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Completely Continuous Multilinear Operators on $C(K)$-Spaces

IGNACIO VILLANUEVA DÍEZ *

Depto. de Análisis Matemático, Universidad Complutense de Madrid,
28040-*Madrid, Spain*

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The purpose of this note is to announce, without proofs, some results concerning vector valued multilinear operators on a product of $C(K)$ spaces.

First we will clear our notation: if K is a compact Hausdorff space, $C(K)$ will be the space of scalar valued continuous functions on K, Σ will denote the σ-algebra of the Borel sets of K, and $B(\Sigma)$ will be the space of Σ- measurable functions on K that are limit of simple functions. If X is a Banach space, X^* denotes its dual. We shall use the symbol $.\overset{[i]}{..}.$ to mean that the i-th coordinate is not involved. For definitions and notations concerning polymeasures see [2] or [1].

As it is well known, the Riesz's representation theorem gives a representation of the operators on $C(K)$ as integrals with respect to Radon measures, and this has been very fruitfully used in the study of the properties of the $C(K)$ spaces and the operators defined on them. In a series of papers (see specially [2], [3]), Dobrakov developed a theory of *polymeasures* , functions defined on a product of σ-algebras which are separately measures, that can be used to obtain a Riesz-style representation theorem for multilinear operators defined on a product of $C(K)$ spaces. Our aim is to exploit both representation theories to study multilinear operators on $C(K)$ spaces. In this note we present some of our first results in this direction. First we will need an extension theorem which can be found in [1].

THEOREM 1. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ be a k linear X-valued operator on $C(K_1)\times\cdots\times C(K_k)$. There is a unique $\tilde{T} \in \mathcal{L}^k(B(\Sigma_1), \dots, B(\Sigma_k), X^{**})$*

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that extends T and is $\omega^* - \omega^*$ separately continuous (the ω^*-topology that we consider in $B(\Sigma_i)$ is the one induced by the ω^*-topology of $C(K_i)^{**}$). Besides, we have

1. $\|T\| = \|\tilde{T}\|$.

2. *For every $(g_1, \overset{[i]}{\ldots}, g_k) \in B(\Sigma_1 \times \overset{[i]}{\ldots} \times B(\Sigma_k)$ there is a unique X^{**}-valued bounded ω^*-Radon measure $\gamma_{g_1, \overset{[i]}{\ldots}, g_k}$ on K_i (i.e., a X^{**}-valued finitely additive bounded vector measure on the Borel subsets of K_i, such that for every $x^* \in X^*$, $x^* \circ \gamma_{g_1, \overset{[i]}{\ldots}, g_k}$ is a Radon measure on K_k), verifying*

$$\int g_i d\gamma_{g_1, \overset{[i]}{\ldots}, g_k} = \tilde{T}(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_k), \quad \forall g_i \in B(\Sigma_i).$$

3. *$\tilde{T}$ is and $\omega^* - \omega^*$ sequentially continuous.*

A multilinear operator $T \in \mathcal{L}^k(E_1, \ldots, E_k; X)$ is said to be completely continuous if, given for every $i = 1 \ldots k$ a weakly Cauchy sequence $(x_i^n)_{n \in \mathbb{N}} \subset E_i$, $T(x_1^n, \ldots, x_k^n)$ is norm convergent. These operators are studied, among other places, in [5], [6] and [7].

Our first statement is a crucial lemma for the proof of the main results. Its proof is easy.

LEMMA 2. *With the notations of theorem 1, if T is completely continuous and for $1 \leq j \leq k$, $j \neq i$, $(f_j^n) \subset C(K_j)$ are weakly Cauchy sequences, then the measures $(\gamma_{f_1^n, \overset{[i]}{\ldots}, f_k^n})_{n \in \mathbb{N}}$, defined as in theorem 1, are uniformly countably additive.*

The next result is also be needed.

LEMMA 3. ([1], Corollary 4) *With the notations of theorem 1, if $\tilde{T}$ is X-valued then $\tilde{T}$ is $\omega^* - \|\cdot\|$ sequentially continuous.*

Using lemmas 2 and 3 and a modification of an idea in [4], we obtain the following result, which is the main one in this note.

THEOREM 4. *Let $K_1, \ldots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \ldots, C(K_k); X)$. Let $\tilde{T}$ be the extension of T given by theorem 1. Then T is completely continuous if and only if $\tilde{T}$ is X-valued.*

The next result is obtained as a direct application of lema 2 and theorem 4.

PROPOSITION 5. *With the notations of theorem 1, T is completely continuous if and only if $\tilde{T}$ is also completely continuous.*

The proof of the previous proposition proves also our last result, which is a strengthening of theorem 1.

PROPOSITION 6. *With the notations of theorem 1, if T is completely continuous and for $1 \leq j \leq k$, $j \neq i$, $(g_j^n) \subset B(\Sigma_j)$ is a weakly Cauchy sequence, then the measures $(\gamma_{g_1^n,\overset{[i]}{\dots},g_k^n})_{n\in\mathbb{N}}$ defined as in 2, theorem 1, are uniformly countably additive.*

It is worth looking at these results from the point of view of polymeasures. For the definition of polymeasure, countably additive polymeasure, semivariation of a polymeasure, integral respect to a polymeasure, etc. see [2], [8] or [1]. We will denote the semivariation of a polymeasure γ by $\|\gamma\|$.

Following is the main representation theorem for multilinear operators to which we refered at the begining of this note. It can be found in [1], theorem 9.

THEOREM 7. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$. Let $\tilde{T}$ be the extension of T given by theorem 1. If we define $\gamma : B(\Sigma_1) \times \cdots \times B(\Sigma_k) \mapsto X^{**}$ as*

$$\gamma(A_1, \dots, A_k) = \tilde{T}(\chi_{A_1}, \dots \chi_{A_k}),$$

then γ is a polymeasure of bounded semivariation that verifies

(a) $\|T\| = \|\gamma\|$.

(b) $T(f_1, \dots, f_k) = \int (f_1, \dots, f_d) d\gamma$ $(f_i \in C(K_i))$

(c) *For every $x^* \in X^*$, $x^* \circ \gamma$ is a separately regular polymeasure and the map $x^* \mapsto x^* \circ \gamma$ is continuous for the topologies $\sigma(X^*, X)$ and $\sigma((C(K_1)\hat{\otimes} \cdots \hat{\otimes} C(K_k))^*, C(K_1)\hat{\otimes} \cdots \hat{\otimes} C(K_k))$.*

Conversely, if $\gamma : B(\Sigma_1) \times \cdots \times B(\Sigma_k) \mapsto X^{**}$ is a polymeasure which verifies (c), *then it has finite semivariation and formula* (b) *defines a k-linear continuous operator from $C(K_1) \times \cdots \times C(K_k)$ into F for which* (a) *holds.*

Therefore the correspondence $T \leftrightarrow \gamma$ is an isometric isomorphism.

It is known that $\tilde{T}$ is X-valued if and only if its representing polymeasure γ is countably additive; this fact can be found in [8] theorem 2.16 or in [3] theorem 6 (in the last reference it is covered only the case when γ is defined on the product of Baire sets of K_i, and T is defined on the product of spaces of Baire-measurable functions on K_i which can be written as the uniform limit of Baire-simple functions).

According to the aforementioned result, theorem 4 can be reformulated in the following way:

THEOREM 8. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$. Let γ be the polymeasure associated to T by theorem 7. Then T is completely continuous if and only if γ is countably additive.*

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