(M,\mathbb{R})}{\check{H}^1(M,\mathbb{Z})} \xrightarrow{\bar{e}^1} \check{H}^1(M,S^1) \xrightarrow{\partial^1} \bar{G} \to 0,$$

(where $\bar{e}^1$ is the natural morphism), that is

$$0 \to G \xrightarrow{\bar{e}^1} \check{H}^1(M,S^1) \xrightarrow{\partial^1} \bar{G} \to 0.$$

Hence $\partial^1\colon \check{H}^1(M,S^1) \to \bar{G}$ is a G-principal bundle of sets.

Next, we are going to construct for each $\sigma \in V$ a bijection ϕ between P and $\check{H}^1(M, S^1)$. Let $\eta\colon V \to P$, $\mu\colon \bar{G} \to \check{H}^1(M, S^1)$ be sections of π and ∂^1 respectively. If $L \in P$, we have that $\pi(L) \in V$. There exists a unique $\bar{g} \in \bar{G}$ such that $\pi(L) = \sigma\bar{g}$. On the other hand, there is a unique $g \in G$ such that $L = \eta(\pi(L))g$, then we define $\phi(L)$ as $\phi(L) := \mu(\bar{g})g$. So, ϕ is a map from P to $\check{H}^1(M, S^1)$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & \check{H}^1(M,S^1) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\partial^1} \\ V & \xrightarrow{\bar{\phi}} & \bar{G} \end{array}$$

where, if $v \in V$, $\bar{\phi}(v) = \bar{h} \in \bar{G}$ is the unique element such that $v = \sigma\bar{h}$. Observe that $\bar{\phi}$ is a bijection and ϕ is a bijection on each fiber, hence ϕ is also a bijection. In addition, ϕ is covariant with respect to the actions of G and $\bar{G}$, that is, if $L \in P$, we have that $\phi(Lg) = \phi(L)g$ and $\bar{\phi}(v\bar{g}) = \bar{\phi}(v)\bar{g}$, for $g \in G$ and $\bar{g} \in \bar{G}$.

Taking into account this bijection, we have proved the following:

THEOREM 5. *Let M be a differentiable manifold and $\bar{\Omega}$ an integer real closed 2-form. The action of the group $\check{H}^1(M, S^1)$ on the set of equivalence classes of complex line bundles with hermitian connection (L, ∇) with curvature $\bar{\Omega}$, is free and transitive.*

COROLLARY 3. *If M is simply connected, then this set of equivalence classes has a unique element (since $\check{H}^1(M, S^1) = 0$, because $\check{H}^1(M, S^1) = \mathrm{Hom}(\pi^1(M), S^1) = 0$).*

4. PREQUANTIZATION

Now, we are ready to start the geometric quantization programme.

Our first goal in the geometric quantization programme is to construct the intrinsic Hilbert space of the system. With this aim, we follow a systematic procedure, starting from the easiest possible model and modifying it in such a way that the situation is adapted to the rules of Definition 4 (as it is done, for instance, in [48], [71] and [92]).

4.1. FIRST ATTEMPTS TO DEFINE QUANTUM STATES AND OPERATORS. Let (M, Ω) be a $2n$-dimensional symplectic manifold. The easiest way of constructing a Hilbert space associated with it is to consider the algebra of complex smooth functions with compact support in M and the inner product defined on it by

$$\langle\varphi_1|\varphi_2\rangle := \int_M \varphi_1\bar{\varphi}_2\Lambda_\Omega \tag{4}$$

for a pair φ_1, φ_2 of such a functions, where Λ_Ω is the *Liouville's volume form* $\Lambda_\Omega := (-1)^{\frac{1}{2}n(n-1)}\frac{1}{n!}\Omega^n$. With respect to this product, this algebra is a pre-Hilbert space. Denote by $\mathcal{C}(M)$ its completion [14].

Our first attempt is to take $\mathcal{C}(M)$ as the intrinsic Hilbert space of the system. Next, we want to define a set of self-adjoint operators $\mathcal{O}(\mathcal{C}(M))$ such that

I) There is a one to one correspondence between the set of classical observables $\Omega^0(M)$ and $\mathcal{O}(\mathcal{C}(M))$.

II) The map $f \mapsto O_f$ satisfies conditions (i-v) of Definition 4.

In order to achieve (I), the simplest way is to construct $\mathcal{O}(\mathcal{C}(M))$ from the set of *Hamiltonian vector fields* in M, $\mathcal{X}_H(M)$. Then, for every $X_f \in \mathcal{X}_H(M)$, we construct an (unique) operator $O_f \in \mathcal{O}(\mathcal{C}(M))$ which is defined as follows:

$$O_f := -i\hbar X_f,$$

where X_f acts linearly on $\mathcal{C}(M)$ as a derivation of (real) functions (i.e., taking into account that $C^\infty(M) = \Omega^0(M) \otimes \mathbb{C}$). Then, if $\psi \in \mathcal{C}(M)$, we have

$$O_f|\psi\rangle := -i\hbar X_f(\psi).$$

Nevertheless, the map

$$\begin{array}{rcccccc} O: & \Omega^0(M) & \longrightarrow & \mathcal{X}_H(M) & \longrightarrow & \mathcal{O}(\mathcal{C}(M)) \\ & f & \mapsto & X_f & \mapsto & O_f \end{array}$$

is not one to one since, if f is a constant function then $df = 0$ and $X_f = 0$, hence functions differing in a constant have the same associated operator. In addition, even though properties (i) and (ii) of Definition 4 are satisfied, (iii) fails to be true. In order to solve this, the simplest correction consists in adding an extra term in the above definition of O_f, writing:

$$O_f := -i\hbar X_f + f,$$

[14] Observe that $\mathcal{C}(M)$ coincides with the set of square integrable smooth complex-valued functions $\mathcal{L}^2(M) \cap C^\infty(M)$.

and so, for every $\psi \in \mathcal{C}(M)$,

$$O_f|\psi\rangle := -i\hbar X_f(\psi) + f\psi.$$

Now the properties (i-iii) hold but not (iv) because

$$\begin{aligned}[O_f, O_g]|\psi\rangle &= \hbar^2[X_f, X_g](\psi) - 2i\hbar\{g, f\}(\psi) \\ &= i\hbar(-i\hbar X_{\{f,g\}} + 2\{f, g\})(\psi) \neq i\hbar O_{\{f,g\}}|\psi\rangle.\end{aligned}$$

Therefore, a new correction is needed. Let $\theta \in \Omega^1(U)$, $(U \subset M)$, be a local *symplectic potential*, i.e., $\Omega = d\theta$ (locally). Then we define

$$O_f := -i\hbar\left(X_f + \frac{i}{\hbar}\langle X_f|\theta\rangle\right) + f, \tag{5}$$

and so, for every $\psi \in \mathcal{C}(M)$,

$$O_f|\psi\rangle := -i\hbar\left(X_f + \frac{i}{\hbar}\langle X_f|\theta\rangle\right)(\psi) + f\psi. \tag{6}$$

Now, (i-iv) hold (see the proof of Theorem 8) and, at the moment, this is the final form of the quantum operator associated with the classical observable f. Observe that this is a local construction and that the expression of this quantum operator depends on the choice of the local symplectic potential.

4.2. SPACE OF STATES: DEFINITIONS AND JUSTIFICATION. The necessity of introducing a local symplectic potential in order to have a correct definition of the quantum operators leads to a new difficulty. In fact, as we have said, the construction of the operators is local. This means that if U, U' are local charts of M and θ, θ' are the corresponding local symplectic potentials, then in the intersection they differ (locally) by an exact one-form, that is, $\theta' = \theta + d\alpha$, for some $\alpha \in \Omega^0(U \cap U')$ [15]. Observe that $O_f|\psi\rangle \neq O'_f|\psi\rangle$. Then, if we want the action of the quantum operator on the vector states to be independent on the choice of θ (that is, to be independent on the local chart), we have to impose that, if $\psi \in \mathcal{C}(M)$, then ψ and $\psi' = e^{\frac{i\alpha}{\hbar}}\psi$ must represent the same vector state of the intrinsic Hilbert space that we want to construct, because

$$\begin{aligned}O'_f|\psi'\rangle &= \left(-i\hbar\left(X_f + \frac{i}{\hbar}\langle X_f, \theta'\rangle\right) + f\right)e^{\frac{i\alpha}{\hbar}}\psi \\ &= e^{\frac{i\alpha}{\hbar}}\left(-i\hbar\left(X_f + \frac{i}{\hbar}\langle X_f, \theta\rangle\right) + f\right)\psi = e^{\frac{i\alpha}{\hbar}}O_f|\psi\rangle,\end{aligned}$$

[15] Physically, to change the local chart means that we are changing the local reference system of the observer, and we are allowing *gauge transformations*.

and then the relation between $O'_f|\psi'\rangle$ and $O_f|\psi\rangle$ is the same as between ψ' and ψ.

The geometrical meaning of this property of invariance is that the vector states $|\psi\rangle$ of the intrinsic Hilbert space cannot be just functions on M, but, according to the results in Section 3.4, sections on a *complex line bundle*, (L, π, M), with structural group $U(1)$. It is relevant to mention that every complex line bundle can be endowed with an hermitian metric h [38], which allows us to define an hermitian inner product in the complex vector space of smooth sections $\Gamma(L)$.

We can summarize this discussion in the following statement [71]:

REQUIREMENT 1. Let (M, Ω) be a symplectic manifold (which represents, totally or partially, the phase space of a physical system). In the geometric quantization programme the space of quantum states $\mathcal{H}_Q$ is constructed starting from the set of (smooth) sections, $\Gamma(L)$, of a *complex line bundle* over M, (L, π, M), with $U(1)$ as structural group.

Then the complex line bundle (L, π, M) is endowed with a smooth hermitian metric, $h\colon \Gamma(L) \times \Gamma(L) \to \mathbb{C}$. The *inner product* of sections (with compact support) is defined by

$$\langle\psi_1|\psi_2\rangle := \left(\frac{1}{2\pi\hbar}\right)^n \int_M h(\psi_1, \psi_2)\Lambda_\Omega. \tag{7}$$

As a consequence, a more subtle study concerning the geometrical structures involved in the quantization procedure is required. In fact, if the quantum states are constructed from sections of a line bundle (L, π, M), and since the quantum operators are defined from Hamiltonian vector fields in M, it is necessary to clarify how these vector fields "act" on sections of L.

It is obvious that the natural way is to introduce a connection ∇ in (L, π, M) and use it to associate a linear differential operator to each vector field on M, acting on sections of L. This operator is just the *covariant derivative* ∇_X. But, which kind of connections are suitable? Taking into account the expression (6) we can conclude that an immediate solution to our problem consists in taking the connection in such a way that, if $\psi \in \Gamma(L)$,

$$\nabla_{X_f}\psi := \left(X_f + \frac{i}{\hbar}\langle X_f, \theta\rangle\right)\psi, \tag{8}$$

and so

$$O_f|\psi\rangle := (-i\hbar\nabla_{X_f} + f)\psi,$$

(observe that the product $f\psi$ is well defined).

Notice that taking into account the expression (2), the equation (8), as a condition on ∇, is equivalent to demand (locally) that $\frac{\theta}{2\pi\hbar}$ is the *connection 1-form* ω of ∇. But, since locally $\Omega = d\theta$, this is equivalent to demanding that

$$\frac{\Omega}{2\pi\hbar} = \operatorname{curv}\nabla \equiv \bar{\Omega}. \tag{9}$$

We will use this condition later, in order to prove that the assignment $f \to O_f$ is a morphism of Lie algebras (see Section 4.3).

Hence, we can establish [71]:

REQUIREMENT 2. In the geometric quantization programme, the complex line bundle (L, π, M) must be endowed with a *connection* ∇ such that condition (9) holds.

In addition, the hermitian metric (introduced in Requirement 1) and the connection have to be *compatible* in the following sense: h is ∇-invariant, that is, if $X \in \mathcal{X}(M)$ and $\psi_1, \psi_2 \in \Gamma(L)$, then

$$X(h(\psi_1, \psi_2)) = h(\nabla_X\psi_1, \psi_2) + h(\psi_1, \nabla_X\psi_2),$$

(i.e., the connection is hermitian in relation to this metric).

We can summarize this discussion in the following:

DEFINITION 14. Let (M, Ω) be a symplectic manifold (which represents, totally or partially, the phase space of a physical system). A *prequantization* of this system is a complex line bundle (L, π, M) endowed with an hermitian metric h and a compatible connection ∇ such that $\frac{\Omega}{2\pi\hbar} = \operatorname{curv}\nabla$. If some prequantization exists for (M, Ω) we say that the system is *prequantizable* and the quintuple $(L, \pi, M; h, \nabla)$ is said to be a *prequantum line bundle* of the system.

At this point, the question is to study whether a classical system (M, Ω) is prequantizable or not. The answer to this question is given by the following theorems [1], [51], [71], [73]:

THEOREM 6. (Weil's theorem [88]) *Let (M, Ω) be a symplectic manifold. Then, the system is prequantizable if, and only if, $[\frac{\Omega}{2\pi\hbar}] \in H^2(M, \mathbb{Z})$, that is, $[\frac{\Omega}{2\pi\hbar}]$ is an integral cohomology class.* [16]

[16] $H^2(M, \mathbb{Z})$ is the image of $\check{H}^2(M, \mathbb{Z})$ by the following composition

$$0 \to \check{H}^2(M, \mathbb{Z}) \to \check{H}(M, \mathbb{R}) \xrightarrow{\sim} H^2(M, \mathbb{R}).$$

In the particular case of M being simply connected, then the connection ∇ and the compatible hermitian metric are unique (except equivalence relations).

Remarks.

- Notice that Ω must be a real form.
- The first part of this theorem is Theorem 2. The second part is a consequence of Section 3.6.
- The condition on the first part means by duality that the integral of $\frac{\Omega}{2\pi\hbar}$ over every integer 2-cocycle in M is integer. This *integrality condition* is called the *Bohr-Sommerfeld quantization condition.*

Another complementary result is the following:

THEOREM 7. *Let M be a manifold and (L, π, M) a complex line bundle endowed with a connection ∇ with local connection 1-form $\frac{\theta}{2\pi\hbar}$. Then, there exists a ∇-invariant hermitian metric in (L, π, M) if, and only if, $\theta - \bar{\theta}$ is an exact 1-form.*

Proof. See [51], p. 110. ∎

A new problem arises now: the integral (7) is not necessarily convergent. This means that $\Gamma(L)$ fails to be a Hilbert space because the norm $\|\psi\|$ is not defined for every $\psi \in \Gamma(L)$ and hence $\Gamma(L)$ cannot be the intrinsic Hilbert space. In order to avoid this difficulty, we can take the subset of $\Gamma(L)$ made of sections with compact support. This subset with the inner product (7) is a *pre-Hilbert space*. Then:

DEFINITION 15. The completion $\mathcal{H}_P$ of the set of sections with compact support in $\Gamma(L)$ is a Hilbert space which is called the *prequantum Hilbert space* [17]. The projective space $P\mathcal{H}_P$ is the *space of prequantum states.*

4.3. PREQUANTIZATION OPERATORS. Now, we can prove that the set of operators O_f given by

$$O_f := -i\hbar\nabla_{X_f} + f = -i\hbar\left(X_f + \frac{i}{\hbar}\langle X, \theta\rangle\right) + f,$$

satisfies part (b) of Definition 4.

[17]Remember that this is also the set of square integrable smooth sections in $\Gamma(L)$.

THEOREM 8. *The assignment* $f \mapsto O_f$ *is a* $\mathbb{R}$*-linear map from the Poisson algebra of the functions* $\Omega^0(M)$ *into the Lie algebra of the self-adjoint operators of* $\mathcal{H}_P$*, satisfying the properties* b(i-iv) *of Definition* 4.

Proof. In order to see that the operator O_f is self-adjoint we calculate:

$$\begin{aligned}
\langle O_f(\psi)|\psi'\rangle &= \int_M h(O_f(\psi),\psi')\Lambda_\Omega = \int_M h((-i\hbar\nabla_{X_f}+f)\psi,\psi')\Lambda_\Omega \\
&= \int_M -i\hbar X_f(h(\psi,\psi'))\Lambda_\Omega + \int_M i\hbar h(\psi,\nabla_{X_f}\psi')\Lambda_\Omega \\
&\quad + \int_M h(f\psi,\psi')\Lambda_\Omega \\
&= \int_M h(\psi,-i\hbar\nabla_{X_f}\psi')\Lambda_\Omega + \int_M h(\psi,f\psi')\Lambda_\Omega \\
&\quad + \int_M -i\hbar X_f(h(\psi,\psi'))\Lambda_\Omega \\
&= \int_M h(\psi,(-i\hbar\nabla_{X_f}+f)\psi')\Lambda_\Omega + \int_M -i\hbar X_f(h(\psi,\psi'))\Lambda_\Omega \\
&= \int_M h(\psi,O_f(\psi'))\Lambda_\Omega + \int_M -i\hbar X_f(h(\psi,\psi'))\Lambda_\Omega \\
&= \langle\psi|O_f(\psi')\rangle + \int_M -i\hbar X_f(h(\psi,\psi'))\Lambda_\Omega,
\end{aligned}$$

and it suffices to show that the last integral vanishes. But observe that this is an integral of the form $\int_M X_f(g)\Lambda_\Omega$, where $g \in C^\infty(M)$ is a function with compact support. We have that $d\Lambda_\Omega = 0$ and $d(g\Lambda_\Omega) = 0$, since they are maximal degree forms. Moreover, $L(X_f)\Omega = 0$, since X_f is a Hamiltonian vector field, hence

$$\begin{aligned}
(X_f g)\Lambda_\Omega &= L(X_f)(g\Lambda_\Omega - g\,L(X_f)\Lambda_\Omega = L(X_f)(g\Lambda_\Omega) \\
&= d\ i(X_f)(g\Lambda_\Omega) + i(X_f)d(g\Lambda_\Omega) = d\ i(X_f)(g\Lambda_\Omega),
\end{aligned}$$

and the integral of $d\,i(X_f)(g\Lambda_\Omega)$ vanishes because g is a function with compact support.

It is quite evident that the assignment is $\mathbb{R}$-linear and that it satisfies the three first properties.

For the last property we have:

$$
\begin{aligned}
[O_f, O_g](\psi) &= (O_f O_g - O_f O_g)(\psi) \\
&= (-i\hbar\nabla_{X_f} + f)(-i\hbar\nabla_{X_g} + g)(\psi) \\
&\quad - (-i\hbar\nabla_{X_g} + g)(-i\hbar\nabla_{X_f} + f)(\psi) \\
&= -\hbar^2(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})(\psi) - i\hbar(\nabla_{X_f}(g\psi) + f\nabla_{X_g}(\psi) + fg\psi \\
&\quad - \nabla_{X_g}(f\psi) - g\nabla_{X_f}(\psi) - gf\psi) \\
&= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}](\psi) - i\hbar(X_f(g)\psi + g\nabla_{X_f}(\psi) + f\nabla_{X_g}(\psi) \\
&\quad - X_g(f)\psi - f\nabla_{X_g}(\psi) - g\nabla_{X_f}(\psi)) \\
&= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}](\psi) + 2i\hbar\{f,g\}\psi \\
&= -\hbar^2\left(\nabla_{[X_f,X_g]} + \frac{i}{\hbar}\Omega(X_f, X_g)\right)(\psi) + 2i\hbar\{f,g\}\psi \\
&= (\hbar^2\nabla_{-[X_f,X_g]} - i\hbar\{f,g\} + 2i\hbar\{f,g\})(\psi) \\
&= (\hbar^2\nabla_{X_{\{f,g\}}} + i\hbar\{f,g\})(\psi) = i\hbar(-i\hbar\nabla_{X_{\{f,g\}}} + \{f,g\})(\psi) \\
&= i\hbar O_{\{f,g\}}(\psi),
\end{aligned}
$$

where we have taken into account the property given by Proposition 5. ∎

Observe that, for every $f \in \Omega^0(M)$, the operator O_f is well defined on the set of compact supported sections of the complex line bundle (L, π, M).

This is the *prequantization procedure* of Kostant, Souriau and Segal [51], [70], [75]. Summarizing, given the symplectic manifold (M, Ω), it consists in constructing a complex line bundle (L, π, M) endowed with an hermitian metric h and a compatible connection ∇ such that $\operatorname{curv}\nabla = \frac{\Omega}{2\pi\hbar}$.

4.4. Examples

4.4.1. Prequantization of cotangent bundles. In the particular case of M being a cotangent bundle T^*Q, we have a natural (global) symplectic potential, which has as local expression $\theta = -p_j dq^j$. The cohomology class of $\frac{\Omega}{2\pi\hbar}$ is zero and we have simply $L \simeq M \times \mathbb{C}$. Then, $\mathcal{H}_P \simeq \mathcal{C}(T^*Q)$ and a natural hermitian metric is defined by

$$h((m, z_1), (m, z_2)) := z_1\bar{z}_2,$$

which is compatible with the connection defined by θ.

4.4.2. PREQUANTIZATION OF THE HARMONIC OSCILLATOR. Next, as a typical example, we consider the n-dimensional harmonic oscillator. In this case we have:

$$M \equiv \{(q^j, p_j) \in \mathbb{R}^{2n}\}, \quad \Omega = dq^j \wedge dp_j, \quad H = \frac{1}{2}(p_j^2 + q^{j^2}).$$

The conditions assuring the prequantization of the system hold since $[\frac{\Omega}{2\pi\hbar}] = 0$ (so it is integer), and $L = M \times \mathbb{C}$, since M is contractible.

The usual metric is the product. If ψ is a section of L, the hermitian connection is given by

$$\nabla_X \psi = X\psi + 2\pi i\omega(X)\psi = X\psi + \frac{i}{\hbar}\theta(X)\psi.$$

If we take as a (local) symplectic potential $\theta = \frac{1}{2}(q^j dp_j - p_j dq^j)$, the quantum operators that we obtain are:

$$O_{q^j} = i\hbar\frac{\partial}{\partial p_j} + \frac{1}{2}q^j, \quad O_{p_j} = -i\hbar\frac{\partial}{\partial q^j} + \frac{1}{2}p_j,$$

and then, with this choice for the symplectic potential,

$$O_H = -i\hbar\left(p_j\frac{\partial}{\partial q^j} - q^j\frac{\partial}{\partial p_j}\right),$$

therefore the action of the operator associated with the energy is

$$O_H(\psi) = i\hbar\{H, \psi\}.$$

Then the classical and prequantum dynamics coincide and this shows that prequantization is not the suitable method for quantizing this classical system, because the spectrum of the energy operator is continuous and this is not true for the quantized harmonic oscillator.

4.4.3. AN ENLIGHTENING CASE. Finally, the following case reveals another problem still remaining at the end of this stage. Consider a cotangent bundle T^*Q, where Q is a compact manifold we have again:

$$\Omega = dq^j \wedge dp_j, \quad \mathcal{H}_P \simeq \mathcal{C}(T^*Q),$$

and we can take $L = T^*Q \times \mathbb{C}$ because $[\frac{\Omega}{2\pi\hbar}] = 0$. Taking as a symplectic potential $\theta = q^j dp_j$, we obtain the following quantum operators:

$$O_{q^j} = -i\hbar\frac{\partial}{\partial p_j}, \quad O_{p_j} = i\hbar\frac{\partial}{\partial q^j} + p_j.$$

Now, if we take the closed subset of $\mathcal{C}(T^*Q)$ made of the functions which are constant on the basis Q, that is, the set $\mathcal{C}_Q(T^*Q) \equiv \{f(p_j)\}$, we have that, for every $f \in \mathcal{C}_Q(T^*Q)$,

$$O_{q^j}(f) = -i\hbar \frac{\partial f}{\partial p_j} \in \mathcal{C}_Q(T^*Q), \quad O_{p_j}(f) = i\hbar \frac{\partial f}{\partial q^j} + p_j f \in \mathcal{C}_Q(T^*Q),$$

hence, $\mathcal{C}_Q(T^*Q)$ is an invariant subspace under the action of this set of operators or, what means the same thing, $\mathcal{H}_P$ is not irreducible by this representation.

Notice that this conclusion can also been reached observing that the operators $\frac{\partial}{\partial q^j}$ and $\frac{\partial}{\partial p_j} - \frac{i}{\hbar} q^j$ commute with O_{q^j} and O_{p_j}. Hence, the last ones do not form a complete set of commuting observables and, as a consequence, there exists some non-empty closed subspace of $\mathcal{H}_P$ different from $\mathcal{H}_P$ which is invariant with respect to their action. So that, the condition (v) of Definition 4 does not hold for these systems.

5. Polarizations

$\mathcal{H}_P$ is not the suitable choice as the intrinsic Hilbert space of the system (or, what is the same thing, $P\mathcal{H}_P$ is not the true space of quantum states). In fact, as we have seen in the last example, in general, in the set $\mathcal{O}(\mathcal{H}_P)$ (whose elements are the operators O_f defined in the equation (5)) we can find that, from a complete set of classical observables, we get a set of quantum observables which does not satisfy condition (v) of Definition 4 (it is not complete). The origin of this problem is the following: if M is the phase space of the classical system and $\dim M = 2n$, then the prequantum states in $P\mathcal{H}_P$ depend on $2n$ variables, but according to Quantum Mechanics, the true quantum states depend just on n variables (the dimension of the configuration space).

In order to solve this problem the idea is to "restrict" the Hilbert space $\mathcal{H}_P$ and the set of quantum operators. Then, a new geometric structure is defined in (M, Ω): *polarizations* [18].

We devote this section to justify, define and develop the main features related to this concept, which are relevant for geometric quantization. A more extensive study on (real and complex) polarizations can be found in [92]. Other interesting references are [73] and [76].

[18] Sometimes, this structure is called *Planck's foliation* [1].

5.1. PREVIOUS JUSTIFICATIONS. In the case $M = T^*\mathbb{R}^n$, (q^i, p_i) is a global set of canonical coordinates. According to Quantum Mechanics, the wave functions depend only on (q^i) (or only on (p_i)). Then we must select n coordinates and remove them in order to obtain the space of wave functions. Next, we are going to generalize this idea.

The first step of the idea lies in selecting n directions in M by means of the choice of a n-dimensional *distribution* $\mathcal{P}$ in TM. Then we will require that the sections of the prequantum line bundle (L, π, M) representing the quantum states are invariant by the transformations induced in M by this distribution, that is,

$$\nabla_{\mathcal{P}}\psi = 0. \tag{10}$$

Now, let $U \subset M$ be an open set in which L trivializes and let $s\colon U \to L$ be a trivializing section. A section $\psi = fs$ satisfying that $\nabla_{\mathcal{P}}\psi = 0$ have to satisfy that $(\mathcal{P}(f) + \frac{i}{\hbar}\langle\mathcal{P}, \theta\rangle)f)s = 0$. If we take the distribution to be *adapted to the connection* ∇, that is, such that there exists a local symplectic potential θ satisfying that $\langle\mathcal{P}, \theta\rangle = 0$, then the above equations reduce to be

$$\mathcal{P}(f) = 0.$$

This is a system of n independent linear partial differential equations and, in order to assure that it is integrable, it suffices to demand that the distribution $\mathcal{P}$ is *involutive*.

So, f is constant along the fibers of $\mathcal{P}$ and therefore ψ is represented as a constant function along the fibers of the distribution. In this way, the states represented by the sections ψ which are solutions of the equations (10) are represented by functions which depend just on n variables, in the following sense: being $\mathcal{P}$ involutive, by the Fröbenius theorem, there exist local coordinate systems $\{x_i, y_i\}_{i=1}^n$ such that $\mathcal{P}$ is spanned by $\left\{\frac{\partial}{\partial x_i}\right\}$; then $\{x_i\}$ are local coordinates of the integral manifolds of $\mathcal{P}$, which are defined by $y_i = \text{const}$. Therefore $\mathcal{P}(f) = 0$ is locally equivalent to $\frac{\partial f}{\partial x_i} = 0$, so f depends only on the variables $\{y_i\}$.

But now, an additional consistency condition is required: equation (10) implies that, if $\frac{\Omega}{2\pi\hbar}$ is the curvature form of the connection, we have

$$0 = [\nabla_{\mathcal{P}}, \nabla_{\mathcal{P}}]\psi = \left(\nabla_{[\mathcal{P},\mathcal{P}]} + \frac{i}{\hbar}\Omega(\mathcal{P}, \mathcal{P})\right)\psi,$$

then, if the distribution is involutive, $\nabla_{[\mathcal{P},\mathcal{P}]}\psi = 0$, and therefore it must be $\Omega(\mathcal{P}, \mathcal{P})\psi = 0$. This last condition is assured if we impose that the distribution is *isotropic* but, since it is n-dimensional, it would be maximal isotropic, that is *Lagrangian*. Thus, at the moment, what we need is an *involutive Lagrangian distribution.*

Suppose $\mathcal{P}$ is spanned by a set of *global Hamiltonian vector fields* $\{Y_{f_j}\}_{j=1}^n$. The isotropy condition $\Omega(Y_{f_l}, Y_{f_j}) = 0$ can be written in an equivalent way as

$$\{f_l, f_j\} = 0. \tag{11}$$

Conversely, if $\{f_j\}$ is a set of n independent global functions such that equation (11) holds, then their Hamiltonian vector fields span an involutive Lagrangian distribution on M. Nevertheless, for many physical systems, to choose a set of n independent global functions satisfying the equation (11) is not always possible and, for this reason, our construction must be more general.

Finally, another consideration is needed: instead of working with real distributions we will consider *complex distributions*, that is, locally spanned by complex vector fields. In other words, we are going to work in the *complexified tangent bundle* $TM^{\mathbb{C}}$ instead of TM [19]. There are mainly two reasons for doing so [20]:

1. First, as we will see later, there is a kind of distributions which are specially interesting for quantization (those which we will call *Kähler polarizations*), and they are complex distributions.
2. Second, complex polarizations are necessary to establish the relationship between geometric quantization and the theory of irreducible unitary representations of Lie groups of symmetries [49], [90].

5.2. Definitions and properties

5.2.1. Complex distributions and polarizations.

Taking into account the above comments, we define:

DEFINITION 16. Let (M, Ω) be a symplectic manifold. A *complex polarization* $\mathcal{P}$ on (M, Ω) is a distribution in $TM^{\mathbb{C}}$ such that:

a) It is *Lagrangian*, that is

[19] Keep in mind that, for every $m \in M$, $T_mM^{\mathbb{C}} \simeq T_mM \otimes \mathbb{C}$.

[20] In addition, in order to make the formalism more coherent, if we are considering complex Hilbert spaces and complex functions in M, it seems reasonable working with $TM^{\mathbb{C}}$, and then taking complex distributions.

i) $\forall m \in M$, $\dim \mathcal{P}_m = n$, (complex dimension)
ii) $\Omega(\mathcal{P}, \mathcal{P}) = 0$ [21].

b) It is *involutive*, that is

iii) $[\mathcal{P}, \mathcal{P}] \subset \mathcal{P}$.

c) It satisfies the additional condition

iv) $\dim(\mathcal{P}_m \cap \bar{\mathcal{P}}_m \cap T_m M)$ is constant for every $m \in M$.

Observe that if $\mathcal{P}$ is a polarization, so is $\bar{\mathcal{P}}$ [22].
In relation to these four conditions we can remark that:

- Condition (i) is stated in order to reduce the number of independent variables just to n.
- Every complex polarization $\mathcal{P}$ induces a real isotropic distribution which is also involutive. In fact, as $\mathcal{P} \cap \bar{\mathcal{P}}$ is a complex distribution invariant by conjugation, then $D := \mathcal{P} \cap \bar{\mathcal{P}} \cap TM$ is a real distribution satisfying that $D^{\mathbb{C}} = \mathcal{P} \cap \bar{\mathcal{P}}$; and it is called the *isotropic distribution*. Furthermore, the fact of $\mathcal{P}$ being involutive (as it is required in condition (iii)) implies that D is also involutive.
- Hence the *Frobenius theorem* assures that D defines a foliation on M and, as a consequence of the isotropy condition (ii) of Definition 16, the tangent bundle D_m of every integrable manifold can be locally spanned by vector fields whose Lagrange brackets are equal to zero. We denote by $\mathcal{D} := M/D$ the space of the integrable manifolds of D and by $\pi_D : M \to \mathcal{D}$ the projection.
- Observe that, due to condition (iv), the leaves of the foliation defined by D have the same dimension. A remaining question is if the space $\mathcal{D}$ is a differentiable manifold.
- Sometimes the involution condition (iii) of Definition 16 is replaced by the following *integrability condition*: there exists a local family of complex functions $\{z_j\}_{j=1}^n \in C^\infty(U)$, $(U \subset M)$, such that $\mathcal{P}$ is locally spanned by the set of locally Hamiltonian vector fields $\bar{X}_{z_j}$ [92].

[21] Ω is extended to $TM^{\mathbb{C}}$ as a $\mathbb{C}$-bilinear form.
[22] Sometimes we commit an abuse of notation denoting also by $\mathcal{P}$ the set of (complex) vector fields taking values on the distribution $\mathcal{P}$.

- As we are going to see in the following subsection, for a certain type of polarizations (Kähler polarizations), there exists a local family of complex functions $\{z_j\}_{j=1}^n \in C^\infty(U)$, $(U \subset M)$, such that $\mathcal{P}$ is locally spanned by the set of vector fields $\frac{\partial}{\partial \bar{z}_j}$ (*Nirenberg-Newlander theorem* [42], [65]).
- Finally, consider the complex distribution $\mathcal{P} + \bar{\mathcal{P}}$. Since it is invariant by conjugation, it can be considered as the complexification of a real distribution E; that is, $E^{\mathbb{C}} = \mathcal{P} + \bar{\mathcal{P}}$; where $E = (\mathcal{P} + \bar{\mathcal{P}}) \cap TM$, which is called the *coisotropic distribution.* Notice that $D^\perp = E$ [23], as it can be easily proved [49].

  Then we define:

DEFINITION 17. Let (M, Ω) be a symplectic manifold. A polarization $\mathcal{P}$ is called *strongly integrable* or *reducible* if:

i) E is involutive.

ii) Both spaces of integral manifolds $\mathcal{D}$ and $\mathcal{E} := M/E$ are differentiable manifolds.

iii) The canonical projection $\pi_{DE} : \mathcal{D} \to \mathcal{E}$ is a submersion.

Given a complex polarization $\mathcal{P}$ on (M, Ω), for every $m \in M$, we can define a (pseudo) hermitian form $\hat{h}_m$ in $\mathcal{P}_m$ by

$$\hat{h}_m(X_m, Y_m) := i\Omega_m(X_m, \bar{Y}_m) \tag{12}$$

for every $X_m, Y_m \in P_m$. We have that:

PROPOSITION 17. $\ker \hat{h} = \mathcal{P} \cap \bar{\mathcal{P}}$.

Proof. In fact $\mathcal{P} \cap \bar{\mathcal{P}} \subset \ker \hat{h}$ because $\mathcal{P}$ is isotropic. On the other side, for every $X \in \mathcal{P}$, if $X \in \ker \hat{h}$, then $\Omega(X, \bar{Y}) = 0$, for all $Y \in \mathcal{P}$, that is, $\Omega(X, Y) = 0$, for all $Y \in \bar{P}$, hence $X \in \bar{\mathcal{P}}$ because $\mathcal{P}$ is Lagrangian and so is $\bar{\mathcal{P}}$. ∎

Consequently, $\hat{h}$ projects onto a non-degenerate form on the quotient $\mathcal{P}/(\mathcal{P} \cap \bar{\mathcal{P}})$. We denote this form by $\bar{h}$. Then:

[23] $D^\perp$ denotes the *orthogonal symplectic complement* of D; that is, $D^\perp := \{X \in \mathcal{X}(M) : \forall Y \in D,\ \Omega(X, Y) = 0\}$.

DEFINITION 18. Let $\mathcal{P}$ be a complex polarization on (M, Ω). $\mathcal{P}$ is said to be *of type* (r, s) if the form $\bar{h}$ has signature (r, s). Then, $\mathcal{P}$ is said to be *positive* if $s = 0$.

Observe that, if $\mathcal{P}$ is of type (r, s), then $\bar{\mathcal{P}}$ is of type (s, r).

Since the complex dimension of $\ker \hat{h}$ is $n - (r + s)$ and $\mathcal{P} \cap \bar{\mathcal{P}}$ is the complexification of the real distribution D, then this number is the dimension of the distribution D in TM. For this reason $n - (r + s) \equiv n - l$ is called the *number of real directions* in $\mathcal{P}$. On the other hand, from the definition and properties of D and E, it can be proved that the dimension of $E = D^{\perp}$ is $n + l$ (and hence $\dim D + \dim E = 2n$).

Furthermore, if $\mathcal{P}$ is strongly integrable then, in every subset $U \subset M$, there is a set of local coordinates $\{x_1, \dots, x_{n-l}; y_1, \dots, y_{n-l}; u_1, \dots, u_l; v_1, \dots, v_l\}$ such that D is generated by the vector fields $\left\{\frac{\partial}{\partial x_j}\right\}_{j=1}^{n-l}$ and $\mathcal{P}$ by $\left\{\frac{\partial}{\partial x_j}\right\}_{j=1}^{n-l}$, $\left\{\frac{\partial}{\partial \bar{z}_k}\right\}_{k=1}^{l}$, with $z_k := u_k + i v_k$ [48].

Next we introduce the condition of $\mathcal{P}$ to be *adapted* to the connection:

DEFINITION 19. Let (M, Ω) be a symplectic manifold and (L, π, M) a complex line bundle endowed with a hermitian connection ∇. A polarization is called *admissible* for the connection ∇ (or *adapted* to the connection) if, in some neighbourhood of each point $m \in M$, there is a symplectic potential θ (i.e., $\omega = 2\pi i \theta$ is a connection form of ∇) such that $\langle \mathcal{P}, \theta \rangle = 0$.

In relation to this concept, we remark here the following result:

PROPOSITION 18. *Let (M, Ω) be a symplectic manifold and (L, π, M) a complex line bundle endowed with a hermitian connection ∇. If $\mathcal{P}$ is a strongly integrable polarization then it is admissible for the connection ∇.*

Proof. See [67]. ∎

Strongly integrable polarizations will be the only ones we are going to consider from now on.

Due to their relevance for quantization, we devote the following subsections to studying two particular cases which are specially interesting.

5.2.2. KÄHLER POLARIZATIONS. The first important case of polarization is the following:

DEFINITION 20. Let (M,Ω) be a symplectic manifold. A polarization $\mathcal{P}$ on M is said to be a *Kähler polarization* if $\mathcal{P}\cap\bar{\mathcal{P}}=0$.

For a Kähler polarization $\mathcal{P}$ we have that $TM^{\mathbb{C}}=\mathcal{P}\oplus\bar{\mathcal{P}}$ and $\pi_{DE}\colon\mathcal{D}=M\to\mathcal{E}$. Hence, if $\mathcal{P}$ is of type (r,s), we have that $r+s=n$ (and, thus, a Kähler polarization is "totally" complex, in the sense that it does not contain any real direction).

In order to analyze the properties of this kind of polarizations, we need to introduce several new concepts (see [40], [43], [66], [92]).

DEFINITION 21. Let M be a differentiable manifold. An *almost-complex structure* in M is a $(1,1)$-tensor field $\mathcal{J}$ on M such that ${\mathcal{J}_m}^2=-Id_{T_mM}$ for every $m\in M$. If $\mathcal{J}$ exists, then $(M,\mathcal{J})$ is said to be an *almost-complex manifold*.

Remark. The existence of an almost-complex structure implies that the dimension of M is even. Hereafter, $\{U;x_1,\dots,x_n,y_1,\dots,y_n\}$ denote a chart of coordinates on M.

$\mathcal{J}_m$ can be extended to a $\mathbb{C}$-linear endomorphism in $T_mM^{\mathbb{C}}$. Consider now the set of complex analytical coordinates $\{z_j:=x_j+iy_j\}$, then a basis of $T_mM^{\mathbb{C}}$ is made of the vectors

$$\left(\frac{\partial}{\partial z_j}\right)_m=\frac{1}{2}\left(\left(\frac{\partial}{\partial x_j}\right)_m-i\left(\frac{\partial}{\partial y_j}\right)_m\right),$$
$$\left(\frac{\partial}{\partial \bar{z}_j}\right)_m=\frac{1}{2}\left(\left(\frac{\partial}{\partial x_j}\right)_m+i\left(\frac{\partial}{\partial y_j}\right)_m\right).$$

Notice that, if $\mathcal{J}$ is an almost-complex structure, then the extension of the endomorphism $\mathcal{J}_m$ to $T_mM^{\mathbb{C}}$ has eigenvalues $\pm i$. Then, $T_mM^{\mathbb{C}}$ can be decomposed into a direct sum:

$$T_mM^{\mathbb{C}}=T_m^{(1,0)}M\oplus T_m^{(0,1)}M,$$

where

$$T_m^{(1,0)}M:=\{X_m\in T_mM^{\mathbb{C}}\colon\ \mathcal{J}_mX_m=+iX_m\},$$
$$T_m^{(0,1)}M:=\{X_m\in T_mM^{\mathbb{C}}\colon\ \mathcal{J}_mX_m=-iX_m\}.$$

Vectors belonging to these sets (and the corresponding vector fields, denoted $\mathcal{X}^{(1,0)}(M)$ and $\mathcal{X}^{(0,1)}(M)$) are called *vectors* (*vector fields*) of $(1,0)$-*type* and *vectors* (*vector fields*) of $(0,1)$-*type*, respectively [24]. Let $\bar{\mathcal{P}}$ and $\mathcal{P}$ be the complex distributions of $TM^{\mathbb{C}}$ defined by $T_m^{(1,0)}M$ and $T_m^{(0,1)}M$ respectively for every $m \in M$.

DEFINITION 22. An almost complex structure in M is said to be a *complex structure* if the complex distributions $\mathcal{P}$ and $\bar{\mathcal{P}}$ are involutive and n-dimensional.

This is equivalent to say ([66, p. 379]) that there are sets of local complex analytical coordinates $(z_j, \bar{z}_j)$ such that, for every $m \in M$,

$$\mathcal{J}_m \left(\frac{\partial}{\partial z_j}\right)_m = i \left(\frac{\partial}{\partial z_j}\right)_m, \quad \mathcal{J}_m \left(\frac{\partial}{\partial \bar{z}_j}\right)_m = -i \left(\frac{\partial}{\partial \bar{z}_j}\right)_m,$$

or, what is equivalent, there exist local charts of coordinates $\{U; x_j, y_j\}$ such that

$$\mathcal{J}_m \left(\frac{\partial}{\partial x_j}\right)_m = \left(\frac{\partial}{\partial y_j}\right)_m, \quad \mathcal{J}_m \left(\frac{\partial}{\partial y_j}\right)_m = -\left(\frac{\partial}{\partial x_j}\right)_m.$$

Then $(M, \mathcal{J})$ is a *complex manifold*.

If $\mathcal{J}$ is a complex structure, then $\left\{\frac{\partial}{\partial z_j}\right\}$ and $\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$ are local basis for $\mathcal{X}^{(1,0)}(M)$ and $\mathcal{X}^{(0,1)}(M)$), that is, for $\bar{\mathcal{P}}$ and $\mathcal{P}$, respectively.

Then we can define:

DEFINITION 23. An *almost-Kähler manifold* is a triple $(M, \Omega, \mathcal{J})$ where:

i) (M, Ω) is a symplectic manifold.
ii) $(M, \mathcal{J})$ is an almost-complex manifold.
iii) The almost-complex structure and the symplectic form are compatible, that is, for every $X, Y \in \mathcal{X}(M)$,

$$\Omega(\mathcal{J}X, \mathcal{J}Y) = \Omega(X, Y).$$

In this case, the distributions $\mathcal{P}$ and $\bar{\mathcal{P}}$ are isotropic, v.g.: if $X; Y \in T_m^{(1,0)}M$ then

$$\Omega(X,Y) = \Omega(\mathcal{J}X.\mathcal{J}Y) = \Omega(iX, iY) = -\Omega(X,Y) \quad \Rightarrow \quad \Omega(X,Y) = 0,$$

and, since $\dim \mathcal{P} + \dim \bar{\mathcal{P}} = 2n$, they are Lagrangian.

[24]Observe that, for every $X \in \mathcal{X}(M)$ we have that $X^+ := \frac{1}{2}(X - i\mathcal{J}X) \in \mathcal{X}^{(1,0)}(M)$ and $X^- := \frac{1}{2}(X + i\mathcal{J}X) \in \mathcal{X}^{(0,1)}(M)$, and hence $X = X^+ + X^-$, so $\mathcal{X}(M) \subset \mathcal{P} \otimes \bar{\mathcal{P}}$.

DEFINITION 24. A *Kähler manifold* is a triple $(M, \Omega, \mathcal{J})$ where:

i) (M, Ω) is a symplectic manifold.
ii) $(M, \mathcal{J})$ is a complex manifold.
iii) The complex structure and the symplectic form are compatible, that is, for every $X, Y \in \mathcal{X}(M)$,

$$\Omega(\mathcal{J}X, \mathcal{J}Y) = \Omega(X, Y).$$

Equivalently, we can say that a Kähler manifold is an almost-Kähler manifold for which the complex distributions $\mathcal{P}$ and $\bar{\mathcal{P}}$ are integrable.

There is a close relation between Kähler manifolds and Kähler polarizations, which justifies the name of these last ones. It is stated in the following proposition:

PROPOSITION 19. *If $(M, \Omega, \mathcal{J})$ is a Kähler manifold, then $\mathcal{P}$ and $\bar{\mathcal{P}}$ are Kähler polarizations, called the holomorphic polarization and the antiholomorphic polarization respectively.*

Conversely, if (M, Ω) is a symplectic manifold which carries a Kähler polarization $\mathcal{P}$, then there is a complex structure $\mathcal{J}$ defined in M which is compatible with Ω, and therefore, $(M, \Omega, \mathcal{J})$ is a Kähler manifold.

Proof. The first part of the proposition is immediate. For the second one we have that if $\mathcal{P}$ is a Kähler polarization, as $T_mM \subset T_mM^{\mathbb{C}} = \mathcal{P}_m \oplus \bar{\mathcal{P}}_m$, then we can define an almost-complex structure $\mathcal{J}$ as follows: for every $X_m \in T_mM$ we can write $X_m = Y_m + \bar{Y}'_m$ for some $Y_m \in \mathcal{P}_m$ and $\bar{Y}'_m \in \bar{\mathcal{P}}_m$, and then

$$\mathcal{J}_m X_m := iY_m - i\bar{Y}'_m.$$

The differentiability of $\mathcal{J}$ follows from its expression in local coordinates. On the other hand, $\mathcal{J}$ is compatible with Ω because, if $X_m, Z_m \in T_mM$, then

$$\begin{aligned}
\Omega_m(\mathcal{J}_m X_m, \mathcal{J}_m Z_m) &= \Omega_m(\mathcal{J}_m(Y_m + \bar{Y}'_m), \mathcal{J}_m(W_m + \bar{W}'_m)) \\
&= \Omega_m(iY_m - i\bar{Y}'_m, iW_m - i\bar{W}'_m) \\
&= \Omega_m(Y_m, \bar{W}'_m) + \Omega_m(\bar{Y}'_m, W_m) \\
&= \Omega_m(Y_m + \bar{Y}'_m, W_m + \bar{W}'_m) = \Omega_m(X_m, Z_m).
\end{aligned}$$

Hence, $(M, \Omega, \mathcal{J})$ is an almost- Kähler manifold. Finally, $\mathcal{P}$ is integrable by definition and then $(M, \Omega, \mathcal{J})$ is a Kähler manifold (for which $\mathcal{P}$ is its holomorphic polarization). ∎

On the other hand, the following property holds:

PROPOSITION 20. *In every Kähler manifold* $(M, \Omega, \mathcal{J})$ *there is a non-degenerate (pseudo) hermitian metric* k *defined on* M *in the following way: For every* $X, Y \in \mathcal{X}(M)$,

$$k(X,Y) := g(X,Y) - i\Omega(X,Y),$$

where g *is a (pseudo) Riemannian metric given by*

$$g(X,Y) := \Omega(X, \mathcal{J}Y).$$

This hermitian metric is "compatible with $\mathcal{J}$*", that is:*

$$k(\mathcal{J}X, \mathcal{J}Y) = k(X,Y).$$

Proof. To prove that this is a non-degenerate hermitian metric is immediate. For the compatibility with $\mathcal{J}$, remember that Ω is compatible with $\mathcal{J}$, hence

$$\begin{aligned} k(\mathcal{J}X, \mathcal{J}Y) &:= g(\mathcal{J}X, \mathcal{J}Y) - i\Omega(\mathcal{J}X, \mathcal{J}Y) = \Omega(\mathcal{J}X, \mathcal{J}\mathcal{J}Y) - i\Omega(X,Y) \\ &= \Omega(X, \mathcal{J}Y) - i\Omega(X,Y) = g(X,Y) - i\Omega(X,Y) := k(X,Y). \end{aligned}$$

∎

Remarks.

- It is interesting to point out that if k is not positive definite then nor is g and therefore the corresponding Kähler polarization $\mathcal{P}$ is of type (r, s), with $s \neq 0$. In the same way k is positive if, and only if, so are g and $\mathcal{P}$. Sometimes, in the literature, the term "Kähler polarization" is applied only when this last condition holds and the other ones are called *pseudo-Kähler*.
- Observe that this proposition allows to state that, in every Kähler manifold, the symplectic form can be taken as the imaginary part of a non-degenerate (pseudo) hermitian metric which is compatible with the complex structure.

In relation to the converse problem we have:

DEFINITION 25. Let $(M, \mathcal{J})$ be an almost-complex manifold endowed with a (pseudo) hermitian metric k defined in M, which is compatible with $\mathcal{J}$. Then we can define a (skew-symmetric) non-degenerate 2-form Ω by

$$\Omega(X, Y) := \operatorname{Img}(k(X, Y)),$$

which is called the *fundamental form* of $(M, \mathcal{J})$.

If Ω is closed (i.e., symplectic), then it is called a *Kähler form*. In this case, $(M, \Omega, \mathcal{J})$ is an almost-Kähler manifold and, in addition, if $\mathcal{J}$ is a complex structure, then $(M, \Omega, \mathcal{J})$ is a Kähler manifold.

It is possible to find a local smooth function $f(z, \bar{z})$ such that the local expression of this Kähler form is

$$\Omega = i\hbar \frac{-\partial^2 f}{\partial \bar{z}_j \partial z_k} d\bar{z}_j \wedge dz_k := i\hbar \bar{\partial} \partial f,$$

(where we have introduced the notation $\partial := \frac{\partial}{\partial z_j} dz_j$, $\bar{\partial} := \frac{\partial}{\partial \bar{z}_k} d\bar{z}_k$ [50], [63], [66] and their composition is in the sense defined in [91]). Then the most natural local symplectic potential is

$$\Theta = \frac{i\hbar}{2}(\partial f - \bar{\partial} f).$$

Taking these local expressions into account, it is immediate to prove that:

PROPOSITION 21. *Let $(M, \Omega, \mathcal{J})$ be a Kähler manifold and (L, π, M, ∇, h) the corresponding prequantum line bundle. Then, there are local symplectic potentials θ and $\bar{\theta}$ adapted to the holomorphic and antiholomorphic polarizations $\bar{\mathcal{P}}$ and P, respectively, (that is, these polarizations are admissible for the connection ∇). Their local expressions are:*

$$\theta = i\hbar \partial f = i\hbar \frac{\partial f}{\partial z_j} dz_j, \quad \bar{\theta} = -i\hbar \bar{\partial} f = -i\hbar \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

5.2.3. REAL POLARIZATIONS. The other interesting case is the opposite one to the above.

DEFINITION 26. Let (M, Ω) be a symplectic manifold. A polarization $\mathcal{P}$ on M is said to be a *real polarization* if $\mathcal{P} = \bar{\mathcal{P}}$.

As you can observe, every real polarization is always positive, as $\hat{h} = 0$; and $D = E$.

Notice that, if $\mathcal{P}$ is a real polarization, it holds that $\mathcal{P} \cap \bar{\mathcal{P}} = \mathcal{P}$. Consider $D \equiv \mathcal{P} \cap TM$, that is, the real elements of $\mathcal{P}$. D is a Lagrangian distribution in TM. Conversely, if D is a Lagrangian distribution of TM, then its complexification $D^{\mathbb{C}}$ is a real polarization. Hence, the fact of considering real polarizations in M is equivalent to take Lagrangian distributions in TM. Therefore, M is foliated by Lagrangian submanifolds.

In addition the following result holds:

PROPOSITION 22. *If $\mathcal{P}$ is a real polarization, then there exist a local basis of D made up of Hamiltonian vector fields.*

Proof. Let $\{X_1, \dots, X_n\}$ be a basis of D in an open set $V \subset M$. The differential 1-forms $i(X_j)\Omega$ $(j = 1, \dots, n)$, are linearly independent and vanish on D over V, since D is Lagrangian. Therefore, in the open set V, the submodule incident to the one generated by these forms is D. Let $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ be a local system of coordinates in $U \subseteq V$ such that

$$\{X_1, \dots, X_n\} \equiv \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\},$$

and let $\{Z_1, \dots, Z_n\}$ be the vector fields defined in U by $i(Z_j)\Omega = dy_j$. The existence of these local systems is guaranteed because D is an involutive distribution. We have that

$$\{dy_1, \dots, dy_n\}' = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\} = \{X_1, \dots, X_n\},$$

(where $\{\ \}'$ denotes the submodule incident to $\{\ \}$), but

$$\{dy_1, \dots, dy_n\} = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}' = \{i(X_1)\Omega, \dots, i(X_n)\Omega\},$$

then, since

$$\{dy_1, \dots, dy_n\} = \{i(Z_1)\Omega, \dots, i(Z_n)\Omega\},$$

it follows that $\{X_1, \dots, X_n\} = \{Z_1, \dots, Z_n\}$ because Ω is symplectic. Hence, the vector fields $\{Z_l\}$ make a basis satisfying the conditions of the statement. ∎

As a consequence of the isotropy condition (ii) of Definition 16; and taking into account the above proposition, you can prove that the Poisson brackets between every two coordinates of the Lagrangian submanifolds which foliate M vanishes.

5.3. THE POLARIZATION CONDITION ON THE STATES

5.3.1. THE GENERAL SITUATION. Once the concept of polarization is established, we use it in order to correctly define the quantum states. Thus, reminding Definition 4, we claim:

REQUIREMENT 3. (and DEFINITION) Let (M, Ω) be a symplectic manifold. Let $(L, \pi, M; h, \nabla)$ be the prequantum line bundle obtained after the prequantization procedure. Let $\mathcal{P}$ be a strongly integrable polarization of (M, Ω). The space of quantum states of the system is constructed starting from the set of smooth sections $\Gamma(L)$ which are *covariantly constant* along $\mathcal{P}$; that is, such that

$$\nabla_{\mathcal{P}} \psi = 0. \tag{13}$$

We call them *polarized sections* (related to $\mathcal{P}$) and denote this set by $\Gamma_{\mathcal{P}}(L)$.

From this condition it can be observed that, if we take local symplectic potentials such that $\langle \mathcal{P}, \theta \rangle = 0$, then the polarized sections are constant along the leaves of the foliation induced in M by D, in the sense which we have discussed in the beginning of this section. In particular, using the local basis of coordinates introduced in the comments after Definition 18, we have that $\mathcal{P}$ is locally spanned by the set of vector fields $\left\{\frac{\partial}{\partial x_j}\right\}_{j=1}^{n-l}$, $\left\{\frac{\partial}{\partial z_k}\right\}_{k=1}^{l}$ (with $z_k := u_k + i v_k$) and then, condition (13) implies that the polarized sections are represented by functions which only depend on the coordinates $\{y_j\}$, $\{\bar{z}_k\}$.

The last requirement introduces new complications in the geometric quantization scheme:

- In general, $\Gamma_{\mathcal{P}}(L) \not\subset \mathcal{H}_P$ since the polarized sections are not necessarily square integrable.
- Even more, if $\psi_1, \psi_2 \in \Gamma_{\mathcal{P}}(L)$, then $h(\psi_1, \psi_2)$ is a function which is constant along the leaves of the foliation induced by D, but the inner product $\langle \psi_1 | \psi_2 \rangle$ is not defined, in general. In fact, since $h(\psi_1, \psi_2)$ defines a function in the quotient manifold $\mathcal{D}$, we would integrate it in $\mathcal{D}$, but this is not possible because, in the general case, we have not a measure defined in $\mathcal{D}$.

The way to solve these problems will be studied in the following section.

5.3.2. QUANTIZATION OF KÄHLER MANIFOLDS. There is a special situation in which the last obstructions are almost overcame: quantization of symplectic manifolds carrying Kähler polarizations or, what is the same thing, quantization of Kähler manifolds. In fact, Kähler manifolds are a distinguished kind of symplectic manifolds for quantization, as it has been studied in [41]. Next, we show the guidelines of this method (following [92]).

Let (M, Ω) be a prequantizable system endowed with a Kähler polarization $\mathcal{P}$, and let (L, π, M) be a prequantum line bundle. We can suppose, without loosing of generality, that we are considering the complex structure in (M, Ω), such that $\mathcal{P}$ is the holomorphic polarization. Remember that the prequantum Hilbert space $\mathcal{H}_P$ is the set of square integrable smooth sections in $\Gamma(L)$.

Now, in order to obtain the polarized sections, we can choose the adapted symplectic potential $\theta = i\hbar\partial f$ of Proposition 21. Then, polarized sections ψ are locally represented by holomorphic functions $F(z)$, that is, those such that $\bar{\partial}F = 0$, Taking another symplectic potential, for instance $\Theta = \frac{i\hbar}{2}(\partial f - \bar{\partial} f)$, the polarized sections are locally given by

$$\Psi(z, \bar{z}) = F(z)e^{-\frac{f(z,\bar{z})}{2}}.$$

Then, you can observe that $\Theta = \theta + d\alpha$ with $\alpha = \frac{i\hbar}{2} f(z, \bar{z})$ and the relation between both representations of polarized sections is $\Psi = \psi e^{\frac{i\alpha}{\hbar}}$, which is consistent with the complex linear bundle structure. In any case, since there exists a local trivialization in which polarized sections are represented by holomorphic functions, we say that they are *holomorphic sections* of the hermitian line bundle (L, π, M).

However, the inner product of these polarized sections are not well defined since they are not necessarily square integrable. Hence, we have to restrict this product to the set of *square integrable holomorphic sections*. We will denote this set by $\Gamma_{hol}(L)$.

Now we are going to prove that $\Gamma_{hol}(L)$ is a closed subspace of $\mathcal{H}_P$. In order to achieve this result, we need the following lemma due to Weil [88] (although the proof we present here is adapted from [77]).

LEMMA 4. *Let ψ a polarized section, that is, $\psi_\alpha = F_\alpha s^\alpha$, with F_α an holomorphic function; and let $\{U_\alpha\}$ be an open covering of M satisfying the following conditions:*

1. *$\{U_\alpha, \phi_\alpha\}$ is a differentiable atlas on M.*
2. *$\{U_\alpha, s^\alpha\}$ is a trivializing covering of L and $h(s^\alpha, s^\alpha) = 1$, for every α.*

3. *For each* U_α *there exists a differentiable function* f_α *such that* $\Omega\mid_{U_\alpha}= i\hbar\frac{-\partial^2 f_\alpha}{\partial\bar{z}_j\partial z_k}d\bar{z}_j\wedge dz_k$.
4. *For each* U_α*, the polarization* $\mathcal{P}$ *is generated by the vector fields* $\left\{\frac{\partial}{\partial\bar{z}_j^\alpha}\right\}$ *and* $\theta = i\hbar\partial f_\alpha$ *is an adapted symplectic potential.*

Then, for each compact $K\subset U_\alpha$*, there exists a constant* $C(K)>0$ *such that, if* $m\in K$*, we have*

$$h_m(\psi_\alpha(m),\psi_\alpha(m)) = F_\alpha(m)\bar{F}_\alpha(m)\leq C(K)\langle\psi\mid\psi\rangle.$$

Proof. Given $K\subset U_\alpha$, consider $\delta>0$ such that, for every $p\in K$, $B'(p,2\delta):=\phi_\alpha^{-1}\Big(B(\phi_\alpha(p),2\delta)\Big)$ where $B(\phi_\alpha(p),2\delta)\subset\phi_\alpha(U_\alpha)\subset\mathbb{R}^{2n}$. Then

$$\begin{aligned}\langle\psi|\psi\rangle &= \int_M h(\psi,\psi)\Lambda_\Omega \geq \int_{U_\alpha} h(\psi,\psi)\Lambda_\Omega = \int_{U_\alpha} F_\alpha\bar{F}_\alpha\Lambda_\Omega\\ &= \int_{\phi_\alpha(U_\alpha)}\phi_\alpha^{-1^*}(F_\alpha\bar{F}_\alpha\Lambda_\Omega)\equiv\int_{\phi_\alpha(U_\alpha)}F'_\alpha\bar{F}'_\alpha\Lambda'_\Omega\geq\int_{B(\phi\alpha(p),\delta)}F'_\alpha\bar{F}'_\alpha\Lambda'_\Omega\\ &\geq C(K')\int_{B(\phi_\alpha(p),\delta)}F'_\alpha\bar{F}'_\alpha dV^{2n} = (*),\end{aligned}$$

where dV^{2n} denotes the Lebesgue measure in $\mathbb{R}^{2n}$, $K':=\overline{\cup_{p\in K}B(\phi_\alpha(p),\delta)}\subset\phi_\alpha(U_\alpha)$ and $C(K')>0$ is a constant such that $\Lambda'_\Omega\geq C(K')dV^{2n}$. Observe that the existence of such a constant is justified since

$$dV^{2n} = g\Lambda'_\Omega\leq\sup_{K'}g\Lambda'_\Omega\ \Rightarrow\ \Lambda'_\Omega\geq C(K')dV^{2n},$$

where $C(K'):=\frac{1}{\sup_{K'}g}$.

$$\begin{aligned}(*) &= C(K')F'_\alpha(\xi)\bar{F}'_\alpha(\xi)\mathcal{V}\Big(B(\phi_\alpha(p),\delta)\Big)\\ &\geq C(K')F_\alpha(\phi_\alpha^{-1}(\xi))\bar{F}_\alpha(\phi_\alpha^{-1}(\xi))\mathcal{V}\Big(B(\phi_\alpha(p),\delta)\Big)\frac{F_\alpha(p)\bar{F}_\alpha(p)}{\sup\left(\sup_{K'}(F_\alpha\bar{F}_\alpha),1\right)}\\ &\equiv C(K')C(\delta)F_\alpha(p)\bar{F}_\alpha(p),\end{aligned}$$

$\mathcal{V}\Big(B(\phi_\alpha(p),\delta)\Big)$ being the volume of the sphere B. The product $C(K')C(\delta)>0$ depends only on K, taking into account the initial choice of δ and the construction of K'. Therefore $F_\alpha(p)\bar{F}_\alpha(p)\leq C(K)\langle\psi|\psi\rangle$; where $C(K)\equiv\frac{1}{C(K')C(\delta)}$. ∎

Then, we have:

PROPOSITION 23. *If $\mathcal{P}$ is a Kähler polarization, then $\Gamma_{hol}(L)$ is a closed subspace of $\mathcal{H}_P$, and hence $\Gamma_{hol}(L)$ is also a Hilbert space.*

Proof. Let $\{\psi_k\}$ be a Cauchy sequence in $\Gamma_{hol}(L)$, that is,

$$\langle \psi_k - \psi_{k'} \mid \psi_k - \psi_{k'} \rangle \xrightarrow{k,k' \to \infty} 0.$$

From the above lemma, $\{\psi_k\}$ is pointwise convergent to a section $\psi \in \Gamma(L)$. Then, since the convergence is uniform in a neighbourhood of every point, and the uniform limit of a sequence of holomorphic functions is also holomorphic, it follows that $\psi \in \Gamma_{hol}(L)$. Furthermore, the sequence $\{\psi_k\}$ converges to an element $\psi' \in \mathcal{H}_P$, as $\mathcal{H}_P$ is a Hilbert space. Moreover, if $K \subset M$ is compact then, using the triangle inequality, it follows that, for each k,

$$\left[\int_K h(\psi - \psi', \psi - \psi')\Lambda_\Omega\right]^{\frac{1}{2}} \leq \left[\int_K h(\psi - \psi_k, \psi - \psi_k)\Lambda_\Omega\right]^{\frac{1}{2}} + \left[(2\pi\hbar)^n \langle \psi' - \psi_k \mid \psi' - \psi_k \rangle\right]^{\frac{1}{2}}.$$

Taking into account the above mentioned convergences, we have that ψ and ψ' differ at most on a zero-measure set, so $\Gamma_{hol}(L)$ is a closed Hilbert subspace for the uniform convergence. Really we have to consider equivalent sections differing just on a zero-measure set, and the induced Hilbert space structure on the corresponding quotient set. ∎

Hence, the quantization procedure would end at this stage. Nevertheless certain problems concerning the spectrum of the quantum operators remains unsolved, as it will see in the examples (see the quantization of the harmonic oscillator).

Using the antiholomorphic polarization, polarized sections would be the antiholomorphic sections, but the procedure is the same as above.

5.4. THE POLARIZATION CONDITION ON THE OPERATORS. To take the polarized sections as the standpoint set for constructing the true quantum states obliges to restrict the set of admissible quantum operators or, what means the same thing, the set of quantizable classical observables. In fact, if we want that condition (v) of Definition 4 holds, we have previously to assure that $\Gamma_{\mathcal{P}}(L)$ is invariant under the action of the quantum operators. This leads to set the following:

DEFINITION 27. The set of polarized quantum operators, $\mathcal{O}(\Gamma_{\mathcal{P}}(L))$, is made of the operators O_f corresponding to functions $f \in C^\infty(M)$ such that, for every $\psi \in \Gamma_{\mathcal{P}}(L)$,

$$O_f|\psi\rangle \in \Gamma_{\mathcal{P}}(L). \tag{14}$$

There are different ways of characterizing the functions f representing the quantizable classical observables. In order to obtain them and establish their equivalence with the above definition we need to prove that:

LEMMA 5. *Let (M,Ω) be a prequantizable system and (L,π,M,h,∇) the corresponding prequantum line bundle. Let $\mathcal{P}$ be a polarization in (M,Ω) and $X \in \mathcal{X}^{\mathbb{C}}(M)$ such that $\nabla_X(\Gamma_{\mathcal{P}}(L)) = 0$. Then $X \in \mathcal{P}$.*

Proof. It suffices to prove it locally. Thus, consider the local set of coordinates $\{x_1,\dots,x_{n-l};y_1,\dots,y_{n-l};u_1,\dots,u_l;v_1,\dots,v_l\}$ introduced in the comments after Definition 18, and a local symplectic potential θ adapted to the polarization $\mathcal{P}$. Since $\mathcal{P}$ is locally spanned by the vector fields $\left\{\frac{\partial}{\partial x_j}\right\}_{j=1}^{n-l}$, $\left\{\frac{\partial}{\partial \bar{z}_k}\right\}_{k=1}^{l}$, (with $z_k := u_k + iv_k$), then a local symplectic potential adapted to $\mathcal{P}$ is given by $\theta = \alpha^j dy_j + \beta^k dz_k$.

Let ψ be a complex function representing a polarized section. Then $\psi \equiv \psi(y_j,z_k)$. Now, if $X \equiv A^j\frac{\partial}{\partial x_j} + B^j\frac{\partial}{\partial y_j} + C^k\frac{\partial}{\partial z_k} + D^k\frac{\partial}{\partial \bar{z}_k}$ is a vector field such that $\nabla_X\psi = 0$, we are going to see that this implies that $B^j = 0$ and $C^k = 0$. In fact, we have that

$$0 = \nabla_X\psi = X(\psi) + \langle X,\theta\rangle = B^j\frac{\partial\psi}{\partial y_j} + C^k\frac{\partial\psi}{\partial z_k} + B^j\alpha_j + C^k\beta_k. \tag{15}$$

Now, the set $\{y_j\}$, $\{z_k\}$ is a basis for the functions representing the polarized sections. The above condition for this set splits into a system of n linear equations (with the coefficients B^j, C^k, as unknowns):

$$\begin{bmatrix} 1+\alpha_1 y_1 & \alpha_2 y_1 & \cdots & \alpha_{n-l} y_1 & \beta_1 y_1 & \cdots & \beta_l y_1 \\ \alpha_1 y_2 & 1+\alpha_2 y_2 & \cdots & \alpha_{n-l} y_2 & \beta_1 y_2 & \cdots & \beta_l y_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1 y_{n-l} & \alpha_2 y_{n-l} & \cdots & 1+\alpha_{n-l} y_{n-l} & \beta_1 y_{n-l} & \cdots & \beta_l y_{n-l} \\ \alpha_1 z_1 & \alpha_2 z_l & \cdots & \alpha_{n-l} z_l & 1+\beta_1 z_l & \cdots & \beta_l z_l \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1 z_l & \alpha_2 z_l & \cdots & \alpha_{n-l} z_l & \beta_1 z_l & \cdots & 1+\beta_l z_l \end{bmatrix} \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^{n-l} \\ C^1 \\ \vdots \\ C^l \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

and we are going to prove that it has only the trivial solution. In fact, this system has the form $TX = -X$, where

$$T = \begin{bmatrix} \alpha_1 y_1 & \alpha_2 y_1 & \dots & \alpha_{n-l} y_1 & \beta_1 y_1 & \dots & \beta_l y_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1 y_{n-l} & \alpha_2 y_{n-l} & \dots & \alpha_{n-l} y_{n-l} & \beta_1 y_{n-l} & \dots & \beta_l y_{n-l} \\ \alpha_1 z_1 & \alpha_2 z_l & \dots & \alpha_{n-l} z_l & \beta_1 z_l & \dots & \beta_l z_l \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1 z_l & \alpha_2 z_l & \dots & \alpha_{n-l} z_l & \beta_1 z_l & \dots & \beta_l z_l \end{bmatrix}$$

and X is the matrix of unknowns. Then, observe that T has rank equal to 1 (each row is a linear combination of other ones), except if $\theta = 0$. Therefore its only eigenvalue is the trace of T, that is, $\sum_j \alpha_j y_j + \sum_k \beta_k z_k$, which is different from -1 (because, if it were constant, then it would be equal to zero: it suffices to calculate it at the point such that the image under the homomorphism of the system is the point zero in $\mathbb{R}^{2n}$). So, -1 is not the eigenvalue of T, therefore the determinant of the system is different from zero and then the solution is $X = 0$, that is, $B^j = 0 = C^k$. Therefore, we have

$$X = A^j \frac{\partial}{\partial x_j} + D^k \frac{\partial}{\partial \bar{z}_k} \in \mathcal{P}.$$

Notice that if $\theta = 0$ this result follows in a straightforward way from (15). ∎

Now, we can prove that:

PROPOSITION 24. *The following statements are equivalent:*

1. $O_f \in \mathcal{O}(\Gamma_{\mathcal{P}}(L))$.
2. O_f *preserves the polarization* $\mathcal{P}$*; in the sense that* $[O_f, \nabla_{\mathcal{P}}]\psi = 0$*, for every* $\psi \in \Gamma_{\mathcal{P}}(L)$.
3. *The function* f*, from which the operator* O_f *is constructed, is such that* $[X_f, \mathcal{P}] \subset \mathcal{P}$.

Proof. The equivalence

$$O_f \in \mathcal{O}(\Gamma_{\mathcal{P}}(L)) \quad \Leftrightarrow \quad [O_f, \nabla_{\mathcal{P}}]\psi = 0$$

is an immediate consequence of the equation (13). Now, we are going to prove that

$$\text{for every } \psi \in \Gamma_{\mathcal{P}}(L), \quad [O_f, \nabla_{\mathcal{P}}]\psi = 0 \quad \Leftrightarrow \quad [X_f, \mathcal{P}] \subset \mathcal{P}.$$

We have that

$$\begin{aligned}
[O_f, \nabla_{\mathcal{P}}]\psi &= O_f \nabla_{\mathcal{P}}\psi - \nabla_{\mathcal{P}} O_f \psi = -\nabla_{\mathcal{P}} O_f \psi = -\nabla_{\mathcal{P}}(-i\hbar\nabla_{X_f} + f)\psi \\
&= i\hbar\nabla_{\mathcal{P}}\nabla_{X_f}\psi - \nabla_{\mathcal{P}}(f\psi) = i\hbar\nabla_{\mathcal{P}}\nabla_{X_f}\psi - \mathcal{P}(f)\psi - f\nabla_{\mathcal{P}}\psi \\
&= i\hbar\nabla_{\mathcal{P}}\nabla_{X_f}\psi - \mathcal{P}(f)\psi = i\hbar(\nabla_{\mathcal{P}}\nabla_{X_f} - \nabla_{X_f}\nabla_{\mathcal{P}})\psi - \mathcal{P}(f)\psi \\
&= i\hbar\left(\nabla_{[\mathcal{P},X_f]} + \frac{i}{\hbar}\Omega(\mathcal{P}, X_f)\right)\psi - \mathcal{P}(f)\psi \\
&= i\hbar\nabla_{[\mathcal{P},X_f]}\psi - (\Omega(\mathcal{P}, X_f) + \Omega(X_f, \mathcal{P}))\psi = i\hbar\nabla_{[\mathcal{P},X_f]}\psi.
\end{aligned}$$

Therefore, if $[X_f, \mathcal{P}] \subset \mathcal{P}$, then $[O_f, \nabla_{\mathcal{P}}]\psi = 0$. Conversely, taking into account the above lemma, we have immediately that

$$[O_f, \nabla_{\mathcal{P}}]\psi = 0 \quad \Rightarrow \quad [X_f, \mathcal{P}] \subset \mathcal{P}.$$

∎

So we have:

REQUIREMENT 4. The set of quantum operators $\mathcal{O}(\Gamma_{\mathcal{P}}(L))$ is made of the operators O_f satisfying condition (14) or, what is equivalent, the conditions (2) or (3) of Proposition 24.

It is evident that, once the complex line bundle is known, the quantization of a system (i.e., the Hilbert space and the set of quantum operators) depend on the choice of the polarization (but it does not on the choice of the symplectic potential). To select a polarization determines the *representation* of the quantum system (see the examples).

In order to compare different representations, that is, quantizations obtained from different choices of polarizations, there is a procedure known as the method of the *Blattner-Kostant-Sternberg kernels*, [14], [15], [16], [40]. In particular, since the condition given in Proposition 24 does not allow to quantize all the classical observables, this method is applied in order to quantize observables which do not satisfy that condition.

5.5. EXAMPLES

5.5.1. COTANGENT BUNDLES: THE SCHRÖDINGER AND THE MOMENTUM REPRESENTATIONS. In the particular case of the Example 1 in Section 4.4, where $M = T^*Q$, there is a very special real polarization: the *vertical polarization*, which is spanned by the vector fields $\left\{\frac{\partial}{\partial p_j}\right\}$. Then, taking the adapted symplectic potential $\theta = -p_j dq^j$, the polarized sections are the functions $\psi \in \mathcal{C}(T^*Q)$ such that $\frac{\partial \psi}{\partial p_j} = 0$; i.e., those being constant along the fibers of T^*Q, that is, $\psi = \psi(q^j)$. The quantum operators corresponding to the observables *positions* and *momenta* are

$$O_{q^j} = q^j, \quad O_{p_j} = -i\hbar \frac{\partial}{\partial q^j},$$

but the energy is not quantizable. When we quantize in this manner, the pair $(\Gamma_{\mathcal{P}}(L), \mathcal{O}(\Gamma_{\mathcal{P}}(L)))$ is known as the *Schrödinger representation* of (T^*Q, Ω).

Using the polarization spanned by the vector fields $\left\{\frac{\partial}{\partial q^j}\right\}$ (which is also real), and taking as adapted symplectic potential $\theta' = q^j dp_j$, we would obtain the functions $\psi = \psi(p_j)$ as the polarized sections and the quantum operators corresponding to the observables position and momenta are now

$$O_{q^j} = i\hbar \frac{\partial}{\partial p_j}, \quad O_{p_j} = p_j.$$

This is the so-called *momentum representation* of (T^*Q, Ω).

Observe that the relation between these representations is the Fourier transform. (See [73] for more information on these topics).

5.5.2. Cotangent bundles: the Bargman-Fock representation. For the case $M = T^*Q$, we have just seen that the Schrödinger and the momentum representations are obtained by choosing the real polarizations spanned by the Hamiltonian vector fields corresponding to the momenta and positions respectively. Next we are going to analyze another typical representation obtained when we use a particular Kähler polarization.

Instead of the canonical coordinates, we introduce the complex ones $\{z_j, \bar{z}_j\}$ where $z_j := p_j + iq^j$. The expression of the symplectic form in these coordinates is

$$\Omega = \frac{i}{2} d\bar{z}_j \wedge dz_j.$$

Now, $(T^*Q, \Omega, \mathcal{J})$ is a Kähler manifold, where the complex structure is given by

$$\mathcal{J}\left(\frac{\partial}{\partial p_j}\right) = \left(\frac{\partial}{\partial q^j}\right), \quad \mathcal{J}\left(\frac{\partial}{\partial q^j}\right) = -\left(\frac{\partial}{\partial p_j}\right).$$

Now, we take the polarization $\mathcal{P}$, spanned by $\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$. As symplectic potentials we can chose

$$\Theta = \frac{i}{4}(\bar{z}_j dz_j - z_j d\bar{z}_j),$$

or the adapted one

$$\theta = \frac{i}{2}\bar{z}_j dz_j.$$

In any case, the polarized sections are the holomorphic sections of the complex line bundle $T^*Q \times \mathbb{C}$, and, using the symplectic potential Θ, the quantum operators are

$$O_{z_j} = -2\hbar\frac{\partial}{\partial \bar{z}_j} + \frac{z_j}{2}, \quad O_{\bar{z}_j} \equiv O^+_{z_j} = 2\hbar\frac{\partial}{\partial z_j} + \frac{\bar{z}_j}{2}, \tag{16}$$

which, in the physical terminology, are called the *creation* and *annihilation* operators, respectively. In this representation, a relevant role is played by the so-called *number operator*:

$$O_{\bar{z}_j z_j} = 2\hbar\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right).$$

This way of quantizing is known as the *holomorphic* or *Bargman-Fock representation* of (T^*Q, Ω). On the contrary, if the polarization used is the holomorphic one $\bar{\mathcal{P}}$, the polarized sections would be the antiholomorphic sections and the representation so obtained is called the *antiholomorphic representation* of (T^*Q, Ω).

5.5.3. THE HARMONIC OSCILLATOR: BARGMANN REPRESENTATION. The next example that we consider is the quantization of the n-dimensional harmonic oscillator, using Kähler polarizations. The prescriptions we are going to follow will lead us to obtain the *holomorphic* or *Bargmann representation* of the harmonic oscillator.

We begin identifying the phase space $M \equiv \mathbb{R}^{2n}$ (which is an Euclidean vector space endowed with the usual metric) with $\mathbb{C}^n$, and introducing the complex analytical coordinates $\{z_j, \bar{z}_j\}$ defined in the above subsection. Hence, the Hamiltonian function and the symplectic form are written

$$H(z_j, \bar{z}_j) = \frac{1}{2}\bar{z}_j z_j, \quad \Omega = \frac{i}{2}d\bar{z}_j \wedge dz_j.$$

$(\mathbb{C}^n, \Omega, \mathcal{J})$ is a Kähler manifold, where the complex structure $\mathcal{J}$ is given in the usual way, and the hermitian metric is given also by the Hamiltonian. Then, we can take the Kähler polarization $\mathcal{P}$ spanned by $\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$, and as symplectic potentials (which are global, in this case) we can chose

$$\Theta = \frac{i}{4}(\bar{z}_j dz_j - z_j d\bar{z}_j),$$

or the adapted one

$$\theta = \frac{i}{2}\bar{z}_j dz_j.$$

Using the first one, the polarized sections (eigenstates of the system) are

$$\psi(z_j, \bar{z}_j) = F(z)e^{-\frac{z_j \bar{z}_j}{4\hbar}},$$

which are holomorphic sections on $\mathbb{C}$, and the inner product is given by

$$\langle \psi_1 \mid \psi_2 \rangle = \left(\frac{1}{2\pi\hbar}\right)^n \int_M F_1(z)\bar{F}_2(z)e^{-\frac{z_j\bar{z}_j}{2\hbar}}\Lambda_\Omega.$$

For the observables, we obtain O_{z_j} and $O_{\bar{z}_j}$ given in (16) and

$$O_H = \frac{1}{2}O_{\bar{z}_j z_j} = \hbar\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right).$$

It can be checked out that these operators satisfy the condition given in Proposition 24, and, so, the Hamiltonian function is quantizable. It is also immediate to observe that

$$\begin{aligned}
O_{z_j}|\psi\rangle &= z_j\psi = z_jF(z)e^{-\frac{z_j\bar{z}_j}{4\hbar}}, \\
O^+_{z_j}|\psi\rangle &= 2\hbar\frac{\partial\psi}{\partial z_j} + \frac{\bar{z}_j}{2}\psi = 2\hbar\frac{\partial F}{\partial z_j}e^{-\frac{z_j\bar{z}_j}{4\hbar}}, \\
O_H|\psi\rangle &= \hbar\left(z_j\frac{\partial\psi}{\partial z_j} - \bar{z}_j\frac{\partial\psi}{\partial\bar{z}_j}\right) = \hbar z_j\frac{\partial F}{\partial z_j}.
\end{aligned}$$

Now, the problem of quantization of the harmonic oscillator is almost solved. It remains to obtain the energy spectrum. In order to make it, we have to solve the eigenvalues equation for the Hamiltonian operator

$$O_H|\psi\rangle = E|\psi\rangle,$$

but, taking into account the expression of O_H, this equation is

$$\hbar z_j\frac{\partial F(z)}{\partial z_j} = EF(z).$$

Therefore, we obtain that the eigenfunctions $F(z)$ are homogeneous polynomials of degree N, $F(z_j) \equiv az_1^{N_1}\dots z_n^{N_n}$ (where $N_j \in \mathbb{Z}$ and $N = N_1 + \cdots + N_n$), and the eigenvalues of the Hamiltonian (i.e., the energy) are $N\hbar$.

As you can observe, if use is made of the adapted symplectic potential θ, the eigenstates are $\psi(z_j, \bar{z}_j) = F(z)$ and the Hamiltonian operator acting on these functions is just $O_H = \hbar z_j \frac{\partial}{\partial z_j}$. Hence the results are the same as above.

Nevertheless these results are not physically correct, since the true eigenvalues equation is really

$$\hbar \left(z_j \frac{\partial}{\partial z_j} + \frac{n}{2} \right) F(z) = EF(z),$$

and the eigenvalues are $(N + \frac{n}{2})\hbar$. This shows that, even in the case of using Kähler polarizations, quantization does not end at this stage and a final step must be done: the so-called *metaplectic correction* [73], [76], [92].

6. Metalinear bundles. Bundles of densities and half-forms. Quantization

As we have already pointed out in the above section, the set $\Gamma_{\mathcal{P}}(L)$ of polarized sections are not necessarily square integrable; and, in addition, if $\psi_1, \psi_2 \in \Gamma_{\mathcal{P}}(L)$, the inner product $\langle \psi_1 | \psi_2 \rangle$ is not defined, in general, because we have not a measure defined in $\mathcal{D}$. Although, in general, the use of Kähler polarizations allows us to overcome these problems, the spectrum of the quantum operators which is obtained is incorrect in certain cases.

The way to solve these problems consists in introducing the *bundles of densities and half-forms*, as new geometrical structures for quantization.

Some basic references on this topic are [30] and [92] (In addition, some interesting and important examples of application of these techniques can be examined in [73], [76], [92], and other references quoted on them.). Finally, in the appendix we give some explanations about bundles associated to group actions, which can be of interest for the understanding of this section.

6.1. Complex metalinear group.

Consider the group $GL(n, \mathbb{C})$. Its subgroup $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ is defined as the group of matrices $A \in SL(n, \mathbb{C})$ with $\det A = 1$. Consider now the following exact sequence of groups:

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{C} \times SL(n, \mathbb{C}) \xrightarrow{p} GL(n, \mathbb{C}) \to 0, \tag{17}$$

where

$$j(k) := \left(\frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} I \right), \quad p(u, A) := e^u A.$$

On the other hand, if $p(z, A) = I$, then $e^z A = I$, hence $A = e^{-z}I$ and therefore $(z, A) = (z, e^{-z}I)$. Since $1 = \det(e^{-z}I) = e^{-zn}$, then $z = \frac{2\pi ik}{n}$, with $k \in \mathbb{Z}$; hence $(z, A) = (\frac{2\pi ik}{n}, e^{-\frac{2\pi ik}{n}}I) = j(k)$. Thus, we have proved that $\ker p = \operatorname{Im} j$. We can also construct the following exact sequence

$$0 \to 2\mathbb{Z} \xrightarrow{j} \mathbb{C} \times SL(n, \mathbb{C}) \xrightarrow{\pi} \frac{\mathbb{C} \times SL(n, \mathbb{C})}{2\mathbb{Z}} \to 0, \tag{18}$$

where now j is restricted to $2\mathbb{Z}$; that is, $j(2k) = \left(\frac{4\pi ik}{n}, e^{-\frac{4\pi ik}{n}}I\right)$.

DEFINITION 28. The quotient group

$$ML(n, \mathbb{C}) := \frac{\mathbb{C} \times SL(n, \mathbb{C})}{2\mathbb{Z}}$$

is called the *complex metalinear group* of dimension n.

Observe that $(z_1, A_1), (z_2, A_2) \in \mathbb{C} \times SL(n, \mathbb{C})$ give equivalent elements in $ML(n, \mathbb{C})$ if and only if there exists $k \in \mathbb{Z}$ such that $z_1 = z_2 + \frac{4\pi ik}{n}$ and $A_1 = e^{-\frac{4\pi ik}{n}}A_2$; in other words, the coset of (z, A) is

$$\overline{(z, A)} = \left\{\left(z + \frac{4\pi ik}{n}, e^{-\frac{4\pi ik}{n}}A\right) : k \in \mathbb{Z}\right\}.$$

From the above definition we have a surjective natural morphism

$$\begin{aligned} \rho : \quad ML(n, \mathbb{C}) &\longrightarrow GL(n, \mathbb{C}) \\ \overline{(z, A)} &\mapsto e^z A \end{aligned}$$

which makes commutative the following diagram

$$\begin{array}{ccc} ML(n, \mathbb{C}) & \xrightarrow{\ \rho\ } & GL(n, \mathbb{C}) \\ & \nwarrow^{\pi} & \uparrow p \\ & & \mathbb{C} \times SL(n, \mathbb{C}) \end{array}$$

If $\rho\overline{(z, A)} = I$ then $e^z A = I$, therefore $e^{nz} = 1$ and $A = e^{-z}I$, hence $z = \frac{2\pi ih}{n}$, for some $h \in \mathbb{Z}$. But z is determined up to an integer multiple of $\frac{4\pi ik}{n}$, therefore

$$\ker \rho \equiv \left\{\left(\frac{2\pi ih}{n} + \frac{4\pi ik}{n}, e^{-\frac{2\pi ih}{n}}e^{-\frac{4\pi ik}{n}}I\right) : \ h, k \in \mathbb{Z}\right\}.$$

Observe that, if h_1, h_2 have the same parity, they give the same element of $ML(n,\mathbb{C})$. Then we have that

$$\begin{aligned}\ker\rho &= \left\{ \overline{\left(\frac{2\pi ih}{n} + \frac{4\pi ik}{n}, e^{-\frac{2\pi ih}{n}} e^{-\frac{4\pi ik}{n}} I\right)} \;:\; h = 0,1,\; k \in \mathbb{Z}\right\} \\ &= \left\{ \overline{(0,I)}, \overline{\left(\frac{2\pi i}{n}, e^{-\frac{2\pi i}{n}} I\right)} \right\} \simeq \mathbb{Z}_2.\end{aligned}$$

Therefore, we have the exact sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{\iota} ML(n,\mathbb{C}) \xrightarrow{\rho} GL(n,\mathbb{C}) \to 0, \tag{19}$$

where $\iota(0) = \overline{(0,I)}$ and $\iota(1) = \overline{\left(\frac{2\pi i}{n}, e^{-\frac{2\pi i}{n}} I\right)}$. Taking into account the natural structures of complex Lie group in $ML(n,\mathbb{C})$ and $GL(n,\mathbb{C})$, this exact sequence characterizes $ML(n,\mathbb{C})$ as a double covering of $GL(n,\mathbb{C})$. Observe that the construction of this double covering arises from the two exact sequences (17) and (18) since

$$\frac{\mathbb{C} \times SL(n,\mathbb{C})}{j(\mathbb{Z})} \simeq GL(n,\mathbb{C}), \quad \frac{\mathbb{C} \times SL(n,\mathbb{C})}{j(2\mathbb{Z})} \simeq ML(n,\mathbb{C}),$$

and taking them together

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \xrightarrow{j} & \mathbb{C} \times SL(n,\mathbb{C}) & \xrightarrow{p} & GL(n,\mathbb{C}) & \to & 0 \\ & & \uparrow & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \rho} & & \\ 0 & \to & 2\mathbb{Z} & \xrightarrow{j} & \mathbb{C} \times SL(n,\mathbb{C}) & \xrightarrow{\pi} & ML(n,\mathbb{C}) & \to & 0 \end{array}$$

Now, ρ is defined in a natural way using these sequences. On the other hand, in order to evaluate $\ker\rho$, it suffices to see that, if $\overline{(z,A)} \in ML(n,\mathbb{C})$ has the same image (by ρ) than $\overline{(0,I)}$, then (z,A) and $(0,I)$ have the same image by p, that is, $(z,A) \in j(\mathbb{Z})$. Therefore, either $(z,A) \in j(2\mathbb{Z})$, and then $\overline{(z,A)} = \overline{(0,I)}$; or $(z,A) \notin j(2\mathbb{Z})$, and hence we have that

$$\ker\rho = \frac{j(\mathbb{Z})}{j(2\mathbb{Z})} \simeq \mathbb{Z}_2,$$

as we have already pointed out.

Notice that the sequence (17) allows us to write the elements of $GL(n,\mathbb{C})$ as

$$\overline{(z,A)} = \left\{ \left(z + \frac{2\pi ih}{n}, e^{-\frac{2\pi ih}{n}} A\right) \;:\; h \in \mathbb{Z} \right\},$$

with $z \in \mathbb{Z}$, $A \in SL(n, \mathbb{C})$.

For every $B \in GL(n, \mathbb{C})$ we can write $B = e^z B_0$, with $\det B_0 = 1$ and $e^{nz} = \det B$. Then, it is clear that z is determined up to a factor $\frac{2\pi i h}{n}$, with $h \in \mathbb{Z}$. Then $B = p(z, B_0)$. In constructing $ML(n, \mathbb{C})$, we split the elements $\overline{(z, B_0)}$ into two classes: the one having h even and the other having h odd.

Observe that, in $GL(n, \mathbb{C})$, the square root of the determinant is not defined as a holomorphic function. However, so is it in $ML(n, \mathbb{C})$. In fact, we have

$$\begin{array}{ccccc} ML(n, \mathbb{C}) & \xrightarrow{\rho} & GL(n, \mathbb{C}) & \longrightarrow & \mathbb{C}^* \\ \overline{(z, A)} & \mapsto & e^z A & \mapsto & \det(e^z A) = e^{nz} \det A = e^{nz} \end{array}$$

and we can construct the function

$$\begin{array}{cccc} \chi\colon & ML(n, \mathbb{C}) & \xrightarrow{\det} & \mathbb{C}^* \\ & \overline{(z, A)} & \mapsto & e^{\frac{nz}{2}} \end{array}$$

which is well defined, since it does not depend on the representative of $\overline{(z, A)}$. In fact, if we consider $\overline{(z, A)} \in ML(n, \mathbb{C})$, then $\chi(\overline{(z, A)}) = e^{nz/2}$ is well defined because $z = z_0 + \frac{4\pi i h}{n}$, and therefore $e^{nz/2} = e^{\frac{nz_0}{2} + 2\pi i h} = e^{\frac{nz_0}{2}}$ which does not depend on h. Moreover, it satisfies that

$$\chi^2(\overline{(z, A)}) = (e^{\frac{nz}{2}})^2 = e^{nz} = (\det \circ \rho)(\overline{(z, A)}),$$

and then $\chi = \sqrt{(\det \circ \rho)}$. That is, we can lift the map *det* to $ML(n, \mathbb{C})$ by means of ρ and obtain a square root of this lifting. Observe that this function cannot be defined in $GL(n, \mathbb{C})$ in this way. In fact, if $B \in GL(n, \mathbb{C})$, according to the above considerations we have $B = e^z B_0$, with $e^{nz} = \det B$. Now, $B = p(z, B_0)$ and writing

$$\begin{array}{ccc} GL(n, \mathbb{C}) & \xrightarrow{\det} & \mathbb{C}^* \\ B = p(z, B_0) & \mapsto & e^{nz} = \det B \end{array}$$

the function $B \to e^{nz/2}$ is not well defined since $z = z_0 + \frac{2\pi i h}{n}$, with $h \in \mathbb{Z}$, and hence $e^{nz/2} = e^{\frac{n}{2}(z_0 + \frac{2\pi i h}{n})} = e^{\frac{nz_0}{2} + \pi i h}$, whose value depends on h.

6.2. Principal metalinear bundles. Classification

Definition 29. Let $p : P \to M$ be a principal fiber bundle with structural group $GL(n, \mathbb{C})$. A *metalinear bundle associated with* (P, p, M) is a principal fiber bundle $\bar{p} : \bar{P} \to M$ with structural group $ML(n, \mathbb{C})$ and a differentiable map $\bar{\rho} : \bar{P} \to P$ between bundles over M giving the identity on M, such that the following diagram commutes

$$\begin{array}{ccc} \bar{P} \times ML(n,\mathbb{C}) & \xrightarrow{\bar{\iota}} & \bar{P} \\ {\scriptstyle \bar{\rho}\times\rho}\downarrow & & \downarrow{\scriptstyle \bar{\rho}} \\ P \times GL(n,\mathbb{C}) & \xrightarrow{\iota} & P \end{array}$$

where $\iota, \bar{\iota}$ are the actions of the groups on the bundles, and such that $\bar{\rho}$ is a double covering.

Now, given a principal fiber bundle (P, p, M) with structural group $GL(n, \mathbb{C})$, we can ask when a metalinear associated bundle exists and, in this case, if it is unique. We will answer these questions afterwards.

Let (U_i, s_i) be a trivial system of P with transition functions $g_{ij} : U_{ij} \to GL(n, \mathbb{C})$. We can construct transition functions of $\bar{P}$, $\bar{g}_{ij} : U_{ij} \to ML(n, \mathbb{C})$, making commutative the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & ML(n,\mathbb{C}) & \xrightarrow{\rho} & GL(n,\mathbb{C}) & \longrightarrow & 0 \\ & & & & & {\scriptstyle \bar{g}_{ij}}\nwarrow & \uparrow{\scriptstyle g_{ij}} & & \\ & & & & & & U_{ij} & & \end{array}$$

In order to construct $\bar{g}_{ij}$, it suffices to take U_{ij} in such a manner that $g_{ij}(U_{ij})$ is contained in a fundamental open set of the covering.

Let us suppose that U_{ij} are contractible. Then the function

$$c_{ijk} := \bar{g}_{ij}\bar{g}_{jk}\bar{g}_{ki} : \ U_{ijk} \to ML(n, \mathbb{C})$$

satisfies that $\rho \circ c_{ijk} = 1$, therefore $c_{ijk} \in \Im\mathbb{Z}_2$, hence c_{ijk} is constant. In this way we have defined a 2- cochain

$$c(P) : (U_i, U_j, U_k) \to c_{ijk},$$

with coefficients in $\mathbb{Z}_2$, which is associated with every trivializing covering of P. It can be proved that

1. $c(P)$ is closed.
2. $[c(P)] \in H^2(M, \mathbb{Z}_2)$ does not depend neither on the local system (U_i, s_i) used, nor on the liftings $\bar{g}_{ij}$ constructed. It is a cohomology class associated with $p \colon P \to M$.

3. A $GL(n, \mathbb{C})$-principal fiber bundle (P, p, M) admits an associated metalinear bundle if, and only if, $c(P) = 0$. Then:

   DEFINITION 30. Let (P, p, M) satisfying that $c(P) = 0$; and $\bar{p}_1 : \bar{P}_1 \to M$ and $\bar{p}_2 : \bar{P}_2 \to M$ two metalinear fiber bundles associated with P, with covering maps $\bar{\rho}_1 : \bar{P}_1 \to P$ and $\bar{\rho}_2 : \bar{P}_2 \to P$. Then $\bar{P}_1$ and $\bar{P}_2$ are *equivalent* if there exists a diffeomorphism $\tau : \bar{P}_1 \to \bar{P}_2$ such that

   a) It is an isomorphism between $\mathbb{Z}_2$-principal fiber bundles over P.
   b) It is an isomorphism between $ML(n, \mathbb{C})$-principal fiber bundles over M.

4. With the above definition of equivalence, the group $H^1(M, \mathbb{Z}_2)$ acts freely and transitively on the set of equivalence classes of metalinear fiber bundles associated with P.

A last question is to find the condition for a function $\bar{f} : \bar{P} \to \mathbb{C}$ to be $\bar{\rho}$-projectable. The necessary and sufficient condition for this is that, if $\bar{a}, \bar{b} \in \bar{P}$ and $\bar{\rho}(\bar{a}) = \bar{\rho}(\bar{b})$, then $\bar{f}(\bar{a}) = \bar{f}(\bar{b})$. Then, you can observe that, if $\bar{\rho}(\bar{a}) = \bar{\rho}(\bar{b})$, then $\bar{p}(\bar{a}) = \bar{p}(\bar{b})$; that is, there exists $\bar{g} \in ML(n, \mathbb{C})$ such that $\bar{b} = \bar{a}\bar{g}$, hence $\rho(\bar{g}) = id$, that is, $\bar{g} \in \ker \rho$. Therefore, the desired condition is that $\bar{a} \in \bar{P}$ and $\bar{g} \in \ker \rho$, then $\bar{f}(\bar{a}) = \bar{f}(\bar{a}\bar{g})$.

6.3. BUNDLES OF DENSITIES. In order to introduce integration in non-integrable manifolds, we need new structures: the so-called *densities*.

6.3.1. BUNDLE OF ORIENTATIONS

DEFINITION 31. Let M be a differentiable manifold with $\dim M = n$, and let (U_α, ψ_α) be an atlas of M. Consider the line bundle L over M defined by taking $\{U_\alpha \times \mathbb{R}\}$ as open sets and

$$f_{\alpha\beta} := \operatorname{sign} \det J(\psi_\alpha \circ \psi_\beta^{-1}) = \operatorname{sign} \det J(g_{\alpha\beta})$$

as transition functions; where $g_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$ are the functions of change of variables in M for the atlas (U_α, ψ_α); and J denotes the Jacobian matrix. L is called the *bundle of orientations* of the manifold M.

Note that, from the condition $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$, we deduce that $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$, and then the family $\{f_{\alpha\beta}\}$ satisfies the cocycle condition and hence the bundle structure is unique.

PROPOSITION 25. *M is orientable if, and only if, L is trivial.*

Proof. ($\Longrightarrow$) If M is orientable then there is an atlas with changes of coordinates with positive jacobians, therefore $f_{\alpha\beta}$ are the identity and, then L trivializes.

($\Longleftarrow$) If L trivializes, there exists a trivializing covering with transition functions $g_{\alpha\beta}$ which are the identity, that is, M is orientable since the signs of the jacobians are positive. ∎

6.3.2. BUNDLE OF DENSITIES

DEFINITION 32. Consider the real line bundle $Q(M) := \Lambda^n(T^*M) \otimes_{\mathbb{R}} L$. Its transition functions

$$h_{\alpha\beta} := \det[{}^tJ(g_{\alpha\beta})]^{-1} \operatorname{sign} \det J(g_{\alpha\beta}) = \left|\det J(g_{\alpha\beta})^{-1}\right|,$$

(U_α, ψ_α) being an atlas of M and $g_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$. $Q(M)$ is called the *bundle of densities* of M. Sections of $Q(M)$ are called *densities* on M.

Now we are going to see how the densities change when the coordinates change. Let $\sigma\colon M \to Q(M)$ be a density on M and let (U_α, ψ_α) be an atlas of M. Put $\psi_\alpha \equiv \{\psi_\alpha^1 \dots \psi_\alpha^n\}$. A basis of the sections of $Q(M)$ in the open set U_α is given by a basis of $\Omega^n(T^*M)\mid_{U_\alpha}$ which is $\{d\psi_\alpha^1 \wedge \dots \wedge d\psi_\alpha^n\}$; and a basis of $L\mid_{U_\alpha}$ which is $\{u_\alpha\} \in \mathbb{R}$. So then $\{\{d\psi_\alpha^1 \wedge \dots \wedge d\psi_\alpha^n\} \otimes u_\alpha\}$ is a basis of the sections of $Q(M)$ in U_α.

Thus, the density σ induces the map $\sigma_\alpha\colon U_\alpha \to U_\alpha \times L$ given by

$$\sigma_\alpha(x) := (x, s_\alpha(x)(d\psi_\alpha^1 \wedge \dots \wedge d\psi_\alpha^n \otimes u_\alpha)_x),$$

that is, in U_α the section is given by $s_\alpha(x)(d\psi_\alpha^1 \wedge \dots \wedge d\psi_\alpha^n \otimes u_\alpha)_x$, and in U_β by $s_\beta(x)(d\psi_\beta^1 \wedge \dots \wedge d\psi_\beta^n \otimes u_\beta)_x$. If we take $x \in U_{\alpha\beta}$ we have $s_\alpha(x) = s_\beta(x) \det\left(J(g_{\alpha\beta}(x))\right)$. Hence, the integration of σ in M has sense, since we have only to restrict to trivializing open sets $\{U_\alpha\}$; and taking a partition of the unity $\{\eta_\alpha\}$,we define

$$\int_M \sigma := \sum_\alpha \int_{U_\alpha} \eta_\alpha \sigma_\alpha,$$

which has the same properties than the usual integration of forms in orientable manifolds.

As final remark, it is usual to consider *complex densities*, that is, to take sections of the fiber bundle $Q(M)\otimes_{\mathbb{R}}\mathbb{C}$, which is a complex line bundle over M. You can observe that a complex density σ is nothing but a sum $\sigma = \sigma_1 + i\sigma_2$; σ_1 and σ_2 being real densities.

6.4. BUNDLES OF HALF-FORMS. Let $p\colon P \to M$ be a principal fiber bundle wit group $GL(n,\mathbb{C})$, which admits an associated metalinear bundle $\bar{p} : \bar{P} \to M$.

DEFINITION 33. 1) Consider the action of $ML(n,\mathbb{C})$ on $\mathbb{C}$

$$\begin{array}{ccc} ML(n,\mathbb{C}) \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ (\bar{g}, z) & \mapsto & \chi(\bar{g})z \end{array}$$

(remember that, if $\bar{g} = \overline{(z, A)}$ with $z \in \mathbb{C}$ and $A \in SL(n,\mathbb{C})$, then $\chi(\overline{(z, A)}) = e^{nz/2}$). The line bundle associated with this action is denoted $N^{1/2}$; that is,

$$N^{1/2} := \bar{P} \times_{ML(n,\mathbb{C})} \mathbb{C} \xrightarrow{\pi_{ML(n,\mathbb{C})}} M,$$

where we remind that $\bar{P} \times_{ML(n,\mathbb{C})} \mathbb{C}$ is defined through the equivalence relation in $\bar{P} \times \mathbb{C}$

$$(\bar{a}, z) \simeq (\bar{b}, w) \;\Leftrightarrow\; \exists \bar{g} \in ML(n,\mathbb{C}) \;\mid\; (\bar{b}, w) = (\bar{a}\bar{g}, g^{-1}z).$$

2) Consider now the action of $GL(n,\mathbb{C})$ on $\mathbb{C}$

$$\begin{array}{ccc} GL(n,\mathbb{C}) \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ (g, z) & \mapsto & \mid \det g \mid z \end{array}$$

The line bundle associated with this action is denoted $|N|$; that is,

$$|N| := P \times_{GL(n,\mathbb{C})} \mathbb{C} \xrightarrow{\pi_{GL(n,\mathbb{C})}} M,$$

where we remind that $P \times_{GL(n,\mathbb{C})} \mathbb{C}$ is defined through the equivalence relation in $P \times \mathbb{C}$

$$(a, z) \simeq (b, w) \;\Leftrightarrow\; \exists g \in GL(n,\mathbb{C}) \;\mid\; (b, w) = (ag, g^{-1}z).$$

Next we study the sections of $N^{1/2}$ and $|N|$.

If $\sigma : M \to N^{1/2}$ is a section of $\pi_{ML(n,\mathbb{C})}$, we can interprete it as a function $\sigma : \bar{P} \to \mathbb{C}$ satisfying the following condition: if $\bar{a}, \bar{b} \in \bar{P}$ with $\bar{p}(\bar{a}) = \bar{p}(\bar{b})$ (and then there exists $\bar{g} \in ML(n,\mathbb{C})$ with $\bar{b} = \bar{a}\bar{g}$), then $\sigma(\bar{a}) = \chi(\bar{g})\sigma(\bar{b})$. In a similar way, If $\mu : M \to |N|$ is a section of $\pi_{GL(n,\mathbb{C})}$, we interprete it as a function $\mu : P \to \mathbb{C}$ satisfying the following condition: if $a, b \in P$ with $p(a) = p(b)$ (that is, there exists $g \in GL(n,\mathbb{C})$ with $b = ga$), then $\mu(a) =\mid \det g \mid \mu(b)$. Bearing this interpretation in mind we have the following result:

PROPOSITION 26. *Let $\sigma, \sigma' \colon M \to N^{1/2}$ be sections of $\pi_{ML(n,\mathbb{C})}$, which we interprete as associated functions $\sigma, \sigma' \colon \bar{P} \to \mathbb{C}$. Then the function $\sigma\sigma'^* : \bar{P} \to \mathbb{C}$ is projectable by $\bar{\rho} : \bar{P} \to P$ and its projected function $\bar{\rho}(\sigma\sigma'^*) : \bar{P} \to \mathbb{C}$ arises from a section of $|N|$. This allows to construct a map*

$$\begin{array}{ccc} \Gamma(N^{1/2}) \times \Gamma(N^{1/2}) & \xrightarrow{\langle,\rangle} & \Gamma(|N|) \\ (\sigma, \sigma') & \mapsto & \bar{\rho}(\sigma\sigma'^*) \end{array}$$

which is sesquilinear with respect to the module structures of the sections of $N^{1/2}$ and $|N|$ on $C^\infty(M) \otimes \mathbb{C}$.

Proof. a) $\sigma\sigma'^*$ is $\bar{\rho}$-projectable:

We have to prove that, if $\bar{a} \in \bar{P}$ and $\bar{g} \in \ker \rho$, then $\sigma\sigma'^*(\bar{a}\bar{g}) = \sigma\sigma'^*(\bar{a})$. In fact, as the action of $\bar{g} = \overline{(z, A)}$ on $\mathbb{C}$ consists in multiplying by $e^{nz/2}$, we obtain that

$$\begin{aligned} \sigma\sigma'^*(\bar{a}\bar{g}) &= \sigma(\bar{a}\bar{g})\sigma'^*(\bar{a}\bar{g}) = \chi(\bar{g})^{-1}\sigma(\bar{a})\left(\chi(\bar{g})^{-1}\right)^*\sigma'(\bar{a})^* \\ &= \chi(\bar{g})^{-1}\left(\chi(\bar{g})^{-1}\right)^*\sigma(\bar{a})\sigma'(\bar{a})^* = \mid \chi(\bar{g}) \mid^{-2} (\sigma\sigma'^*)(\bar{a}) \\ &= \mid \chi(\bar{g})^2 \mid^{-1} (\sigma\sigma'^*)(\bar{a}) = (\sigma\sigma'^*)(\bar{a}). \end{aligned}$$

(Notice that $\chi(\bar{g})^2 = \det \rho(g) = \det I = 1$.) So, the result is proved.

b) $\rho(\sigma\sigma'^*)$ arises from a section of $|N|$:

We have to prove that, if $a \in P$ and $g \in GL(n, \mathbb{C})$, then

$$\bar{\rho}(\sigma\sigma'^*)(ag) = \mid \det g \mid^{-1} \bar{\rho}(\sigma\sigma'^*)(a).$$

In order to see that, consider $\bar{a} \in \bar{P}$ and $\bar{g} \in ML(n, \mathbb{C})$ with $\bar{\rho}(\bar{a}) = a$ and $\rho(\bar{g}) = g$. Then we have that

$$\begin{aligned} \bar{\rho}(\sigma\sigma'^*)(ag) &= (\sigma\sigma'^*)(\bar{a}\bar{g}) = \chi(\bar{g})^{-1}(\chi(\bar{g})^{-1})^*(\sigma\sigma'^*)(\bar{a}) = \mid \chi(\bar{g}) \mid^{-2} (\sigma\sigma'^*)(\bar{a}) \\ &= \mid \det \rho(\bar{g}) \mid^{-1} (\sigma\sigma'^*)(\bar{a}) = \mid \det g \mid^{-1} (\sigma\sigma'^*)(\bar{a}), \end{aligned}$$

as we wanted to prove. ∎

Now we introduce the Lie derivative of sections of $N^{1/2}$ and $|N|$.

Let $X \in \mathcal{X}(P)$ be a $GL(n, \mathbb{C})$-invariant vector field. Since $\bar{\rho} : \bar{P} \to P$ is a local diffeomorphism, we can lift X to a vector field $\bar{X} \in \mathcal{X}(\bar{P})$ which is $ML(n, \mathbb{C})$-invariant (because X is $GL(n, \mathbb{C})$-invariant and it has coverings). Let $\sigma : M \to N^{1/2}$ be a section. If we consider it as a function from $\bar{P}$ to $\mathbb{C}$, it makes sense to calculate $L(\bar{X})\sigma$. In the same way, if $\mu : M \to |N|$ is a section, it makes sense to calculate $L(X)\mu$, when we interprete it as a function from P to $\mathbb{C}$. Then the following property holds:

PROPOSITION 27. *Let* $X \in \mathcal{X}(P)$ *be a* $GL(n,\mathbb{C})$*-invariant vector field and* $\bar{X} \in \mathcal{X}(\bar{P})$ *its* $ML(n,\mathbb{C})$*-invariant lifting. Consider* $\sigma, \sigma' : M \to N^{1/2}$*. Then*

a) $L(\bar{X})\sigma,\ L(\bar{X})\sigma'$ *are sections of* $N^{1/2}$.

b) $L(X)\langle\sigma,\sigma'\rangle$ *is a section of* $|N|$.

c) $L(X)\langle\sigma,\sigma'\rangle = \bar{\rho}((L(\bar{X})\sigma)\sigma'^*) + \bar{\rho}(\sigma\, L(\bar{X})\sigma'^*)$.

Proof. a) We have to prove that, if $\bar{a} \in \bar{P}$ and $\bar{g} \in ML(n,\mathbb{C})$, then

$$(L(\bar{X})\sigma)(\bar{a}\bar{g}) = \chi(\bar{g})^{-1}(L(\bar{X})\sigma)(a).$$

In fact, if $\bar{X}$ is $ML(n,\mathbb{C})$-invariant, this means that $\bar{X}_{\bar{a}\bar{g}} = T_{\bar{a}}\mu(\bar{g})\bar{X}_{\bar{a}}$, where

$$\begin{array}{rccc} \mu(\bar{g}) : & \bar{P} & \longrightarrow & \bar{P} \\ & \bar{x} & \mapsto & \bar{x}\bar{g} \end{array}$$

therefore

$$\begin{aligned}(L(\bar{X})\sigma)(\bar{a}\bar{g}) &= \bar{X}_{\bar{a}\bar{g}}\sigma = (T_{\bar{a}}\mu(\bar{g})\bar{X}_{\bar{a}})\sigma = \bar{X}_{\bar{a}}(\sigma\circ\mu(\bar{g})) = \bar{X}_{\bar{a}}(\chi(\bar{g})^{-1}\sigma) \\ &= \chi(\bar{g})^{-1}\bar{X}_{\bar{a}}\sigma = \chi(\bar{g})^{-1}(L(\bar{X}_{\bar{a}})\sigma)(\bar{a}),\end{aligned}$$

since X is $\mathbb{C}$-linear and $(\sigma\circ\mu(\bar{g}))(\bar{x}) = \sigma(\bar{x}\bar{g}) = \chi(\bar{g})^{-1}\sigma(\bar{x})$.
b) We have to see that, if $a \in P$ and $g \in GL(n,\mathbb{C})$, then

$$(L(X)\langle\sigma,\sigma'\rangle)(ag) = |\det g|^{-1}\,(L(\bar{X})\langle\sigma,\sigma'\rangle)(ag).$$

In general, this condition holds for every section $\alpha : M \to |N|$; in fact:

$$\begin{aligned}(L(X)\alpha)(ag) &= X_{ag}\alpha = (T_a\mu(g)X_a)\alpha = X_a(\alpha\circ\mu(g)) = X_a(\,|\det g|^{-1}\,\alpha) \\ &= |\det g|^{-1}\,X_a\alpha = |\det g|^{-1}\,L(X)\alpha\end{aligned}$$

by the same reasons than in the above paragraph.
c) Consider $a \in P$ and $\bar{a} \in \bar{P}$ such that $\bar{\rho}(\bar{a}) = a$. We have that

$$\begin{aligned}(L(X)\langle\sigma,\sigma'\rangle)(a) &= X_a\langle\sigma,\sigma'\rangle = X_a\bar{\rho}(\sigma\sigma'^*) = (T_{\bar{a}}\bar{\rho}\bar{X}_{\bar{a}})(\bar{\rho}(\sigma\sigma'^*)) \\ &= \bar{X}_{\bar{a}}(\bar{\rho}(\sigma\sigma'^*)\circ\bar{\rho}) = \bar{X}_{\bar{a}}(\sigma\sigma'^*) = (\bar{X}_{\bar{a}}\sigma)\sigma'^*(\bar{a}) + \sigma(\bar{a})\bar{X}_{\bar{a}}\sigma'^* \\ &= (L(\bar{X})\sigma)(\bar{a})\sigma'^*(\bar{a}) + \sigma(\bar{a})(L(\bar{X})\sigma'^*)(\bar{a}) \\ &= (L(\bar{X})\sigma\sigma')(\bar{a}) + (\sigma\, L(\bar{X})\sigma'^*)(\bar{a}) \\ &= [\bar{\rho}((L(\bar{X})\sigma)\sigma'^*) + \bar{\rho}(\sigma\, L(\bar{X})\sigma'^*)](a)\end{aligned}$$

∎

Observe that, in general, the equality

$$L(X)\langle\sigma,\sigma'\rangle = \langle L(\bar{X})\sigma,\sigma'\rangle + \langle\sigma, L(\bar{X})\sigma'\rangle$$

does not hold since $L(\bar{X})\sigma'^* \neq (L(\bar{X})\sigma')^*$, except if X is real. In this case the true relation is

$$L(X)\langle\sigma,\sigma'\rangle = \langle L(\bar{X})\sigma,\sigma'\rangle + \bar{\rho}(\sigma\, L(\bar{X})\sigma'^*),$$

and it is easy to prove that the function $\sigma\, L(\bar{X})\sigma'^*$ is $\bar{\rho}$-projectable and, hence, the expression makes sense. It can also be written as

$$L(X)\langle\sigma,\sigma'\rangle = \langle L(\bar{X})\sigma,\sigma'\rangle + \langle\sigma, L(\bar{X}^*)\sigma'\rangle,$$

where $\bar{X}^*$ is the conjugate vector field of $\bar{X}$.

In general, we are going to be interested in deriving with respect to vector fields which are tangent to M. In order to do it we must have a procedure for lifting them to vector fields tangent to P being $GL(n,\mathbb{C})$-invariant. In addition, if we restrict to real vector fields, then

$$L(X)\langle\sigma,\sigma'\rangle = \langle L(\bar{X})\sigma,\sigma'^*\rangle + \langle\sigma, L(\bar{X})\sigma'\rangle.$$

6.5. Some particular cases

6.5.1. Frame bundle of a differentiable manifold. Let M be a differentiable manifold with $\dim M = n$ and $p_*\colon P_* \to M$ the bundle of covariant complex references of M. P_* is a principal fiber bundle with structural group $GL(n,\mathbb{C})$. Suppose that P_* admits an associated metalinear bundle $\bar{p}_*\colon \bar{P}_* \to M$ and let $|N|$ be the associated bundle.

Proposition 28. *$|N|$ is diffeomorphic to the bundle of complex densities of M.*

Proof. The bundle of complex densities of M is $Q(M)\otimes_{\mathbb{R}}\mathbb{C}$. In order to prove that they are diffeomorphic, we must see that they have the same transition functions.

Observe that $Q(M)$ and $Q(M)\otimes_{\mathbb{R}}\mathbb{C}$ have the same transition functions, since $\mathbb{C}$ is the trivial bundle; then its transition functions are the identity. On the other hand, the transition functions of $Q(M)$ are the absolute value of those of $\Lambda^*(T^*M)$, therefore, the transition functions of $Q(M)$ are $|\det^t[Tg_{\alpha\beta}]^{-1}|$. The transition functions of $|N|$ are the same than the ones of P_*, which are $(Tg_{\alpha\beta})^*$, and act on $\mathbb{C}$ in order to obtain $|N|$ as $|\det(Tg_{\alpha\beta}^{-1})^*|$. Hence, they have the same transition functions. ∎

DEFINITION 34. With the above considerations, the sections of the bundle $N^{1/2}$ are called *uniformly complex sections*.

According to the above results, if $\sigma, \sigma' \in \Gamma(N^{1/2})$, then $\langle \sigma, \sigma' \rangle$ is a complex density on M.

As an example, consider the bundle $p^*: P^* \to M$ of contravariant complex references of M, and suppose that it admits an associated metalinear bundle $\bar{p}^*: \bar{P}^* \to M$. In this case and, in the same way, the sections of the associated bundle $|N|$ can be identified with the complex multiples of the absolute values of the skew symmetric contravariant tensors of higher degree.

Now, let X be a real vector field in M. It can be lifted to a vector field in $\bar{P}^*$ in the following way: Let τ_t be a local uniparametric group associated with X. This group acts on the references of X by means of the differential, and so we obtain a vector field $\tilde{X} \in \mathcal{X}(P^*)$. In addition, $\tilde{X}$ is $GL(n,\mathbb{C})$-invariant, that is, if $\{u_1, \dots, u_n\}$ is a complex reference of T_xM and $g \in GL(n,\mathbb{C})$, then

$$T\tau_t((u_1, \dots, u_n)g) = (T\tau_t(u_1, \dots, u_n))g.$$

Since $\tilde{X}$ is $GL(n,\mathbb{C})$-invariant and $\bar{P}^*$ is a covering of P^*, the vector field $\tilde{X}$ lifts to a vector field $\bar{X}$ in $\bar{P}^*$. In order to construct $\bar{X}$ it suffices to do it in basic open sets of a covering. Observe that the so-obtained vector field $\bar{X}$ is $ML(n,\mathbb{C})$-invariant by construction.

If X is a complex vector field, it suffices to take its real and imaginary parts in order to obtain the associated vector field $\bar{X} \in \mathcal{X}(\bar{P}^*)$ which is $ML(n,\mathbb{C})$-invariant.

6.5.2. FRAME BUNDLE OF A REAL POLARIZATION IN A SYMPLECTIC MANIFOLD. (See Section 5.2 for notation.) Let (M,Ω) be a symplectic manifold with $\dim M = 2n$, and let $\mathcal{P}$ be a real polarization. Let $p: P \to M$ be the frame bundle of $\mathcal{P}$ which is a principal bundle with structural group $GL(n,\mathbb{C})$, and we suppose that it admits a metalinear bundle $\bar{P}$ associated with P. Let $|N|$ and $N^{1/2}$ the line bundles associated with P and $\bar{P}$. The elements of $\Gamma(M,|N|)$ and $\Gamma(M,N^{1/2})$ are called *complex densities* and *complex half-forms* on M *associated with the polarization* $\mathcal{P}$.

Consider $X \in \mathcal{X}^{\mathbb{C}}(M)$ such that lets the polarization $\mathcal{P}$ invariant; that is, $[X,\mathcal{P}] \subset \mathcal{P}$. Denote by $\mathcal{X}^{\mathbb{C}}(\mathcal{P})$ the set of these vector fields. If τ_t is a local uniparametric group associated with X and $T\tau_t$ its lifting to $TM^{\mathbb{C}}$, we can obtain a vector field $\tilde{X} \in \mathcal{X}(TM^{\mathbb{C}})$ associated with the group $T\tau_t$. Since X lets invariant the polarization, the transformations $T\tau_t$ change references of $\mathcal{P}$

into references of $\mathcal{P}$, then $\tilde{X}$ can be interpreted as a vector field tangent to P. This is equivalent to saying that the trajectories of $T\tau_t$ in $TM^{\mathbb{C}}$ are contained in P or that $\tilde{X}$ is tangent to P.

In addition, $\tilde{X}$ is $GL(n,\mathbb{C})$-invariant since if $p \in P$ and $A \in GL(n,\mathbb{C})$ we have $T_x\tau(pA) = (T_x\tau(p))A$; as a consequence of the associativity of the product of matrices. Therefore, $\tilde{X}$ lifts to a vector field $\bar{X}$ which is $ML(n,\mathbb{C})$-invariant, using the local diffeomorphism $\bar{\rho} : \bar{P} \to P$ and the action of $\mathbb{Z}_2$ on $ML(n,\mathbb{C})$.

We can define the action of $\mathcal{X}^{\mathbb{C}}(\mathcal{P})$ on the densities and half-forms associated with the polarization $\mathcal{P}$. Let $X \in \mathcal{X}^{\mathbb{C}}(\mathcal{P})$; and $\alpha \in \Gamma(M, |N|)$ and $\beta \in \Gamma(M, N^{1/2})$. Since α can be considered as a function $\alpha : P \to \mathbb{C}$, we can write

$$L(X)\alpha := \tilde{X}(\alpha),$$

and $L(X)\alpha \in \Gamma(M,|N|)$, since $\tilde{X}$ is a $GL(n,\mathbb{C})$-invariant vector field. In the same way, since β can be interpreted as a function $\beta : \bar{P} \to \mathbb{C}$, we can write

$$L(X)\beta := \bar{X}(\beta),$$

and $L(X)\beta \in \Gamma(M, N^{1/2})$, since $\bar{X}$ is a $ML(n,\mathbb{C})$-invariant vector field.

Now, consider $X \in \mathcal{X}^{\mathbb{C}}(M)$, its extension $\tilde{X}$ to P and a section $\alpha \in \Gamma(M,|N|)$. Suppose that it is a Hamiltonian vector field (according to the Proposition 22), and denote it by X_f. If X_g is another Hamiltonian vector field in D [25], we have that $i([X_f, X_g])\Omega = 0$; and taking into account that Ω in non-degenerate we conclude that $[X_f, X_g] = 0$. Therefore, X_g is invariant under the action of the uniparametric group τ_t generated by X_f; that is, $T\tau_t(X_{g_x}) = X_{f_{\tau_t}}(x)$. Then, let $\{X_{f_1}, \dots, X_{f_n}\}$ be a basis of $\mathcal{X}(\mathcal{P})$ in an open set $U \subset M$ made of Hamiltonian vector fields. We can consider the following functions in U associated with the section α

$$\begin{aligned} F_\alpha(x) &:= \left(L(X)\alpha\right)(X_{f_1}(x), \dots, X_{f_n}(x)), \\ G_\alpha(x) &:= \left[X\left(\alpha(X_{f_1}, \dots, X_{f_n})\right)\right](x), \end{aligned}$$

for a given X in D. Therefore:

PROPOSITION 29. *If $X \equiv X_f$ is a Hamiltonian vector field then $F_\alpha = G_\alpha$*

[25]Remember that, for a real polarization, $D = E = \mathcal{P} \cap TM$.

Proof. If $\tilde{X}$ is the canonical lift of X to P, we have that

$$\begin{aligned} F_\alpha(x) &:= \big(L(X)\alpha\big)(X_{f_1}(x),\dots,X_{f_n}(x)) = \\ &= \lim_{t\to 0}(1/t)\Big(\alpha(T_x\tau_t X_{f_1}(x),\dots,T_x\tau_t X_{f_n}(x)) - \alpha(X_{f_1}(x),\dots,X_{f_n}(x))\Big) \\ &= \lim_{t\to 0}(1/t)\Big(\alpha(X_{f_1}(\tau_t(x)),\dots,X_{f_n}(\tau_t(x))) - \alpha(X_{f_1}(x),\dots,X_{f_n}(x))\Big), \end{aligned}$$

since the vector fields are invariant under the action of τ_t. On the other hand,

$$\begin{aligned} G_\alpha(x) &:= [X(\alpha(X_{f_1},\dots,X_{f_n}))](x) \\ &= \lim_{t\to 0}(1/t)(\alpha(X_{f_1},\dots,X_{f_n}))(\tau_t(x)) - \alpha(X_{f_1},\dots,X_{f_n})(x) \\ &= \lim_{t\to 0}(1/t)\Big(\alpha(X_{f_1}(\tau_t(x)),\dots,X_{f_n}(\tau_t(x))) - \alpha(X_{f_1}(x),\dots,X_{f_n}(x))\Big). \end{aligned}$$ ∎

Taking into account the above considerations we have:

PROPOSITION 30. *Consider $x \in U \subset M$. Let V be the integral manifold of D passing through x; and the section*

$$\begin{array}{rccc} \sigma\colon & U\cap V & \longrightarrow & P \\ & x & \mapsto & (X_{f_1},\dots,X_{f_n}) \end{array}$$

where $\{X_{f_1},\dots,X_{f_n}\}$ is a basis of D in U made of Hamiltonian vector fields. Then, $\tilde{X}_f$ is tangent to the image of σ, for every $f \in C^\infty(M)$.

Proof. If τ is the local uniparametric group associated with X_f, then $T\tau$ lets invariant the vector fields X_{f_i}, hence the integral curves of $\tilde{X}$ are contained in $\sigma(U\cap V)$ and therefore it is tangent to the image of σ. ∎

Let $\pi\colon M \to \mathcal{D}$ be the projection defined by the integral manifolds of the distribution D. Let $\{\bar{\alpha}\}$ be the set of complex densities of $\mathcal{D}$. We are going to define a map φ from $\{\bar{\alpha}\}$ to $\Gamma(M,|N|)$ which allows us to integrate sections of $|N|$. Consider $x \in M$ and $X_x \in D_x$, then $i(X_x)\Omega\colon T_xM \to \mathbb{C}$ is $\mathbb{R}$-linear and vanishes on D_x, hence it passes to the quotient T_xM/D_x which is canonically isomorphic to $T_{\pi(x)}\mathcal{D}$. Let $\Phi(X_x)$ the map induced in $T_{\pi(x)}\mathcal{D}$. Then, we have a $\mathbb{R}$-linear map

$$\begin{array}{rccc} \Phi\colon & D_x & \longrightarrow & \mathrm{Hom}_{\mathbb{R}}(T_{\pi(x)}\mathcal{D},\mathbb{C}) \\ & X_x & \mapsto & \Phi(X_x) \end{array}$$

which is an isomorphism because Ω is a symplectic form. If $\{X_x^1,\dots,X_x^n\}$ is a $\mathbb{C}$-basis of D_x, then $\{\Phi(X_x^1),\dots,\Phi(X_x^n)\}$ is a basis of $(T^*_{\pi(x)}\mathcal{D})^{\mathbb{C}}$. Let $\{\bar{X}_x^1,\dots,\bar{X}_x^n\}$ be the dual $\mathbb{C}$-basis. Then:

PROPOSITION 31. *For every complex density* $\bar{\alpha} \in \mathcal{D}$, *let* $\{X_x^1, \dots, X_x^n\}$ *be a reference of* D_x *(that is a point of* P*). Define a map* $\varphi\colon \{\bar{\alpha}\} \to \Gamma(M, |N|)$, *such that* $\alpha := \varphi(\bar{\alpha})$, *as follows*

$$\alpha(X_x^1, \dots, X_x^n) := \bar{\alpha}(\bar{X}_x^1, \dots, \bar{X}_x^n).$$

Therefore:

i) φ *is well defined.*

ii) φ *is injective.*

iii) $\operatorname{Im}\varphi = \{\alpha\colon P \to \mathbb{C} : \alpha \in \Gamma(M, |N|),\ L(X)\alpha = 0,$ *for every* X *in* D *(loc. hamiltonian)*$\}$.

Proof. i) We have to see that the so-defined function satisfies that

$$\alpha(Rg) = g^{-1}\alpha(R) = |\det g|^{-1}\alpha(R),$$

where $R \in P$ and $g \in GL(n, \mathbb{C})$. Let $R = (v^1, \dots, v^n)$ be a reference of D_x and $g = (g_{ij})$ a matrix of $GL(n, \mathbb{C})$; then

$$Rg = \left(\sum g_{i1}v^i, \dots, \sum g_{in}v^i\right) = (u^1, \dots, u^n),$$

therefore

$$\Phi(u^j) = \Phi\left(\sum g_{ij}vi\right) = \sum g_{ij}\Phi(v^i),$$

and hence $\Phi(Rg) = \Phi(R)g$. Now, we are going to see the relation between the dual basis of $\Phi(R)$ and $\Phi(R)g$. Denoting by $\overline{\Phi(R)}$ the dual basis $\Phi(R)$ we have that ${}^t\overline{\Phi(R)} \cdot \Phi(R) = I$, and hence ${}^t\overline{\Phi(R)g} \cdot \Phi(R)g = I$. But $\overline{\Phi(R)g} = \overline{\Phi(R)}h$, for some $h \in GL(n, \mathbb{C})$, then ${}^th\,{}^t\overline{\Phi(R)} \cdot \Phi(R)g = I$; and thus $h = {}^tg^{-1}$, therefore we obtain $\overline{\Phi(R)g} = \overline{\Phi(R)} \cdot {}^tg^{-1}$, so we have that

$$\begin{aligned} \alpha(Rg) &= \bar{\alpha}(\overline{\Phi(Rg)}) = \bar{\alpha}(\overline{\Phi(R)g}) = \bar{\alpha}(\overline{\Phi(R)} \cdot {}^tg^{-1}) \\ &= |\det\ {}^tg^{-1}|\bar{\alpha}(\overline{\Phi(R)}) = |\det g|^{-1}\alpha(R). \end{aligned}$$

ii) If $\varphi(\bar{\alpha}) = 0$ then $\bar{\alpha}$ vanishes in a reference of $(TM)^{\mathbb{C}}$, thus it is null.

iii) Let $\alpha \in \operatorname{Im}\varphi$ with $\alpha = \varphi(\bar{\alpha})$, we have to see that $L(X)\alpha = 0$ for every locally Hamiltonian vector field X in D (it suffices to prove it for the vector fields in D). Observe that it suffices to see that, for all $x \in M$, it holds $(L(X)\alpha)(X_x^1, \dots, X_x^n) = 0$ for a reference of D_x. Let $\{X_x^1, \dots, X_x^n\}$ be a reference of D and $\{X_{f_1}, \dots, X_{f_n}\}$ a basis made of Hamiltonian vector fields in an open set $U \subset M$ with $x \in U$, which prolongs the above reference. Let $\tilde{X}$ be the extension of X to P. If V_x is the integral manifold of D passing through x, $\tilde{X}$ is tangent to $\sigma(U \cap V_x)$, being $\sigma(x) = (X_{f_1}(x), \dots, X_{f_n}(x))$, therefore, in order to see that $L(X)(X_x^1, \dots, X_x^n) = 0$, it suffices to see that α is constant on $\sigma(U \cap V_x)$. In fact, consider $\bar{x} = \pi(x) \in \mathcal{D}$ and let $\bar{U} \subset \mathcal{D}$ be an open set with $\pi(U) \subset \bar{U}$. Since X_{f_j} is in D, the functions f_j are constant along the integral submanifolds of D. In fact, $i(X_{f_j})\Omega = df_j$ and $df_j(X_{f_i}) = 0$, $\forall i$; then $df_j = 0$, when restricted to the integral submanifolds of D. Therefore, there exist functions $\bar{f}_j \colon \bar{U} \to \mathbb{C}$ such that $\pi^* \bar{f}_j = f_j$. Now, consider $y \in U \cap V_x$, hence $\pi(y) = \bar{x}$ and we have that

$$\Phi(X_{f_j}(y)) = df_j(x) \mid_{T_{\bar{x}}\mathcal{D}} = d\bar{f}_j(\bar{x}),$$

therefore the dual basis of $\{\Phi(X_{f_1}(y)), \dots, \Phi(X_{f_n}(y))\}$ is $\left\{ \frac{\partial}{\partial \bar{f}_1}\Big|_{\bar{x}}, \dots, \frac{\partial}{\partial \bar{f}_n}\Big|_{\bar{x}} \right\}$ and thus

$$\alpha(X_{f_1}(y), \dots, X_{f_n}(y)) = \bar{\alpha}(\bar{X}_{f_1}(\bar{x}), \dots, \bar{X}_{f_n}(\bar{x})) = \bar{\alpha}\left(\frac{\partial}{\partial \bar{f}_1}\Big|_{\bar{x}}, \dots, \frac{\partial}{\partial \bar{f}_n}\Big|_{\bar{x}} \right),$$

which is constant when y varies in $U \cap V_x$, as we wanted to prove.

Conversely, consider $\alpha \in \Gamma(M, |N|)$, $\alpha \colon P \to \mathbb{C}$, satisfying that $L(X)\alpha = 0$ for every locally Hamiltonian vector field X in D. We are going to construct a density $\bar{\alpha}$ on $\mathcal{D}$ such that $\alpha = \varphi(\bar{\alpha})$. In order to do this, it suffices to give its action on a local basis in $\mathcal{D}$. Thus, consider $\bar{x} \in \mathcal{D}$ and let $\{\bar{f}_1, \dots, \bar{f}_n\}$ be local coordinates in $\mathcal{D}$. Put $f_j = \pi^* \bar{f}_j$ which are defined in $U = \pi^{-1}(\bar{U})$ and let $X_{f_1}, \dots, X_{f_n}$ be their associated Hamiltonian vector fields; then they are a local basis of D made of Hamiltonian vector fields and, for every $y \in V = \pi^{-1}(\bar{x})$ (V being an integral submanifold of D), $\alpha(X_{f_1}(y), \dots, X_{f_n}(y))$ is constant. In fact, since V is connected, it suffices to see that, if $X_x \in T_x V$ then $X_x(\alpha(X_{f_1}, \dots, X_{f_n})) = 0$. Let $X \in \mathcal{X}(M)$ be a Hamiltonian vector field which prolongs X_x locally; according to Proposition 29 we have

$$0 = \left(L(X)\alpha \right)_x (X_{f_1}, \dots, X_{f_n}) = X_x(\alpha(X_{f_1}, \dots, X_{f_n})),$$

hence $\alpha(X_{f_1}, \dots, X_{f_n})$ is constant on V.

Now we can define the density $\bar{\alpha}$ in $\mathcal{D}$ given by

$$\bar{\alpha}\left(\frac{\partial}{\partial \bar{f}_1}\Big|_{\bar{x}}, \dots, \frac{\partial}{\partial \bar{f}_n}\Big|_{\bar{x}} \right) := \alpha(X_{f_1}, \dots, X_{f_n})\Big|_V,$$

(since we can take any point of V). In this way we have defined locally α. Now we can define it globally. Let $\{\bar{g}_1, \dots, \bar{g}_n\}$ be another local coordinate system in a neighbourhood $\bar{V}$ of $\bar{x}$. Then

$$\bar{\alpha}\left(\frac{\partial}{\partial \bar{g}_1}\Big|_{\bar{x}}, \dots, \frac{\partial}{\partial \bar{g}_n}\Big|_{\bar{x}}\right) = \alpha(X_{g_1}, \dots, X_{g_n}) \mid_V .$$

If $\bar{g} = \psi(\bar{f})$ and $(J\psi)$ denotes the Jacobian matrix of ψ (the change of coordinates), then, if α must be a density, it must satisfy that

$$\left(\bar{\alpha}\left(\frac{\partial}{\partial \bar{f}_1}\Big|_{\bar{x}}, \dots, \frac{\partial}{\partial \bar{f}_n}\Big|_{\bar{x}}\right)\right)_x = |\det J\psi| \left(\bar{\alpha}\left(\frac{\partial}{\partial \bar{g}_1}\Big|_{\bar{x}}, \dots, \frac{\partial}{\partial \bar{g}_n}\Big|_{\bar{x}}\right)\right)_x ,$$

since $\frac{\partial}{\partial \bar{f}} = \frac{\partial}{\partial \bar{g}}(J\psi)$. In fact, we have that

$$d\bar{g} = d\bar{f}(J\psi) \quad \Rightarrow \quad X_{g_i} = X_{f_i}(J\psi),$$

since the map $\bar{f} \mapsto X_f$ is $\mathbb{C}$-linear. Therefore

$$\begin{aligned}\bar{\alpha}\left(\frac{\partial}{\partial \bar{g}_1}, \dots, \frac{\partial}{\partial \bar{g}_n}\right)\Big|_x &= \alpha(X_{g_1}, \dots, X_{g_n}) \mid_V = \alpha((X_{f_1}, \dots, X_{f_n})(J\psi)) \mid_V \\ &= |\det J\psi|^{-1} \alpha(X_{f_1}, \dots, X_{f_n}) \mid_V \\ &= |\det J\psi|^{-1} \bar{\alpha}\left(\frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_n}\right)\Big|_{\bar{x}},\end{aligned}$$

as we wanted to prove. Hence $\bar{\alpha} \in \{\bar{\alpha}\}$. ∎

In relation with densities over $\mathcal{D}$ and $N^{1/2}$ we have:

PROPOSITION 32. *Let σ, σ' be two sections of $N^{1/2}$ such that $L(X)\sigma = 0$, $L(X)\sigma' = 0$, for every locally Hamiltonian vector field X in D. Then $\langle \sigma, \sigma' \rangle$ is a complex density on $\mathcal{D}$ (that is, $\langle \sigma, \sigma' \rangle$ belongs to $\operatorname{Im} \phi$).*

Proof. If X is locally Hamiltonian then

$$L(X)\langle \sigma, \sigma' \rangle = \langle L(X)\sigma, \sigma' \rangle + \langle \sigma, L(X)\sigma' \rangle = 0,$$

and the result follows as a consequence of the above proposition. ∎

6.6. QUANTIZATION: SPACE OF QUANTUM STATES. OPERATORS. Let (M, Ω) be a symplectic manifold, $[\frac{\Omega}{2\pi\hbar}] \in H^2(M, \mathbb{R})$ being an integer class. Let $\pi\colon L \to M$ be a complex line bundle endowed with a hermitian connection ∇ with curvature 2-form $\frac{\Omega}{2\pi\hbar}$. Suppose that M is also endowed with a polarization $\mathcal{P}$ and that the frame bundle of $\mathcal{P}$, $p\colon P \to M$, admits a metalinear bundle $\bar{p}\colon \bar{P} \to M$. Let $N^{1/2}$ and $|N|$ be the line bundles of half-forms and densities associated with $\bar{P}$.

We are going to define the space of quantum states in the following way: We consider the line bundle $p\colon L \otimes N^{1/2} \to M$ and, in the set of sections $\Gamma(M, L \otimes N^{1/2})$, we denote by $H^{\mathcal{P}}$ the $\mathbb{C}$-vector space generated by the sections $s \otimes \sigma$, where $s \in \Gamma(M, L)$ and $\sigma \in \Gamma(M, N^{1/2})$, with compact support and such that $\nabla_X s = 0$ and $L(X)\sigma = 0$, for all locally Hamiltonian vector field $X \in \mathcal{P} \cup \bar{\mathcal{P}}$; that is, the sections s and σ are invariant by the vector fields of $\mathcal{P} \cup \bar{\mathcal{P}}$. Observe that, if $s \otimes \sigma$ and $s' \otimes \sigma'$ are elements of $H^{\mathcal{P}}$, then $\langle s, s' \rangle$ is a projectable function on $\mathcal{D} = M/D$. On the other hand, $\langle \sigma, \sigma^* \rangle$ is identified with a complex density on $\mathcal{D}$. Therefore, $\langle s, s' \rangle \langle \sigma, \sigma^* \rangle \in \{\bar{\alpha}\}_{\mathbb{C}}$, (where $\{\bar{\alpha}\}_{\mathbb{C}}$ denotes the set of complex densities on $\mathcal{D}$ with compact support). This allows us to define a pre-Hilbert product in the following way:

$$(s \otimes \sigma, s' \otimes \sigma') := \int_{\mathcal{D}} \langle s, s' \rangle \langle \sigma, \sigma' \rangle, \tag{20}$$

which can be extended linearly to $H^{\mathcal{P}}$. Then:

REQUIREMENT 5. In the geometric quantization programme, the completion $\mathcal{H}^{\mathcal{P}}$ of the set $H^{\mathcal{P}}$, endowed with the hermitian product (20), is the intrinsic Hilbert space $\mathcal{H}_Q$ and the projective space $P\mathcal{H}^{\mathcal{P}}$ is the space of quantum states $P\mathcal{H}_Q$ of the Definition 4.

Now, as it was stated in Section 2, we must represent the Poisson algebra of (M, Ω) in $\mathcal{H}^{\mathcal{P}}$. Nevertheless, according to the Proposition 24, it is not possible to represent all its elements but only those belonging to $A^{\mathcal{P}} \equiv \{f \in C^\infty(M) \,:\, [X_f, \mathcal{P}] \subset \mathcal{P}\}$ (the set of observables whose Hamiltonian vector fields preserve the polarization). Therefore:

REQUIREMENT 6. (AND DEFINITION) In the geometric quantization programme, the quantum operator associated with the classical observable $f \in A^{\mathcal{P}}$, is the operator ${}^{\mathcal{P}}O_f$ defined in $\mathcal{H}^{\mathcal{P}}$, with values in $\Gamma(M, L \otimes N^{1/2})$, which is defined by

$${}^{\mathcal{P}}O_f(s \otimes \sigma) := (O_f s) \otimes \sigma + is \otimes L(X_f)\sigma,$$

where O_f is the prequantization operator defined by (5).

And then we can prove:

THEOREM 9. *The map*

$$\begin{array}{rccc} \mathcal{Q}\colon & A^{\mathcal{P}} & \longrightarrow & \text{self-adjoint operators in } \mathcal{H}^{\mathcal{P}} \\ & f & \mapsto & {}^{\mathcal{P}}O_f \end{array}$$

is well defined and satisfies the conditions (b-iii) *and* (b-iv) *of the Definition* 4.

Proof. In order to see that $\mathcal{Q}$ is well defined we have to prove that

1. $\mathcal{Q}(f)(s\otimes\sigma) \in \mathcal{H}^{\mathcal{P}}$, $\forall s\otimes\sigma \in H^{\mathcal{P}}$,
2. $\mathcal{Q}(f)$ is self-adjoint.

Therefore, let U be an open set of M and X_g in $\mathcal{P}\cap\bar{\mathcal{P}}$ a locally Hamiltonian vector field for $g\in C^\infty(U)$; since $\mathcal{Q}(f)(s\otimes\sigma) = O_f s\otimes\sigma + is\otimes L(X_f)\sigma$, we can easily prove that

$$\begin{aligned} \nabla_{X_g}(O_f s) = \nabla_{X_g}(\nabla_{X_f} s - 2\pi i f s) = 0, \\ L(X_g)\,L(X_f)\sigma = 0, \end{aligned}$$

and the part (1) follows. The proof of the part (2) is a simple matter of calculation.

Finally the verification of the conditions (b-iii) and (b-iv) of the Definition 4 is immediate. ∎

6.7. EXAMPLE: SIMMS QUANTIZATION OF THE HARMONIC OSCILLATOR. The physical system we are considering is specified by the following features:

- Phase space: $M = \{(q,p)\in\mathbb{R}^2 - \{0\}\} \cong \mathbb{C}^*$.
- Symplectic form: $\Omega = dp\wedge dq$,

  Symplectic potential: $\theta = 1/2(pdq - qdp)$.
- Hamiltonian function: $H = \frac{1}{2}(p^2+q^2)$.

We take as line bundle the trivial bundle $L = M\times\mathbb{C}$, with the usual metric $h((x,z),(x,z')) = zz'$ and the hermitian connection given by

$$\nabla_X f = Xf + 2\pi i\theta(X) f,$$

where $f\colon M \to \mathbb{C}$ is a section of L. Consider in M the real polarization $\mathcal{P}$ determined by the circumferences with center at the origin of $\mathbb{R}^2$. In this case, $\mathcal{P}$ admits a global basis given by the vector field

$$X_H = -p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p} \equiv \frac{\partial}{\partial \Theta},$$

which is just the Hamiltonian vector field associated with H. Therefore, the $GL(1,\mathbb{C})$-principal fiber bundle, P, of references of $\mathcal{P}$ is a trivial bundle and a global trivialization is given by

$$\begin{array}{ccc} M \times GL(1,\mathbb{C}) & \longrightarrow & P \\ (x,z) & \mapsto & zX_H(x) \end{array}$$

which is a diffeomorphism. (Observe that $(x,1) \mapsto X_H(x)$). On the other hand, according to the definitions we have that $H^2(M,\mathbb{Z}_2) = 0$ and $H^1(M,\mathbb{Z}_2) = \mathbb{Z}_2$, (both because S^1 is a deformation retrac of M), and hence P admits two equivalence classes of associated metalinear bundles. We are going to describe and quantify them.

First case: Trivial metalinear bundle.

In this case we have $\bar{P}_1 = M \times ML(1,\mathbb{C})$, and the covering map $\bar{\rho}\colon P_1 \to P$ is given by $\bar{\rho}(x,\bar{g}) := ((x,\rho(\bar{g}))$, where $\rho\colon ML(1,\mathbb{C}) \to GL(1,\mathbb{C})$ is the natural covering.

In order to identify the sections of the bundle $L \otimes N^{1/2} \to M$ you can observe that, if $s_0\colon M \to M \times L$ is the unit section and $\sigma_0 \in \Gamma(M, N^{1/2})$ satisfies that $\sigma_0(x,1) = 1$, (we consider it as a function $\sigma_0\colon \bar{P}_1 \to \mathbb{C}$), then the sections of $L \otimes N^{1/2}$ have the form $fs_0 \otimes \sigma_0$, where f is a function of M in $\mathbb{C}$.

For constructing $H^{\mathcal{P}}$ we have to consider the sections $fs_0 \otimes \sigma_0$ such that $L(X_H)\sigma_0 = 0$, $\nabla_{X_H}(fs_0) = 0$, since the polarization $\mathcal{P}$ is real and X_H generates $\mathcal{P}$ globally. The first condition holds trivially and, for the second one, we have that

$$\begin{aligned} 0 = \nabla_{X_H} f s_0 &= (X_H f)s_0 + 2\pi i f \theta(X_H) s_0 \\ &= \frac{\partial f}{\partial \Theta} s_0 + 2\pi i f\left(-\frac{1}{2}r^2\right) s_0 = \left(\frac{\partial f}{\partial \Theta} - \pi i r^2 f\right) s_0, \end{aligned}$$

and the condition for f is

$$\frac{\partial f}{\partial \Theta} - \pi i r^2 f = 0, \tag{21}$$

besides of having compact support. This equation has no solution. In fact: its general solution should be $f(r,\Theta) = C(r)e^{\pi i r^2 \Theta}$, but it must satisfy that $f(r,\Theta) = f(r, \Theta + 2\pi n)$, $n \in \mathbb{Z}$, and hence $e^{\pi i r^2 2\pi n} = 1$, that is $\pi r^2 \in \mathbb{Z}$, which is absurd if f depends differentiabily on r. Therefore $H^{\mathcal{P}} = 0$ and hence this system cannot be quantified.

Nevertheless, instead of ordinary functions we can take distribution sections of L. This consists of taking sections of the form fs_0 where f is a distribution in M. In this way, we will arrive now to the same equation (21) which has solution. In fact, we can prove a solution of the form $f(r,\Theta) = g(r)e^{ik\Theta}$, with $k \in \mathbb{R}$ and $g(r)$ being a distribution in $\mathbb{R}^+$. We have that

$$ikg(r)e^{ik\Theta} - \pi i r^2 g(r) e^{ik\Theta} = 0,$$

therefore $(k - \pi r^2)g(r) = 0$. Thus $g(r)$ has to be a multiple of the distribution $\delta(r - \sqrt{\frac{k}{\pi}})$ and, in addition, k has to be a non-negative integer since $f(r,\Theta) = f(r, \Theta + 2\pi h)$, $h \in \mathbb{Z}$. Therefore we obtain that a set of non-vanishing vectors of $\mathcal{H}^{\mathcal{P}}$ is given by $e^{ik\Theta}\delta(r - \sqrt{\frac{k}{\pi}})s_0 \times \sigma_0$, with $k \in \{0\} \cup \mathbb{Z}$. In this space we can apply the quantization procedure previously described for the case of functions. The result is that these vectors are eigenvectors of the operator O_H corresponding to the Hamiltonian function. Moreover, since X_H belongs to the polarization, as a consequence of the corollary of the quantization theorem, we have that O_H consists in multiplying by the function $2\pi H = 2\pi r^2$.

Second case: Non trivial metalinear bundle.

In this case, in order to construct the bundle $\bar{p}\colon \bar{P}_2 \to P$, we consider the set $\mathbb{R}^+ \times \mathbb{R} \times ML(1,\mathbb{C})$. We take on it the equivalence relation defined by

$$(r, \Theta, \lambda) \sim (r, \Theta + 2\pi h, \varepsilon^h \lambda), \;\; h \in \mathbb{Z},$$

where ε is the non trivial element of the ker of the morphism $\rho\colon ML(1,\mathbb{C}) \to GL(1,\mathbb{C})$. Let $[r,\Theta,\lambda]$ be the equivalence class of (r,Θ,λ) and let $\bar{P}_2$ the quotient set. We have the natural map

$$\begin{array}{rccc} \bar{p}\colon & \bar{P}_2 & \longrightarrow & P \\ & [r,\Theta,\lambda] & \mapsto & (r, e^{i\Theta}, p(\lambda)) \end{array}$$

It is clear that $\bar{P}_2$ is not the trivial bundle $\bar{P}_1$, since the points we have identified when we construct $\bar{P}_2$ are not the same as in $\bar{P}_1$.

In order to construct a trivial system in $\bar{P}_2$, observe that M is the quotient of $\mathbb{R}^+ \times \mathbb{R}$, by the same equivalence relation, but restricted to this set, that is,

$$(r,\Theta) \sim (r, \Theta + 2\pi h), \;\; h \in \mathbb{Z}.$$

Now, we take the following open sets in M:

$$U_1 \equiv \{re^{i\Theta} : r \in \mathbb{R}^+, \ \Theta \in (0, 2\pi)\}, \quad U_2 \equiv \{re^{i\Theta} : r \in \mathbb{R}^+, \ \Theta \in (-\pi, \pi)\},$$

and, from the natural projection $\bar{p}\colon \bar{P}_2 \to M$, the following sections:

$$\begin{array}{cccc} \bar{s}_1\colon & U_1 & \longrightarrow & \bar{P}_2 \\ & re^{i\Theta} & \mapsto & [r, \Theta, 1] \end{array} \qquad \begin{array}{cccc} \bar{s}_2\colon & U_2 & \longrightarrow & \bar{P}_2 \\ & re^{i\Theta} & \mapsto & [r, \Theta, 1] \end{array}$$

Then, $\{U_i, \bar{s}_i\}$ is a trivial system of the bundle $\bar{p}\colon \bar{P}_2 \to M$. The transition function $c_{12} = \bar{s}_2 \circ \bar{s}_1^{-1}$ is

$$c_{12}([r, \Theta, 1]) = (\bar{s}_2 \circ \bar{s}_1^{-1})[r, \Theta, 1] = \begin{cases} \bar{s}_2(re^{i\Theta}), & \Theta \in (0, \pi) \\ \bar{s}_2(re^{i(\Theta - 2\pi)}), & \Theta \in (\pi, 2\pi) \end{cases}$$
$$= \begin{cases} [r, \Theta, 1], & \Theta \in (0, \pi) \\ [r, \Theta - 2\pi, 1] = [r, \Theta, \varepsilon], & \Theta \in (\pi, 2\pi) \end{cases}$$

(remember that $U_{12} = (0, \pi) \cup (\pi, 2\pi)$). Let $\sigma\colon M \to \bar{P}_2$ be a section of $\bar{p}$, then $\bar{p} \circ \sigma\colon M \to P$ is a section of p. For $\bar{s}_1$ and $\bar{s}_2$ we have

$$(p \circ \bar{s}_1)(re^{i\Theta}) = \bar{p}([r, \Theta, 1]) = (re^{i\Theta}, 1), \quad \text{in } U_1,$$
$$(p \circ \bar{s}_2)(re^{i\Theta}) = \bar{p}([r, \Theta, 1]) = (re^{i\Theta}, 1), \quad \text{in } U_2.$$

Observe that, according to the global trivialization defined in P, $X_H(re^{i\Theta})$ corresponds to $(re^{i\Theta}, 1)$, and then both sections $p \circ \bar{s}_1$ and $p \circ \bar{s}_2$ give X_H in their domains.

In order to quantize the non trivial bundle, we start describing the bundle $N^{1/2} \to M$. In U_1 and U_2 we take the sections $\sigma_i\colon U_i \to N^{1/2}$ as follows: we define σ_i as a function $\sigma_i\colon \bar{p}^{-1}(U_i) \to \mathbb{C}$ and we work at a point (taking into account that the invariance condition must be satisfied). If $x \in U_1$, we define $\sigma_1(x) \in N_x^{1/2}$ such that $\sigma_1(x)([\bar{s}_1(x)]) = 1$, and if $x \in U_2$, we define $\sigma_2(x)$ as $\sigma_2(x)([\bar{s}_2(x)]) = 1$. The relation between both sections is as follows: if $\Theta \in (0, 2\pi)$, then $\bar{s}_1(re^{i\Theta}) = \bar{s}_2(re^{i\Theta})$, therefore $\sigma_1(re^{i\Theta}) = \sigma_2(re^{i\Theta})$. If $\Theta \in (\pi, 2\pi)$, then $\bar{s}_1(re^{i\Theta}) = \bar{s}_2(re^{i\Theta})\varepsilon$, and hence

$$\sigma_1\Big((\bar{s}_1(re^{i\Theta}))\Big) = \sigma_1(\bar{s}_2(re^{i\Theta})) = -\sigma_2(\bar{s}_2(re^{i\Theta})).$$

and, in this case, $\sigma_1 = -\sigma_2$.

The sections of $L \otimes N^{1/2}$ are studied in the following way: let $s_0 \colon M \to L$ be the unit section. The elements of $H^{\mathcal{P}}$ are of the form $f_1 s_0 \otimes \sigma_1$ in U_1, and $f_2 s_0 \otimes \sigma_2$ in U_2; and since X_H is a global generator for $\mathcal{P}$, we have that

$$\begin{aligned} L(X_H)\sigma_1 &= 0, \quad \text{in } U_1, & L(X_H)\sigma_2 &= 0, \quad \text{in } U_2, \\ \nabla_{X_H}(f_1 s_0) &= 0, \quad \text{in } U_1, & \nabla_{X_H}(f_2 s_0) &= 0, \quad \text{in } U_2. \end{aligned}$$

Now we can see that the conditions on σ_i are automatically satisfied. In fact, consider $\bar{x} \in \bar{P}_2$ with $\bar{\rho}(\bar{x}) = x$, then there exists $\lambda \in ML(1, \mathbb{C})$ with $\bar{x} = \bar{s}_1(x)\lambda$, therefore

$$\big(L(X_H)\sigma_1\big)(\bar{x}) = \big(L(X_H)\sigma_1\big)(\bar{s}_1(x)\lambda) = \lambda^{-1}\big(L(X_H)\sigma\big)(\bar{s}_1(x)),$$

where the invariance condition of $\bar{X}_H$ (the lifting of X_H to $\bar{P}_2$) is taken into account. Now, if τ_t is a uniparametric group of $\bar{X}_H$, we have that

$$\big(L(X_H)\sigma\big)(\bar{s}_1(x)) = \lim_{t \to 0} \frac{1}{t}\Big(\sigma\Big(\bar{\tau}_t(\bar{s}_1(x))\Big) - \sigma(\bar{s}_1(x))\Big),$$

and using the corresponding local diffeomorphisms we have that

$$\begin{aligned} \bar{\tau}_t(\bar{s}_1(x)) &= \bar{\rho}^{-1}(T\tau_t\bar{p}(\bar{s}_1(x)) = \bar{\rho}^{-1}(T\tau_t X_H(x)) \\ &= \bar{\rho}^{-1}\Big(X_H(\tau_t(x))\Big) = \bar{\rho}^{-1}\Big(\bar{p} \circ \bar{s}_1(\tau_t(x))\Big) = \bar{s}_1(\tau_t(x)), \end{aligned}$$

where we have taken into account that $\bar{p} \circ s_1 = X_H$; and hence

$$\begin{aligned} \big(L(X_H)\sigma_1\big)(\bar{s}_1(x)) &= \lim_{t \to 0} \frac{1}{t}\Big((\sigma_1 \circ \bar{s}_1)(\tau_t(x)) - (\sigma_1 \circ \bar{s}_1)(x)\Big) \\ &= \big(X_H(\sigma_1 \circ \bar{s}_1)\big)(x) = 0, \end{aligned}$$

since $\sigma_1 \circ \bar{s}_1$ is the unit constant function. In the same way we obtain that $L(X_H)\sigma_2 = 0$ in U_2.

The conditions on $f_i s_0$ give again the equation (21), therefore $H^{\mathcal{P}} = 0$ unless we work with distribution sections as in the above case.

For other examples see [73], [76], [92].

7. Some ideas on geometric quantization of constrained systems

The geometric quantization programme which we have explained along the above sections deals with *regular* dynamical systems; that is, systems whose classical phase space is represented (totally or partially) by a symplectic manifold. Nevertheless, Dirac and Bergmann started early the study of *singular* (or *constrained*) dynamical systems and their quantization [11], [25], [26]. Hence we expect that the methods of geometric quantization can be applied for quantizing these systems (perhaps after some minor modifications). Geometrically this means to quantize presymplectic manifolds. Thus, in this brief section we make an introduction on how the geometric quantization programme is applied to singular systems and the problems arising in this procedure. Representative references on this subject are [8], [18], [19], [33], [36],[57], [61], [74], [78]; and we refer to them for a more detailed expositions on the topics of this section. Survey expository works on the geometrical description of classical singular systems are, for instance, [12], [20], [35], [37], [64].

7.1. GENERAL SETTING AND ARISING PROBLEMS. As it is known, the phase space of a constrained system is a presymplectic manifold (C, ω_C). But, in order to apply the geometric quantization procedure we need a symplectic manifold. Then, there are three kinds of symplectic manifolds associated with (C, ω_C), namely: some extended phase space (M, Ω), the reduced phase space $(\tilde{C}, \tilde{\Omega})$, and some "gauge fixed manifold" (P, Ω_P). The first one can be constructed, in general, by applying the *coisotropic imbedding theorem* [34], [60]. Then M is a tubular neighbourhood of C, considered as the zero section of the dual bundle $(\ker\, \omega_C)^*$ (where $\ker\, \omega_C$ denotes the characteristic bundle of C); and M is unique up to local symplectomorphisms between tubular neighbourhoods. In the second one, $\tilde{C}$ is the quotient $C/\ker\, \omega_C$ (which is assumed to be a differentiable manifold). Finally, a gauge fixed manifold is a global section of the canonical projection $\rho\colon C \to \tilde{C}$. Thus, the quantization of a constrained systems is based on the quantization of some of these symplectic manifolds.

Geometric quantization via gauge fixing conditions is not the most suitable way. The most usual forms of quantizing a constrained system consist in using the reduced or some extended phase space, and these are the only cases that we discuss here (comments and results on the geometric quantization via gauge fixing conditions can be found in [74]).

As it is known the true space of physical states of a constrained systems is the reduced phase space $(\tilde{C}, \tilde{\Omega})$. Hence, the most reasonable thing seems to quantize it directly. Nevertheless, in general, this is a difficult task because:

- The topology of $\tilde{C}$ is very complicated.
- Covariance is lost in the reduction procedure.

- Additional assumptions on C are required in order to $\tilde{C}$ have a suitable structure for quantization.
- $\tilde{C}$ has singularities, etc.

In spite of these problems, the geometric quantization of the reduced phase space can be successfully carried out in a significant number of cases [8], [33], [83]; and then we will denote $(\tilde{\mathcal{H}}, \mathcal{O}(\tilde{\mathcal{H}}))$ the Hilbert space and the set of operators so obtained. In these cases, the advantage of the method is that, both constraints and gauge symmetries are incorporated and divided out at the classical level and they have not to be considered at the quantum level: $\tilde{\mathcal{H}}$ is the intrinsic Hilbert space and $\mathcal{O}(\tilde{\mathcal{H}})$ is the set of quantum operators corresponding to the constrained system.

Although it is not the true physical phase space of a constrained systems, quantization of the extended phase space is more natural by several reasons:

- The topology and geometry of M are simpler than those of $\tilde{C}$.
- No additional requirements must be assumed for C.
- In many cases, the extended phase space M is an initial datum of the problem and it has not to be constructed; although, in these cases, C must be a coisotropic submanifold of M. If C is not coisotropically imbedded in M, then second class constraints must be previously removed in order to achieve a consistent quantization; and then the Poisson bracket must be replaced by a new operation called *Dirac bracket*. In geometrical terms, this correspond to take a symplectic manifold where the submanifold C is coisotropically imbedded, and then use the Poisson bracket operation defined on it by the corresponding symplectic structure [72].

This method involves two steps: starting from (M, Ω), we first obtain the corresponding pair $(\mathcal{H}^{ext}, \mathcal{O}(\mathcal{H}^{ext}))$. But $\mathcal{H}^{ext}$ is not the true intrinsic Hilbert space of the constrained systems, since constraints have not been taken into account. Then, they have to be enforced at the quantum level. The way to implement this is known as the *Dirac's method* of quantization of constrained systems, whose guidelines are the following:

- It is assumed that the final constraint submanifold C is defined in M as the zero set of a family of constraint functions $\{\zeta\}$, and that these constraints are quantizable, that is, $\{O_\zeta\} \subset \mathcal{O}(\mathcal{H})$.
- Then, constraints are enforced at the quantum level, demanding that the set of admissible quantum states is $\mathcal{H}^C := \{|\psi\rangle \in \mathcal{H} \colon \; O_\zeta|\psi\rangle = 0\}$.

The translation of this procedure in terms of geometric quantization was carried out mainly by Gotay *et al* [34], [36]. This method leads to consistent results only if it allows us to obtain a representation of the Lie algebra of the gauge group on $\mathcal{H}^{ext}$ and, in order to achieve it, some previous requirements are needed (see [36]). When the Dirac's method goes on, a subset $\mathcal{H}^C \subset \mathcal{H}^{ext}$ is obtained and it contains the physical quantum wave functions of the constrained system.

Then, a subsequent problem to be taken into account is that $\mathcal{H}^C$ is not always a Hilbert space, and the way to make it into a Hilbert space $\mathcal{H}_0$ is not clear in general. When this is possible, $\mathcal{H}_0$ and $\mathcal{O}(\mathcal{H}_0))$ are the intrinsic Hilbert space and the quantum operators of the constrained system.

7.2. COMPARISON BETWEEN METHODS. In this way, several new problems arise in relation to the geometric quantization of a constrained system. In fact:

1. In some cases, to quantize both the reduced and the extended phase space simultaneously is not always possible, since:
   a) Sometimes the reduced phase space $(\tilde{C}, \tilde{\Omega})$ cannot be quantized.
   b) Some (or none) extended phase space (M, Ω) is not quantizable, because the Dirac's method is not applicable.
   c) The set $\mathcal{H}^C$ (if it exists) is no longer a Hilbert space.
2. If both methods of quantization go on, it is expected to be equivalent[26], that is, the Hilbert spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}_0$ would be unitarily isomorphic; and the same for the corresponding sets of quantum operators. Unfortunately, as it was analyzed first in [7], this is not the case because there are two kind of obstructions:
   a) The geometrical structures needed for the quantization of (M, Ω) (hermitian line bundles, polarizations, metalinear bundles) could not be invariant under the action of the gauge group and then they cannot be ρ-projectable in order to obtain compatible geometrical structures for the quantization of $(\tilde{C}, \tilde{\Omega})$. Thus, both quantization procedures are incompatible.

[26]In fact, at the classical level it is equivalent to construct the reduced phase space $(\tilde{C}, \tilde{\Omega})$ or, starting from the extended phase space (M, Ω), to carry out the constraints and divide the symmetries out. Hence we can expect the same equivalence to be true at the quantum level.

b) Both quantization procedures can be compatible in the above sense, but a second obstruction can appear when we introduce the inner product on $\mathcal{H}^{ext}$ and $\tilde{\mathcal{H}}$. In fact, $\mathcal{H}_0$ could not inherit an inner product from $\mathcal{H}^{ext}$ or, if does it, the equivalence between both quantizations could not be extended to an unitary isomorphism between $\mathcal{H}_0$ and $\tilde{\mathcal{H}}$.

In the literature, these questions are referred as the noncommutativity of the procedures of *reducing* (or *constraining*) and *quantizing*; and they constitute one of the main problems in the quantization of constrained systems.

A lot of works have been devoted to discuss these topics:

For example, geometric quantization of constrained systems is studied in [33] for the particular case when the extended symplectic manifold is a cotangent bundle; showing that, under some general hypothesis, reduction and quantization commute. In relation to this last case, in [28], some examples for which these hypothesis do not hold are discussed (and then, the problem of quantization is solved in the ambient of the BRST theory).

Another more generic example of noncommutativity between reduction and quantization is given in [57], where the quantization of the extended phase space leads to non-equivalent different possibilities, in general.

[36] is another classical reference, where geometric quantization via coisotropic imbedding is studied. After giving the conditions for which quantization is independent on the choice of the ambient symplectic manifold other problems concerning this way of quantizing are commented; namely: how the quantization procedure depends on the choice of a basis of constraints defining C in M, or what happens if some of these constraint functions is not quantizable or, even, how to quantize when there are not constraints defining C in M.

Finally, [8] is mainly devoted to the study and comparison of polarizations in both ways of quantization.

As a final remark, it is important to point out that many of the problems concerning geometric quantization of singular systems are solved using the more recent techniques of the BRST theory. The explanation of this method goes far from the aim of this work. Some references on this topic are [4], [5], [27], [43], [53], [58], [79], [80] (as it is obvious, this list is far to be complete).

Appendix. Bundles associated with group actions

Let $p : P \to M$ be a principal fiber bundle with structural group G and let Q be a differentiable manifold with a differentiable left action of G in Q. Consider the set $P \times Q$ and the action of G defined by

$$\begin{array}{ccc} G \times (P \times Q) & \longrightarrow & P \times Q \\ g, (a, q) & \mapsto & (ag, g^{-1}q) \end{array}$$

Let $P \times_G Q$ be the orbit space of this action. This is the quotient of $P \times Q$ by the equivalence relation $(a, q) \sim (r, s) \Leftrightarrow \exists g \in G \mid (ag, g^{-1}q) = (r, s)$ We have the natural projection

$$\begin{array}{cccc} \pi : & P \times_G Q & \longrightarrow & M \\ & \overline{(a, q)} & \mapsto & p(a) \end{array}$$

Let $\{U_\alpha, \psi_\alpha\}$ be a trivializing system of P. Then $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a diffeomorphism and $\psi_\alpha \circ \psi_\beta^{-1}$ means to multiply by an element of G in the fiber. We can construct the diffeomorphisms

$$p^{-1}(U_\alpha) \times Q \xrightarrow{\psi_\alpha \times id} U_\alpha \times G \times Q$$

which allow us to define the bijections

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times Q \\ \overline{(a, q)} & \mapsto & (p(a), \bar{\varphi}_\alpha(a, q)) \end{array}$$

where $\bar{\varphi}_\alpha(a, q)$ is the unique element of Q such that $(p(a), e, \bar{\varphi}_\alpha(a, q)) \sim (\psi_\alpha(a), q)$. Notice that φ_α is well defined; in fact, consider $(a, q) \sim (a_1, q_1)$, that is, $\exists g \in G \mid (a_1 g, g^{-1} q_1) = (a, q)$ then

$$\begin{aligned} (\psi_\alpha \times id)(a, q) &= (\psi_\alpha \times id)(a_1 g, g^{-1} q_1) = (\psi_\alpha(a_1 g), g^{-1} q_1) \\ &= (\psi_\alpha(a_1 g) g^{-1}, q_1) = (\psi_\alpha(a_1), q_1) = (\psi_\alpha \times id)(a_1, q_1). \end{aligned}$$

Imposing the condition of φ_α to be diffeomorphisms, we obtain on $\pi^{-1}(U_\alpha)$ a structure of differentiable manifold which, on its turn, gives a fiber bundle structure on $P \times_G Q$ with the projection $\pi : P \times_G Q \to M$, with fiber Q. So we have that $\pi : P \times_G Q \to M$ is a fiber bundle over M with fiber Q which is say to be *associated* with P by the action of G in Q. If $\{U_\alpha, \psi_\alpha\}$ is a trivializing system of P, then we can construct another one of $P \times_G Q$ by means of

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times Q \\ \overline{(a, q)} & \mapsto & (p(a), \bar{\varphi}_\alpha(a, q)) \end{array}$$

where $(\psi_\alpha(a), q) \sim (p(a), 1, \bar{\varphi}_\alpha(a, q))$.

Now, let $\sigma : M \to P \times_G Q$ be a section of π. We can interprete σ as a map $\tilde{\sigma} : P \to Q$ in the following way: let $a \in P$ and $x = p(a)$, we take $\sigma(x) = \overline{(a, z)}$ and then we define $\tilde{\sigma}(a) := z$, that is, we take $\tilde{\sigma}(a)$ in such a way that $\sigma(x) = \overline{(a, \tilde{\sigma}(a))}$. You can observe that there is only one element of the class of $\sigma(x)$ whose representative has $a \in P$ as the first element, that is, such that $\sigma(p(a)) = \overline{(a, \tilde{\sigma}(a))}$. It is easy to prove that $\tilde{\sigma}$ is differentiable.

Thus we have a map

$$\begin{array}{rccc} \eta : & \Gamma(P \times_G Q) & \longrightarrow & C^\infty(P, Q) \\ & \sigma & \mapsto & \tilde{\sigma} \end{array}$$

Now we can ask for the image of η, that is, given a map ϕ, which condition satisfies ϕ in order to be the map associated with a section of π? Let σ be a section of π and $a, b \in P$ such that $p(a) = p(b)$, then, if $b = ag$, we have

$$\sigma(p(a)) = \sigma(p(b)) = \overline{(a, \tilde{\sigma}(a))} = \overline{(b, \tilde{\sigma}(b))} = \overline{(ag, \tilde{\sigma}(b))} = \overline{(a, g\tilde{\sigma}(b))}.$$

That is, if $b = ag$, then

$$\tilde{\sigma}(a) = g\tilde{\sigma}(b) \ \Rightarrow \ \tilde{\sigma}(ag) = g^{-1}\tilde{\sigma}(a).$$

This condition characterizes the image of η. In fact, let $\phi : P \to Q$ be a map satisfying that $\phi(a) = g\phi(b)$ if $b = ag$. We construct a section $\tilde{\phi} : M \to P \times_G Q$ in the following way: $\tilde{\phi}(x) = \overline{(a, \phi(a))}$, a being any element of P such that $\pi(a) = x$. Obviously $\eta(\tilde{\phi}) = \phi$.

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