EXTRACTA MATHEMATICAE Vol. **12**, Núm. 3, 231 – 241 (1997)

A 2-Dimensional Cohomology with Coefficients in Categorical Groups

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(Research paper presented by A.M. Cegarra)

AMS *Subject Class.* (1991): 18D10, 18G55, 55U10 *Received January* 13, 1997

One of the applications of singular cohomology in algebraic topology is given by Eilenberg-MacLane's classification theorem, which states that when Y is an Eilenberg-MacLane space $K(\pi, n)$, i.e., $\pi_i(Y) = 0$ for all $i \neq n$, and X is an arbitrary CW-complex, then the set $[X, Y]$, of homotopy classes of continous maps from X to Y, is in one-to-one correspondence with the group $H^n(X, \pi)$ of singular cohomology of X with coefficients in $\pi = \pi_n(Y)$.

In this paper we extend this cohomology classification theorem to spaces Y with $\pi_i(Y) = 0$ for all $i \neq 1, 2$. In order to do this, since the homotopy type of these spaces is not given by a group but by a categorical group, we introduce a 2-cohomology set, $\mathbf{H}^2(\underline{\mathbb{E}}_\bullet)$, of a cosimplicial complex of categorical groups $\underline{\mathbb{E}}_\bullet$, which is inspired from Ulbrich's cohomology definition [11]. When $\underline{\mathbb{E}}_\bullet = \underline{\mathbb{E}}^{X_\bullet}$, $X_\bullet$ a simplicial set and $\underline{\mathbb{E}}$ a categorical group, we obtain the 2-cohomology set of $X_\bullet$ with coefficients in $\underline{\mathbb{E}}$, $\mathbf{H}^2(X_\bullet, \underline{\mathbb{E}})$, which gives the cohomology classification theorem.

This 2-cohomology set coincides with the classical H^2 when abelian groups are taken as coefficents, with Ulbrich cohomology when $X_\bullet$ is $K(G, 1)$, G a group, and $\underline{\mathbb{E}}$ is a Picard category and with Dedecker's cohomology $\mathbb{H}^2_{Deck}(G, \Phi)$ of G with coefficients in a crossed module Φ,when $X_\bullet = K(G, 1)$ and $\underline{\mathbb{E}}$ is a strict categorical group with Φ its associated crossed module.

Let us recall that a categorical group $\underline{\mathbb{E}} = (\mathbb{E}, \otimes, a, I, l, r)$ consists of a groupoid $\mathbb{E}$, a functor $\otimes : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$, an object I of $\mathbb{E}$ and natural morphisms

$$a = a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

[1]Supported by DGICYT: PB94-0823

$$l = l_A : I \otimes A \to A \qquad r = r_A : A \otimes I \to A$$

which satisfy coherence conditions [8], and such that for each object A there is an object A^o and a morphism $A^o \otimes A \to I$. We refer to [8], [6],[10] for the background about monoidal categories.

1. Definition of $\mathbf{H}^2$.

For the definition of the cohomology we consider a cosimplicial complex of categorical groups and monoidal functors

$$\underline{\mathbb{E}}_\bullet \;=\; \begin{tikzcd} \underline{\mathbb{E}}_0 \arrow[r, shift left, "\varepsilon_0"] \arrow[r, shift right, "\varepsilon_1"'] & \underline{\mathbb{E}}_1 \arrow[l, bend right, "\delta_0"'] \arrow[r, shift left=2, "\varepsilon_0"] \arrow[r] \arrow[r, shift right=2, "\varepsilon_2"'] & \underline{\mathbb{E}}_2 \arrow[l, bend right, "\delta_0"'] \arrow[l, bend right=40, "\delta_1"'] \arrow[r, shift left=3, "\varepsilon_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "\varepsilon_3"'] & \underline{\mathbb{E}}_3 \cdots \arrow[l, bend right, "\delta_0"'] \arrow[l, bend right=40] \arrow[l, bend right=50, "\delta_2"'] \end{tikzcd} \tag{1}$$

which satisfies the usual cosimplicial identities [1]. A *2-cocycle of* $\underline{\mathbb{E}}_\bullet$ is a pair (P, g) consisting of an object P of $\mathbb{E}_1$ and an arrow $g : \varepsilon_0(P) \otimes \varepsilon_2(P) \to \varepsilon_1(P)$ in $\mathbb{E}_2$ such that the following diagram (*cocycle condition*) is commutative:

$$\begin{tikzcd} \varepsilon_0\varepsilon_0(P) \otimes \varepsilon_0\varepsilon_2(P) \otimes \varepsilon_3\varepsilon_2(P) \arrow[r, "\varepsilon_0(g)\otimes 1"] \arrow[d, "1\otimes\varepsilon_3(g)"'] & \varepsilon_0\varepsilon_1(P) \otimes \varepsilon_3\varepsilon_2(P) \arrow[d, "\varepsilon_2(g)"] \\ \varepsilon_0\varepsilon_0(P) \otimes \varepsilon_3\varepsilon_1(P) \arrow[r, "\varepsilon_1(g)"] & \varepsilon_2\varepsilon_1(P) \end{tikzcd} \tag{2}$$

where we have omitted some canonical associativity morphisms and some canonical structure morphisms of the ε_j's.

The *groupoid of 2-cocycles* of $\underline{\mathbb{E}}_\bullet$, $\mathbf{Z}^2(\underline{\mathbb{E}}_\bullet)$, has as objects the 2-cocycles and as arrows, $f : (P, g) \to (P', g')$, arrows $f : P \to P'$ in $\mathbb{E}_1$ such that the diagram

$$\begin{tikzcd} \varepsilon_0(P) \otimes \varepsilon_2(P) \arrow[r, "g"] \arrow[d, "\varepsilon_0(f)\otimes\varepsilon_2(f)"'] & \varepsilon_1(P) \arrow[d, "\varepsilon_1(f)"] \\ \varepsilon_0(P') \otimes \varepsilon_2(P') \arrow[r, "g'"] & \varepsilon_1(P') \end{tikzcd} \tag{3}$$

is commutative.

Let $\mathbb{Z}^2(\underline{\mathbb{E}}_\bullet)$ denote the set of connected components of $\mathbf{Z}^2(\underline{\mathbb{E}}_\bullet)$. Then we can define an action of $\pi_0(\underline{\mathbb{E}}_0)$, the group of connected components of $\mathbb{E}_0$, on $\mathbb{Z}^2(\underline{\mathbb{E}}_\bullet)$ as follows: for each object Q of $\mathbb{E}_0$ and each 2-cocycle (P,g), using the density of the functor $-\otimes\varepsilon_1(Q)$, we choose an object ${}^QP$ and an isomorphism $f_{Q,P} : {}^QP\otimes\varepsilon_1(Q)\to\varepsilon_0(Q)\otimes P$ in $\mathbb{E}_1$. Then the action is given by:

$$\begin{array}{ccc} \pi_0(\mathbb{E}_0)\times\mathbb{Z}^2(\underline{\mathbb{E}}_\bullet) & \longrightarrow & \mathbb{Z}^2(\underline{\mathbb{E}}_\bullet) \\ & & \\ ([Q],[(P,g)]) & \longmapsto & {}^{[Q]}[(P,g)] = \left[({}^QP,{}^Qg)\right] \end{array} \tag{4}$$

where ${}^Qg : \varepsilon_0({}^QP)\otimes\varepsilon_2({}^QP)\to\varepsilon_1({}^QP)$ is given by the commutativity of the diagram:

$$\begin{tikzcd} \varepsilon_0({}^QP)\otimes\varepsilon_2({}^QP)\otimes\varepsilon_1\varepsilon_1(Q) \arrow[r, "{}^Qg\otimes 1"] \arrow[d, "1\otimes\varepsilon_2(f)"] & \varepsilon_1({}^QP)\otimes\varepsilon_1\varepsilon_1(Q) \arrow[dd, "\varepsilon_1(f)"] \\ \varepsilon_0({}^QP)\otimes\varepsilon_2\varepsilon_0(Q)\otimes\varepsilon_2(P) \arrow[d, "\varepsilon_0(f)\otimes 1"] & \\ \varepsilon_0\varepsilon_0(Q)\otimes\varepsilon_0(P)\otimes\varepsilon_2(P) \arrow[r, "1\otimes g"] & \varepsilon_1\varepsilon_0(Q)\otimes\varepsilon_1(P) \end{tikzcd} \tag{5}$$

(where we have again omitted some canonical morphisms).

It is straightforward to see that (4) is a well define map. Moreover if $h : S\otimes\varepsilon_1(Q)\to\varepsilon_0(Q)\otimes P$ is another isomorphism en $\mathbb{E}_1$, with $S\in Obj(\mathbb{E}_1)$, then we get another 2-cocycle (S,s) connected with $({}^QP,{}^Qg)$ by a 2-cocycle morphism $t : ({}^QP,{}^Qg)\to(S,s)$ defined by the commutativity of the diagram:

$$\begin{tikzcd} {}^QP\otimes\varepsilon_1(Q) \arrow[r, "f"] \arrow[d, "t\otimes 1"'] & \varepsilon_0(Q)\otimes P \arrow[d, equal] \\ S\otimes\varepsilon_1(Q) \arrow[r, "h"] & \varepsilon_0(Q)\otimes P \end{tikzcd} \tag{6}$$

We may now stablish the following definition:

Definition. Let $\underline{\mathbb{E}}_\bullet$ be a cosimplicial complex of categorical groups. The 2-cohomology set of $\underline{\mathbb{E}}_\bullet$, $\mathbf{H}^2(\underline{\mathbb{E}}_\bullet)$, is the set of orbits of $\mathbb{Z}^2(\underline{\mathbb{E}}_\bullet)$ by the above action of $\pi_0(\mathbb{E}_0)$.

EXAMPLE 1. Let $A_\bullet = A_0 \rightrightarrows A_1 \Rrightarrow A_2 \Rrightarrow\!\!\!\!\to A_3 \cdots$ be a cosimplicial complex of abelian groups. The addition in each A_n defines in the groupoid $\underline{A_n} = (A_n \rightrightarrows 0)$ (with only one object and group of arrows A_n) a structure of strict categorical group; so we have a cosimplicial complex of categorical groups $\underline{A}_\bullet$ as in (1). It is easy to see that

$$\begin{aligned}\mathbf{H}^2(\underline{A}_\bullet) &\cong \mathbf{H}^2(\ \cdots \longrightarrow A_1 \xrightarrow{\ \partial\ } A_2 \xrightarrow{\ \partial\ } A_3 \longrightarrow \cdots\ )\\ &\cong \mathbf{H}^2(NA_\bullet) = \pi^2(A_\bullet)\end{aligned}$$

where $\partial = \sum(-1)^i\varepsilon_i$ and $NA_\bullet$ is the normalized chain subcomplex of $(A_\bullet, \partial)$, i.e., $NA_n = A_n \cap Ker\delta_0 \cap \cdots \cap Ker\delta_{n-1}$ and $\pi^2(A_\bullet)$ is the cohomotopy group [1].

Any set X can be considered as a discrete category (in which the objects are the elements of X and any arrow is an identity); if $\underline{\mathbb{E}}$ is a categorical group, the category, $\underline{\mathbb{E}}^X$, of functors from X to $\underline{\mathbb{E}}$, inherits the categorical group structure from $\underline{\mathbb{E}}$. Then for any simplicial set $X_\bullet$ and any categorical group $\underline{\mathbb{E}}$, we can obtain a cosimplicial complex of categorical groups (with strict morphisms)

$$\begin{tikzcd}[column sep=large] \underline{\mathbb{E}}^{X_\bullet} = \underline{\mathbb{E}}^{X_0} \arrow[r, shift left, "\varepsilon_0"] \arrow[r, shift right, "\varepsilon_1"'] & \underline{\mathbb{E}}^{X_1} \arrow[l, bend right=40, "\delta_0"'] \arrow[r, shift left=2, "\varepsilon_0"] \arrow[r] \arrow[r, shift right=2, "\varepsilon_2"'] & \underline{\mathbb{E}}^{X_2} \arrow[l, bend right=30, "\delta_0"'] \arrow[l, bend right=50, "\delta_1"'] \arrow[r, shift left=3, "\varepsilon_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "\varepsilon_3"'] & \underline{\mathbb{E}}^{X_3} \cdots \arrow[l, bend right=20, "\delta_0"'] \arrow[l, bend right=35] \arrow[l, bend right=50, "\delta_2"'] \end{tikzcd} \tag{7}$$

where $\varepsilon_i = d_i^*$ and $\delta_j = s_j^*$ are the induced morphisms by the face maps d_i and degeneracy maps s_j, respectively. Then we define:

DEFINITIONS. Let $X_\bullet$ be a simplicial set and $\underline{\mathbb{E}}$ a categorical group. The 2-cohomology set of $X_\bullet$ with coefficients in the categorical group $\underline{\mathbb{E}}$ is the 2-cohomology set of $\underline{\mathbb{E}}^{X_\bullet}$, that is

$$\mathbf{H}^2(X_\bullet, \underline{\mathbb{E}}) = \mathbf{H}^2(\underline{\mathbb{E}}^{X_\bullet}).$$

Let X be a topological space. We define the singular 2-cohomology set of X with coefficients in $\underline{\mathbb{E}}$ by

$$\mathbf{H}^2(X, \underline{\mathbb{E}}) = \mathbf{H}^2(\mathcal{S}(X), \underline{\mathbb{E}})$$

where $\mathcal{S}(X)$ is the total singular complex of X.

EXAMPLE 2. Let X be a topological space and A an abelian group, then considering the strict categorical group $\mathbb{E} = (A \rightrightarrows 0)$, $\mathbf{H}^2(X, A \rightrightarrows 0)$ is just the usual simplicial 2-cohomology group of X with coefficients in A, [9].

EXAMPLE 3. If $X_\bullet$ is $K(G,1)$ with G an arbitrary group and $\mathbb{E}$ is a symmetric categorical group, that is a categorical group $\mathbb{E}$ together with a braiding $c : A \otimes B \to B \otimes A$ such that $c^2 = 1$, then $\mathbf{H}^2(X_\bullet, \mathbb{E})$ is just Ulbrich's cohomology group $H^1(G, \mathbb{E})$ of G with coefficients in the symmetric categorical group $\mathbb{E}$ with trivial G-module structure [11]. And in the particular case of $\mathbb{E} = (A \rightrightarrows 0)$, then $\mathbf{H}^2(X_\bullet, \mathbb{E})$ is the classical Eilenberg-MacLane cohomology group $\mathbf{H}^2(G, A)$.

On the other hand if $X_\bullet$ is $K(G,1)$, and $\mathbb{E}$ is a strict categorical group, then $\mathbf{H}^2(X_\bullet, \mathbb{E})$ coincides with $\mathbb{H}^2_{Deck}(G, \Phi)$, Dedecker's cohomology group of G with coefficients in Φ, where Φ is the crossed module associated to $\mathbb{E}$, [5].

Let $\mathbb{E}_\bullet$ be a cosimplicial complex of categorical groups, as in (1), a normalized 2-cocycle of $\mathbb{E}_\bullet$ is a 2-cocycle (P, g) of $\mathbf{Z}^2(\mathbb{E}_\bullet)$ such that:

1. $\delta_0(P) = I$
2. $\delta_0(g) : \delta_0(\varepsilon_0(P) \otimes \varepsilon_2(P)) \xrightarrow{(1_P \otimes \Phi_0) \circ \Phi} P \otimes I \xrightarrow{r_P} P = \delta_0 \varepsilon_1(P)$
3. $\delta_1(g) : \delta_1(\varepsilon_0(P) \otimes \varepsilon_2(P)) \xrightarrow{(\Psi_0 \otimes 1_P) \circ \Psi} I \otimes P \xrightarrow{l_P} P = \delta_1 \varepsilon_1(P)$

where (Φ, Φ_0) and (Ψ, Ψ_0) are the structure morphisms of δ_0 and δ_1, respectively.

We denote $\mathbf{Z}^2_N(\mathbb{E}_\bullet)$ the groupoid of normalized 2-cocycles whose morphisms are those morphisms $f \in \mathbf{Z}^2(\mathbb{E}_\bullet)$ such that $\delta_0(f) = 1_I$.

The following theorem stablishes the necessary conditions in order the inclusion functor $J : \mathbf{Z}^2_N(\mathbb{E}_\bullet) \hookrightarrow \mathbf{Z}^2(\mathbb{E}_\bullet)$ be an equivalence.

THEOREM. *Let $\mathbb{E}_\bullet$ be a cosimplicial complex of categorical groups such that the morphism $\delta_0 : \mathbb{E}_1 \to \mathbb{E}_0$ is a fibration of groupoids (i.e., for each object P of $\mathbb{E}_1$ and each morphism in $\mathbb{E}_0$, $f : \delta_0(P) \to Q$, there exists a morphism $g : P \to P'$ with $\delta_0(g) = f$), then the inclusion functor $J : \mathbf{Z}^2_N(\mathbb{E}_\bullet) \hookrightarrow \mathbf{Z}^2(\mathbb{E}_\bullet)$ is an equivalence of groupoids. Therefore, the corresponding sets of connected components are isomorphic, $\mathbb{Z}^2_N(\mathbb{E}_\bullet) \cong \mathbb{Z}^2(\mathbb{E}_\bullet)$.*

Proof. We will give the proof only when the morphisms ε_i and δ_j are strict morphisms (note that is the case when $\mathbb{E}_\bullet = \mathbb{E}^{X_\bullet}$, for $X_\bullet$ a simplicial set and $\mathbb{E}$ a categorical group (7)).

Let (P, g) be a 2-cocycle, applying δ_0^2 to g, we get a morphism $\delta_0^2(g)$: $\delta_0(p) \otimes \delta_0(p) \to \delta_0(p)$, and then, using that $\delta_0(P) \otimes -$ is an equivalence of categories, a morphism $\bar{t} : \delta_0(P) \to I$ in $\mathbb{E}_0$ .Then since δ_0 is a fibration, there exists a morphism in $\underline{\mathbb{E}}_1$, $t : P \to P'$, such that $\delta_0(t) = \bar{t}$; in particular, $\delta_0(P') = I$. Moreover, we obtain a 2-cocycle (P', g'), where g' is defined by the conmutativity of the following diagram:

$$\begin{tikzcd}
\varepsilon_0(P) \otimes \varepsilon_2(P) \arrow[r, "g"] \arrow[d, "\varepsilon_0(t)\otimes\varepsilon_2(t)"'] & \varepsilon_1(P) \arrow[d, "\varepsilon_1(t)"] \\
\varepsilon_0(P') \otimes \varepsilon_2(P') \arrow[r, dotted, "g'"] & \varepsilon_1(P')
\end{tikzcd}$$

and $t : P \to P'$ defines a morphism in $\mathbf{Z}^2(\underline{\mathbb{E}}_\bullet)$ from (P, g) to (P', g').

Let us now suppose a 2-cocycle (P, g) with $\delta_0(p) = I$.Then the composition $f = \delta_0(g) \circ r_P^{-1}$ is an automorphism of P and we can define a 2-cocycle (P, g') , where $g' = \varepsilon_1(f) \circ g \circ (\varepsilon_0(f) \otimes \varepsilon_2(f))^{-1}$.

It is straightforward to see that (P, g') is a normalized 2-cocycle and, since $f : (P, g) \to (P, g')$ is a morphism of 2-cocycles, we conclude that J is a dense functor.

Clearly J is a faithfull functor and for the fully, let $f : (P, g) \to (P', g')$ be a morphism between normalized 2-cocycles . Applying δ_0 to the diagram (3) we get the commutative diagram:

$$\begin{tikzcd}
P \otimes I \arrow[r, "r_p"] \arrow[d, "f\otimes\varepsilon_1\delta_0(f)"'] & P \arrow[d, "f"] \\
P' \otimes I \arrow[r, "r_{P'}"] & P'
\end{tikzcd}$$

(note that in the strict case, the normalization conditions for g reduce to $\delta_0(g) = r_p$ and $\delta_1(g) = l_p$), then, by the naturality of r, $f \otimes \varepsilon_1\delta_0(f) = f \otimes 1_I$ and therefore $\varepsilon_1\delta_0(f) = 1_I$, that is, $\delta_0(f) = 1_I$. ∎

Remark. If $\underline{\mathbb{E}}_\bullet = \underline{\mathbb{E}}^{X_\bullet}$, whith $X_\bullet$ a simplicial set and $\underline{\mathbb{E}}$ a categorical group, it is easy to prove that $\delta_0 = s_0^* : \underline{\mathbb{E}}^{X_1} \to \underline{\mathbb{E}}^{X_0}$ is a fibration of groupoids and so, by the above theorem, we can use normalized cocycles to calculate the 2-cohomology set of $X_\bullet$ with coefficients in $\underline{\mathbb{E}}; \mathbf{H}^2(X_\bullet, \underline{\mathbb{E}})$.

2. CLASSIFICATION THEOREM.

It is well known that, in the abelian case, $K(A,n)$-complexes give a homotopy representation theorem for singular cohomology. In this section we obtain an analogous theorem for the cohomology $\mathbf{H}^2(X_\bullet, \underline{\mathbb{E}})$ we have defined, and, as an application, we obtain the classification theorem for the set $[X,Y]$, of homotopy classes of continous maps from X to Y, where Y is a space with the homotopy type of a CW-complex with only two homotopy groups π_1 and π_2.

The complex we use for representation theorem is *the nerve of a categorical group* defined by Carrasco-Cegarra, in [4], as follows: Given $\underline{\mathbb{E}}$ a categorical group, $Ner_2(\underline{\mathbb{E}})$ is the 3-coskeleton of the 3-truncaded simplicial set given by: $(Ner_2(\underline{\mathbb{E}}))_0 = \{I\}$, $(Ner_2(\underline{\mathbb{E}}))_1 = Obj(\mathbb{E})$, $(Ner_2(\underline{\mathbb{E}}))_2 = \big\{(x; A_0, A_1, A_2) \in Mor(\mathbb{E}) \times Obj(\mathbb{E})^3 / x : A_0 \otimes A_2 \to A_1\big\}$ and $(Ner_2(\underline{\mathbb{E}}))_3$ the set of commutative diagrams in $\mathbb{E}$ such as

$$\begin{tikzcd}(A \otimes B) \otimes C \arrow[rr, "a"] \arrow[d, "x_0 \otimes 1"'] & & A \otimes (B \otimes C) \arrow[d, "1 \otimes x_3"] \\ D \otimes C \arrow[r, "x_2"'] & F & A \otimes E \arrow[l, "x_1"]\end{tikzcd}$$

The face and degeneracy operators are given by: $d_i(x; A_0, A_1, A_2) = A_i, 0 \leq i \leq 2$ and $d_j(x_0, x_1, x_2, x_3) = x_j, 0 \leq j \leq 3$, $s_0(A) = (r_A; A, A, I)$, $s_1(A) = (l_A; I, A, A)$ and, for $x : A_0 \otimes A_2 \to A_1$ of $\mathbb{E}$, $s_0(x) = (x, x, r_{A_1}, r_{A_2})$, $s_1(x) = (r_{A_0}, x, x, l_{A_2})$ and $s_2(x) = (l_{A_0}, l_{A_1}, x, x)$.

$Ner_2(\underline{\mathbb{E}})$ is a Kan complex (even more, is a 2-hypergroupoid in the sense of Duskin-Glenn [7]) whose homotopy groups $\pi_i(Ner_2(\underline{\mathbb{E}}))$ are trivial for $i \neq 1, 2$, while $\pi_1(Ner_2(\underline{\mathbb{E}}))$ is isomorphic to $\pi_0(\underline{\mathbb{E}})$, the group of connected components of $\mathbb{E}$, and $\pi_2(Ner_2(\underline{\mathbb{E}})) \cong Aut_{\mathbb{E}}(I)$, the group of automorphisms in I. Then we have:

THEOREM. (REPRESENTATION THEOREM) *Let $\underline{\mathbb{E}}$ be a categorical group and let $X_\bullet$ be a simplicial set. Then there exists a natural bijection*

$$Simpl(X_\bullet, Ner_2(\underline{\mathbb{E}})) \cong Obj(\mathbf{Z}^2_N(X_\bullet, \underline{\mathbb{E}}))$$

between the set of simplicial maps from $X_\bullet$ to $Ner_2(\underline{\mathbb{E}})$ and the set of normalized 2-cocycles of $X_\bullet$ with coefficients in $\underline{\mathbb{E}}$.

Moreover, two simplicial maps are homotopic if, and only if, their corresponding 2-cocycles determine the same element in $\mathbf{H}^2(X_\bullet, \underline{\mathbb{E}})$. Consequently the above bijection induces another one

$$[X_\bullet, Ner_2(\underline{\mathbb{E}})] \cong \mathbf{H}^2(X_\bullet, \underline{\mathbb{E}})$$

between the set of homotopy classes of simplicial maps from $X_\bullet$ to $Ner_2(\underline{\mathbb{E}})$ and the 2 cohomology set of $X_\bullet$ with coefficients in $\underline{\mathbb{E}}$.

Proof. Since $Ner_2(\underline{\mathbb{E}})$ is a reduced 2-hypergroupoid, to give a simplicial morphism from $X_\bullet$ to $Ner_2(\underline{\mathbb{E}})$ is equivalent to give a pair of maps $P = f_1 : X_1 \to Ob(\mathbb{E})$ and $f_2 : X_2 \to Ner_2(\underline{\mathbb{E}})$ such that

- $d_i f_2 = f_1 d_i$, $0 \leq i \leq 2$, that is, for any $x_2 \in X_2$,

$$f(x_2) = (g(x_2); Pd_0(x_2), Pd_1(x_2), Pd_2(x_2))$$

  for some morphism $g(x_2) : Pd_0(x_2) \otimes Pd_2(x_2) \to Pd_2(x_2)$ in $\mathbb{E}$.

- $f_1 s_0 = I$ and $f_2 s_k = s_k f_1$, $k = 0, 1$. That is, for any $x_0 \in X_0$ $ps_0(x_0) = I$ and for any $x_1 \in X_1$, $gs_0(x_1) = r_{P(x_1)}$ and $gs_1(x_1) = l_{P(x_1)}$.

and finally

- The pair of maps f_1, f_2 can be extended to a simplicial morphism from $X_\bullet$ to $Ner_2(\underline{\mathbb{E}})$, which translates into the cocycle condition for the pair (P, g).

Consequently to give a simplicial morphism from $X_\bullet$ to $Ner_2(\underline{\mathbb{E}})$ is equivalent to give a normalized 2-cocycle on $X_\bullet$ with coefficients in $\underline{\mathbb{E}}$.

Now let $h = (h^i_j)$ be a homotopy between the simplicial morphisms associated to the 2-cocycles (P, g) and (P', g') by the above bijection. As before, this homotopy is determined by the maps $Q = h^0_0 : X_0 \to Obj(\mathbb{E})$ and $h^i_1 : X_1 \to (Ner_2(\underline{\mathbb{E}}))_2, i = 0, 1$. The simplicial identities with respect to the face operators allows us to express h^0_1 and h^0_1 as follows: $h^0_1 = \left(\varphi_0; P, d_1 h^0_1 = d_1 h^1_1, Qd_1\right)$ and $h^1_1 = \left(\varphi_1; Qd_0, d_1 h^0_1, P'\right)$,where $\varphi_0, \varphi_1 : X_1 \to Mor(\mathbb{E})$ are maps such that $\varphi_0(x) : P(x) \otimes Qd_1(x) \to d_1 h^0_1(x)$ and $\varphi_1(x) : Qd_0(x) \otimes P'(x) \to d_1 h^1_1(x)$.

It is easy to see that the maps $Q = h^0_0, h^0_1$ and h^1_1 can be extended to a homotopy, between the associated simplicial morphisms to (P, g) and (P', g') if, and only if, the diagram in $\mathbb{E}^{X_2}$:

$$\begin{tikzcd}
Pd_0 \otimes Pd_2 \otimes Qd_1d_1 \arrow[rr, "g\otimes 1"] \arrow[d, "1\otimes(\varphi_1^{-1}\varphi_0)d_2"] & & Pd_1 \otimes Qd_1d_1 \arrow[dd, "(\varphi_1^{-1}\varphi_0)d_1"] \\
Pd_0 \otimes Qd_0d_2 \otimes P'd_2 \arrow[d, "(\varphi_1^{-1}\varphi_0)d_0\otimes 1"] & & \\
Qd_0d_0 \otimes P'd_0 \otimes P'd_2 \arrow[rr, "1\otimes g'"] & & Qd_0d_0 \otimes P'd_1
\end{tikzcd}$$

is commutative.

Then, considering the composition $f = \varphi_1^{-1}\varphi_0 : P \otimes Qd_1 \to Qd_0 \otimes P'$, it turns out that, by the commutativity of the above diagram, (P, g) is in the orbit of (P', g'), that is, ${}^{[Q]}[(P', g')] = [(P, g)]$ (see (5)), and therefore (P', g') and (P, g) define the same element in $\mathbf{H}^2(X_\bullet, \underline{\mathbb{E}})$.

Conversely, let $Q : X_0 \to Obj(\underline{\mathbb{E}})$ be an object of $\mathbb{E}^{X_1}$ and let $\psi : (P, g) \to ({}^QP', {}^Q g')$ be a morphism of normalized 2-cocycles. If $f : {}^QP' \otimes Qd_1 \to Qd_0 \otimes P'$ is the defining map of ${}^QP'$ (which, since (P', g') is a normalized 2-cocycle, can be chosen such that ${}^QP's_0(x) = I$, for all $x \in X_0$, and $fs_0(x) = r_{Q(x)}^{-1} \circ l_{Q(x)} : I \otimes Q(x) \to Q(x) \otimes I$), we define a truncated homotopy by $h_0^0 = I \otimes Q$, $h_1^0 = \Big((r_P \otimes 1_{Qd_1}) \circ a_{P,I,Qd_1}^{-1}, P, P \otimes Qd_1, I \otimes Qd_1)\Big)$ and $h_1^1 = \big((\psi \otimes 1_{Qd_1})^{-1} \circ f^{-1} \circ (l_{Qd_0} \otimes 1_{P'}), I \otimes Qd_0, P \otimes Qd_1, P'\big)$, which extends to a homotopy between the simplicial mophisms associated to (P, g) and (P', g'). ∎

In [4] it is proved the existence of an adjoint situation

$$(\mathbf{Gr-categories})_* \underset{Ner_2}{\overset{\wp_2}{\underset{\sim}{\rightleftarrows}}} \text{reduced Kan Simplicial Sets} \tag{8}$$

where $(\mathbf{Gr-categories})_*$ denotes the category of categorical groups and monoidal functors which are strict on the unit object, and the functor $\wp_2$ applies a pointed Kan simplicial set (not necessarily reduced) to the fundamental groupoid of its loop complex. The counit morphisms of this adjunction are isomorphisms while the unit morphisms are weak 2-equivalences. This allows them to show a new way of how categorical groups give algebraic models for connected homotopy 2-types. Concretely if $X_\bullet$ is a connected Kan simplicial set with trivial homotopy groups at any dimension but 1 and 2, and $* \in X_0$, then $X_\bullet$ is homotopy equivalent to $Ner_2(\wp_2(X, *))$.

Then, as a consequence of above theorem, we may conclude:

THEOREM. *Let X be any space and let Y be pointed CW-complex with $\pi_i(Y) = 0$ for all $i \neq 1, 2$. There exists a bijection*

$$[X, Y] \cong \mathbf{H}^2(X, \wp_2(Y))$$

between the set of homotopy classes of continous maps from X to Y and the singular 2-cohomology set of X with coefficients in $\wp_2(Y)$, the categorical group associated to the pointed Kan simplicial set $S(Y)$.

Remark. It is well known that any categorical group is monoidal equivalent to a strict one. Then, if X is $K(G, 1)$-space, that is, an aspherical space with fundamental group G and Φ is the crossed module associated to a strict categorical group monoidal equivalent to $\wp_2(Y)$, the above theorem particularizes to

$$[X, Y] \cong \mathbb{H}^2_{Deck}(G, \Phi)$$

(see example 3). So we obtain, as a particular case, the classification theorem given by Bullejos-Cegarra in [2].

ACKNOWLEDGEMENTS

We wish to thank A.M. Cegarra since many of the basic ideas behind this paper were developed in collaboration with him.

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