EXTRACTA MATHEMATICAE Vol. **12**, Núm. 2, 145 – 164 (1997)

# Separating Polynomials on Banach Spaces *

R. Gonzalo and J.A. Jaramillo

*Dpto. Análisis Matemático, Fac. Ciencias Matemáticas, Universidad Complutense, 28040-Madrid, Spain. e-mail: jaramil@eucmax.sim.ucm.es*

AMS *Subject Class.* (1991): 46B20, 46E30, 46G05

## 1. Introduction and basic properties

In this paper we survey some recent results concerning separating polynomials on real Banach spaces. By this we mean a polynomial which separates the origin from the unit sphere of the space, thus providing an analog of the separating quadratic form on Hilbert space.

This kind of polynomials were first used by Kurzweil [18,19] in connection with the problem of uniform approximation of an arbitrary continuous function by real-analytic ones in separable Banach spaces. As we will see, the existence of a separating polynomial on a Banach space is a quite restrictive property, which can be seen as a strong form of smoothness. Conversely, results of Deville et al. [5,8] show that combining high order of smoothness with some kind of convexity gives in fact a separating polynomial on the space (see also [10]). This is presented in Section 2. We also give there some geometric and structural conditions satisfied by spaces with a separating polynomial, especially the result of Deville [4] that such a space always contains a copy of $\ell_p$ with $p$ an even integer (see also [13]).

Section 3 is devoted to separating polynomials on symmetric spaces, in both the discrete and the continuous cases. That is, either in spaces with symmetric basis or in rearrangement invariant function spaces. A symmetrization procedure is described which allows us to construct symmetric separating polynomials in this case and, using this, results of Gonzalo, González and Jaramillo [11,13] are given which exactly describe whose of these spaces admit a separating polynomial in the separable case.

Finally, in Section 4 the existence of separating polynomials on spaces $L^p(L^q)$, and in some special subspaces, is studied. In this way we provide some further examples of spaces with separating polynomials.

*Supported in part by DGICYT PB93/0452.

Throughout, $X$ will stand for a real Banach space. Recall that a mapping $P : X \to \mathbb{R}$ is said to be a continuous $k$-homogeneous polynomial if there is a continuous $k$-linear form $A : X \times \cdots \times X \to \mathbb{R}$ such that for all $x \in X$,

$$P(x) = A(x, \ldots, x).$$

A polynomial $P$ from $X$ into $\mathbb{R}$ is a finite sum $P = P_0 + P_1 + \cdots + P_m$ where $P_0$ is constant and each $P_k$ is a continuous, $k$-homogeneous polynomial on $X$ for $k = 1, \ldots, m$. The degree of $P$ is the maximum degree of its summands. Note that we always consider continuous polynomials. For an extensive treatment of polynomials on Banach spaces we refer to the books by Dineen [9] or Mujica [27].

We say that $P$ is a separating polynomial on $X$ if $P(0) = 0$ and

$$\inf\{P(x) : \|x\| = 1\} > 0.$$

In other words, if $P$ separates 0 from the unit sphere of $X$. Note that, since the unit sphere is connected, if there is a polynomial $P$ on $X$ such that $P(0) = 0$ and

$$\inf\{|P(x)| : \|x\| = 1\} > 0.$$

then either $P$ or $-P$ is a separating polynomial on $X$.

Next we present some basic facts about separating polynomials.

1.1. As it is easily seen, every finite dimensional space admits a separating polynomial.

1.2. The nicest example of an infinite dimensional space which admits a separating polynomial is Hilbert space. Indeed, if $B$ is the bilinear form defining the scalar product of a Hilbert space $H$ then $P(x) = B(x, x) = \|x\|^2$ defines a separating polynomial on $H$. Conversely, supose that $X$ is a Banach space which admits an homogeneous separating polynomial $P$ of degree 2. Let $A$ be the bilinear symmetric form associated to $P$, and let $\alpha := \inf\{P(x) : \|x\| = 1\} > 0$. By homogeneity,

$$\alpha\|x\|^2 \leq P(x) = A(x, x) \leq \|A\|\|x\|^2.$$

This means that the expression

$$\|x\| = (P(x))^{1/2}$$

is a Hilbertian equivalent norm on $X$.

1.3. Another examples of spaces with separating polynomials are the spaces $L_p(\mu)$ for any measure $\mu$ and $p = 2n$ an even integer. Indeed, the expression

$$P(f) = \|f\|^{2n} = \int f^{2n}(t)dt \qquad f \in L_{2n}(\mu),$$

defines a $(2n)$-homogeneous separating polynomial on such space. In the case of $\ell_{2n}$, if $\{e_j\}$ denotes the usual basis of the space, the expression

$$P(\sum_{j=1}^{\infty} x_j e_j) = \| \sum_{j=1}^{\infty} x_j e_j \|^{2n} = \sum_{j=1}^{\infty} x_j^{2n}$$

defines a $2n$-homogeneous polynomial on $\ell_{2n}$.

1.4. Infinite-dimensional spaces $L_p(\mu)$, where $p \neq 2k$, do not admit a separating polynomial [2]. In fact, as it is shown in [15], something stronger can be said: if either $n \neq kp$ or $p$ is not an even integer and $P$ is an $n$-homogeneous polynomial on $\ell_p$, then for each infinite-dimensional subspace $Z$ of $\ell_p$ and each $\varepsilon > 0$ there exists an infinite-dimensional subspace $Y \subset Z$ such that $|P(y)| < \varepsilon$ for all $y \in Y$ with $\|y\| = 1$. This means, in particular, that there is no homogeneous separating polynomial on $\ell_p$ whenever $p$ is not an even integer. Also, if $p$ is an even integer and there is a $k$-homogeneous separating polynomial on $\ell_p$ then $k$ must be a multiple of $p$.

1.5. The space $c_0$ does not admit a separating polynomial. This follows from the fact proved by Pelczynski [29], that every polynomial on $c_0$ is weakly sequentially continuous. Therefore, if $P$ is a polynomial on $c_0$ such that $P(0) = 0$ and $\{e_j\}$ is the usual basis, then

$$\inf_{j \in \mathbb{N}} |P(e_j)| = 0$$

and $P$ cannot be separating.

1.6. If $X$ admits a separating polynomial $P$, then $Q(x) = \frac{1}{2}(P(x)+P(-x))$ is also a separating polynomial on $X$ and in addition $Q$ is an even function. Note that $Q = Q_2 + Q_4 + \cdots + Q_{2m}$, where each $Q_{2k}$ is $2k$-homogeneous.

1.7. If $X$ admits a separating polynomial $P$, then there also exists an homogeneous separating polynomial $Q$ on $X$ (see [10]). Indeed, let $P = P_0 + P_1 + \cdots + P_m$ be a separating polynomial, where each $P_k$ is $k$-homogeneous and $P_0$ is constant. Note that $P_0 = 0$. Now define

$$Q = P_1^{2(m!)} + P_2^{2(m!)/2} + \cdots + P_m^{2(m!)/m}.$$

Then $Q$ is a $2(m!)$-homogeneous separating polynomial on $X$.

1.8. Suppose that $X$ admits a separating polynomial $P = P_1 + \cdots + P_m$, where each $P_k$ is $k$-homogeneous, and let $\alpha := \inf\{P(x) : \|x\| = 1\} > 0$. Then the polynomial

$$Q(x) = \left(\frac{m}{\alpha}\right)^2 ((P_1(x))^2 + \cdots + (P_m(x))^2)$$

satisfies that $Q(x) \geq 1$ whenever $\|x\| \geq 1$.

1.9. A finite family of polynomials $\{P_1, \dots, P_n\}$ on $X$ is said to be a separating family if, for all $x \in X$ with $\|x\| = 1$, we have

$$\max_{1 \leq i \leq n} \{|P_i(x)|\} \geq 1.$$

Of course, if there is a separating family of polynomials $\{P_1, \dots, P_n\}$ on $X$, then the space admits a separating polynomial. Indeed, it is enough to consider $P(x) = (P_1(x))^2 + \cdots + (P_n(x))^2$. What is not clear is whether under the assumption of having a separating family of polynomials of degree at most $m$, a separating polynomial of degree at most $m$ may be found.

1.10. The property of having a separating polynomial is invariant under isomorphisms; in other words, if there is an isomorphism between two Banach spaces and one of them admits a separating polynomial, then so the other does. A finite product of spaces with separating polynomial also admits a separating polynomial. Therefore, if a finite-codimensional subspace admits a separating polynomial, then the whole space also does.

1.11. To have a separating polynomial is a hereditary property. As a consequence of this, spaces with separating polynomial do not contain isomorphic copies of $c_0$. For quotient spaces the situation is quite different. Indeed, in [21.I 4.d.10] a quotient of $\ell_2 \oplus \ell_4$ is constructed with the property of containing isomorphic copies of $\ell_r$ for $2 \leq r \leq 4$. Therefore, such quotient space does not admit a separating polynomial. On the other hand, as mentioned in [21.I 2.d.7], $\ell_{4/3}$ contains a subspace isomorphic to $\left(\bigoplus_{n=1}^{\infty} \ell_{5/4}^{(n)}\right)_{\ell_{4/3}}$, and therefore $\ell_4$ has a quotient isomorphic to $\left(\bigoplus_{n=1}^{\infty} \ell_5^{(n)}\right)_{\ell_4}$, but this space does not admit a separating polynomial. We are grateful to Manuel González for providing us these examples.

1.12. Recall that a property $\mathcal{P}$ of Banach spaces is said to be a three space property when given a Banach space $X$ and a closed subspace $Y$ of $X$, if both $Y$ and $X/Y$ have $\mathcal{P}$, then also $X$ has $\mathcal{P}$. We note that to admit a separating polynomial is not a three space property. This result can be deduced from [6 V.1.10], and it is also obtained in [11].

1.13. We say that $X$ has a polynomial norm $\|.\|$ if, for some even integer $k$, we have that $P(x) = \|x\|^k$ is a $k$-homogeneous polynomial on $X$. Of course, if $X$ admits an equivalent polynomial norm then it has a separating polynomial. In fact, we will see that all known examples of spaces which admit a separating polynomial actually have an equivalent polynomial norm. This suggests the following question:

QUESTION 1. Is there a Banach space with separating polynomial but with no equivalent polynomial norm?

Actually, once one can find a convex $k$-homogeneous separating polynomial $P$ on $X$, then the expression $\|x\| = (P(x))^{1/k}$ defines an equivalent polynomial norm on $X$. So, the problem is reduced to find from a separating polynomial a convex separating polynomial.

1.14. We finish this Section with a short comment about separating polynomials and ultrapowers. We refer to [16] for the definition and basic properties of ultraproducts of Banach spaces. Let $\mathcal{U}$ be a nontrivial ultrafilter on a given set $I$. For a Banach space $X$, we denote by $X_{\mathcal{U}}$ the corresponding ultrapower. It is possible to extend each polynomial $P$ on $X$ to a polynomial $P_{\mathcal{U}}$ on $X_{\mathcal{U}}$ by defining

$$P_{\mathcal{U}}((x_i)_{i\in I}) = \lim_{\mathcal{U}} P(x_i).$$

For details see [22], where this and other extensions are studied, and some applications are given. It is easily seen that if $P$ is a separating polynomial, then so is $P_{\mathcal{U}}$. Thus if $X$ admits a separating polynomial (respectively a polynomial norm), then every ultrapower $X_{\mathcal{U}}$ also admits a separating polynomial (resp. a polynomial norm).

Recall that a Banach space $Y$ is said to be finitely representable in $X$ if, for every $\varepsilon > 0$ and every finite-dimensional subspace $Y_0$ of $Y$, there exist a finite-dimensional subspace $X_0$ of $X$ and an isomorphism $T : Y_0 \to X_0$ such that

$$\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon.$$

It is well known that $Y$ is finitely representable in $X$ if, and only if, $Y$ is isometric to a subspace of some ultraproduct $X_{\mathcal{U}}$ (see e.g. [16]). Therefore we obtain that if $X$ admits a separating polynomial (respectively a polynomial norm), and $Y$ is finitely representable in $X$, then also $Y$ admits a separating polynomial (resp. a polynomial norm). This was essentially observed in [31, 2.3].

## 2. Separating polynomials and smoothness

As we mentioned before, the notion of separating polynomial was introduced in 1954 by Kurzweil [18], in order to obtain the following remarkable approximation result:

THEOREM 2.1. *Suppose that $X$ is a separable Banach space which admits a separating polynomial. Then, for every $\varepsilon > 0$, every Banach space $Y$, and every continuous function $f : X \to Y$, there exits a real-analytic function $g : X \to Y$ such that*

$$\sup_{x \in X} \|f(x) - g(x)\| < \varepsilon.$$

In fact, Kurzweil proved that it is possible to obtain not only uniform approximation but also approximation for the fine topology. In [19] a converse of this result is given for uniformly convex spaces $X$. It follows from Deville's results (see Theorem 2.2 below) that a converse of Theorem 2.1 actually holds for spaces $X$ not containing isomorphic copy of $c_0$. The case of $c_0$ is then specially relevant, and it would be very nice to know the following:

QUESTION 2. Is it possible to approximate every continuous real function on $c_0$, uniformly on $c_0$, by real-analytic functions?

Concerning Theorem 2.1, there is also another interesting question:

QUESTION 3. Does Theorem 2.1 hold for nonseparable spaces $X$?

The existence of a separating polynomial on a Banach space $X$ has strong connections with the smoothness of the space. As usual, we say that a real function on $X$ is $C^k$-smooth if it is $k$-times Fréchet differentiable and the $k$-th derivative is continuous. Recall that a function $b : X \to \mathbb{R}$ is said to be a bump function if it has nonempty bounded support, and $X$ is said to be $C^k$-smooth if it admits a $C^k$-smooth bump function. $C^\infty$-smoothness is defined in the same way. For an extensive treatment of smoothess properties of Banach spaces we refer to [6].

Now we point out that if $X$ admits a separating polynomial then it is $C^\infty$-smooth. In order to see it, let $P$ be a polynomial on $X$ such that $P(0) = 0$ and $P(x) \geq 1$ whenever $\|x\| \geq 1$, and consider a $C^\infty$-smooth function $\phi : \mathbb{R} \to \mathbb{R}$

verifying that $\phi(0) = 1$ and $\phi(t) = 0$ if $|t| \geq 1$. Then $\phi \circ P$ is a $C^\infty$-smooth bump function on $X$. Note that, in addition, $\phi \circ P$ has uniformly continuous derivative. It is well known that superreflexive spaces can be characterized as those spaces which admit a $C^1$-bump function with uniformly continuous derivative (see e.g. [6 V.3.2]). In this way we obtain that if $X$ admits a separating polynomial then $X$ is superreflexive.

The typical example of $C^\infty$-smooth space without a separating polynomial is $c_0$. The following result, due to Deville [5], shows that this is essentially the only obstruction.

THEOREM 2.2. *Let $X$ be a Banach space without any isomorphic copy of $c_0$. The following are equivalent:*

(i) *$X$ admits a separating polynomial.*

(ii) *$X$ is $C^\infty$-smooth.*

Combining Theorems 2.1 and 2.2 we obtain:

COROLLARY 2.3. *Let $X$ be a separable Banach space without any isomorphic copy of $c_0$. Then the following statements are equivalent:*

(i) *$X$ admits a separating polynomial.*

(ii) *$X$ is $C^\infty$-smooth.*

(iii) *For every $\varepsilon > 0$ and every continuous function $f : X \to \mathbb{R}$, there exists a real analytic function $g : X \to \mathbb{R}$ such that*

$$\sup_{x \in X} |f(x) - g(x)| < \varepsilon.$$

(iv) *For every Banach space $Y$, every $\varepsilon > 0$ and every continuous function $f : X \to Y$, there exists a real analytic function $g : X \to Y$ such that*

$$\sup_{x \in X} \|f(x) - g(x)\| < \varepsilon.$$

In the sequel we study how combining high order of smoothness with certain properties of convexity we obtain the existence of a separating polynomial.

We say that a function $f : X \to \mathbb{R}$ is $T^p$-smooth, where $1 \leq p < \infty$, if it has a Taylor expansion of order $p$ around each point. That is, if for each

$x \in X$ there exists a polynomial $P$ on $X$ of degree $\leq [p]$, where $[p]$ denotes the integer part of $p$, satisfying that $P(0) = 0$ and

$$|f(x+h) - f(x) - P(h)| = o(\|h\|^p).$$

Note that if $f$ is $m$-times Fréchet differentiable on $X$, then Taylor's theorem gives that $f$ is $T^p$-smooth for $1 \leq p \leq m$. As usual, we say that the space $X$ is $T^p$-smooth if there is a $T^p$-smooth bump function on $X$.

Recall that the modulus of convexity of a norm $\|.\|$ is defined by

$$\delta(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}$$

for $0 \leq \varepsilon \leq 2$.

From Pisier [30] (see also [6 IV.4.8]), every superreflexive space admits an equivalent norm with modulus of convexity of power type $q$, for some $q \geq 2$. This means that there exists a constant $K > 0$ such that $\delta(\varepsilon) \geq K\varepsilon^q$ for all $0 \leq \varepsilon \leq 2$.

Using these properties of smoothness and convexity, the following result is obtained in [8].

THEOREM 2.4. *Suppose that, for some $p \geq 2$, $X$ is $T^p$-smooth and admits an equivalent norm with modulus of convexity of power type $p$. Then $X$ admits a separating polynomial of degree $\leq p$.*

In particular, if $X$ admits a separating polynomial, the above Theorem shows that the polynomial can be chosen to have degree at most the power estimate of the modulus of convexity of any equivalent renorming of $X$.

Theorem 2.4 is similar to Theorem 1 in [10], where a different kind of smoothness on the space is used, and the convexity condition of the norm is substituted by a differentiability condition of the dual norm. In this sense, note that the space $X$ admits a norm with modulus of convexity of power type $p$ if, and only if, the dual space $X^*$ admits an equivalent norm whose derivative is uniformly $(p-1)^{-1}$-Hölder on the unit sphere (see e.g. [6 IV.1.12 and IV.5.1]) if, and only if, $X^*$ admits an equivalent norm whose derivative is pointwise $(p-1)^{-1}$-Hölder on the unit sphere [7].

We recall now a related kind of smoothness. For $1 \leq p < \infty$ we say, according to Meshkov [26], that a function $f : X \to \mathbb{R}$ is $H^p$-smooth if $f$ is $m$ times Fréchet differentiable, where $m$ is the largest integer strictly less than $p$, and the $m$-th derivative is locally uniformly $(p-m)$-Hölder. That is, for each $x \in X$ there exist a neighbourhood $V(x)$ and a constant $M > 0$ such that

$$\|f^{(m)}(y) - f^{(m)}(z)\| \leq M\|y-z\|^{p-m},$$

for all $y, z \in V(x)$.

Note that if $f$ is $H^p$-smooth then it is $T^q$-smooth for $1 \leq q < p$.

As usual, we say that $X$ is $H^p$-smooth if there is a $H^p$-smooth bump function on $X$. Deville obtained in [5] the existence of a separating polynomial using $H^p$-smoothness and a kind of convexity condition formulated in terms of cotype. Recall that a Banach space $X$ is said to have cotype $q$, for $2 \leq q < \infty$, if there exists a constant $C > 0$ such that, for any finite family $x_1, \ldots, x_n \in X$,

$$(\sum_{i=1}^n \|x_i\|^q)^{\frac{1}{q}} \leq C \frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \|\sum_{i=1}^n \epsilon_i x_i\|.$$

Note that if $X$ has modulus of convexity of power type $q$ then $X$ has cotype $q$, but the converse is not true in general (see e.g. [21.II 1.e.16]).

More precisely, the following result is due to Deville [5].

THEOREM 2.5. *Suppose that $X$ is $H^p$-smooth and has cotype $q$ where $p > q$. Then $X$ admits a separating polynomial.*

The existence of a separating polynomial on a Banach space has strong geometrical implications, as was shown by Deville in [4]:

THEOREM 2.6. *Suppose that $X$ admits a separating polynomial. Then $X$ has exact cotype $2k$ and contains an isomorphic copy of $\ell_{2k}$, for some integer $k$.*

This implies that spaces with a separating polynomial are saturated with $\ell_{2k}$ subspaces, since the property of having a separating polynomial is hereditary.

The proof of Theorem 2.6 uses the following result, that relates separating polynomials to cotype, and which requires the use of probabilistic techniques.

THEOREM 2.7. *Suppose that there is a finite family of homogeneous polynomials of degree $\leq m$ on $X$ which is a separating family. Then, $X$ has cotype $m$.*

In the sequel, we describe the isomorphic copies of $\ell_{2k}$ in spaces with separating polynomial. In order to do it, we denote

$$\tilde{\cot}(X) = \inf\{q : X \text{ has cotype } q\},$$

and

$$\tilde{c}(X) = \inf\{\tilde{\cot}(Y) : Y \subset X,\ \dim(Y) = \infty\}.$$

It is clear that $\tilde{\cot}(X) \geq \tilde{c}(X)$.

On the other hand, the following indices are introduced in [14], related to different properties of weak summability of sequences:

$$l(X) = \sup\{p > 1 : X \text{ has } S_p - \text{ property}\} \in [1, \infty]$$

and

$$u(X) = \inf\{p > 1 : X \text{ has } T_p - \text{ property}\} \in [1, \infty].$$

Recall that, for $1 < p < \infty$, a Banach space $X$ is said to have $S_p$-property (respectively $T_p$-property) if every weakly null normalized sequence in $X$ has a subsequence with an upper $\ell_p$-estimate (respectively a lower $\ell_p$-estimate). It is clear that if $X$ is not a Schur space then $l(X) \leq u(X)$.

We have that $u(X) \leq \tilde{\cot}(X)$ (see [14]), and they do not coincide in general: consider, for instance, the space $X = \left(\bigoplus_{n=1}^{\infty} \ell_4^{(n)}\right)_{\ell_2}$; this space has exact cotype 4, and $u(X) = 2$, since every normalized weakly null sequence has a subsequence equivalent to the unit vector basis of $\ell_2$.

On the other hand, $l(X) \leq \tilde{c}(X)$, and they do not coincide in general, as the example $X = \ell_{3/2}$ easily shows.

Finally, note that for simple examples such as $X = \ell_2 \oplus \ell_4$ we may obtain that $\tilde{c}(X) = 2 < 4 = u(X)$.

In the following Theorem we summarize the various results concerning the existence of isormorphic copies of $\ell_{2k}$ is spaces admiting separating polynomials.

THEOREM 2.8. *Let $X$ be a Banach space with separating polynomial. Then:*

- (i) *There is an integer $n$ such that $\tilde{\cot}(X) = u(X) = 2n$ and $X$ contains an isomorphic copy of $\ell_{2n}$.*
- (ii) *There is an integer $m$ such that $\tilde{c}(X) = l(X) = 2m$ and $X$ contains an isomorphic copy of $\ell_{2m}$.*
- (iii) *If $\ell_{2k} \subset X$, then $2m \leq 2k \leq 2n$.*

In [6 V.4 and V.5] it is shown that a space with separating polynomial contains an isomorphic copy of $\ell_{\tilde{\cot}(X)}$ and $\ell_{\tilde{c}(X)}$. On the other hand, and with different techniques, in [13] it is proved the existence of isomorphic copies of $\ell_{l(X)}$ and $\ell_{u(X)}$ in such spaces.

Note that from the above results it follows that there is no universal space for separable spaces with separating polynomials. Indeed, if $\mathcal{X}$ were a Banach space containing all separable spaces with separating polynomial, then $\mathcal{X}$ would contain $\ell_{2k}$ for every integer $k$, and also would have finite cotype, but this is not possible. Nevertheless, we may ask the following:

QUESTION 4. For a fixed integer $k$, is there an universal Banach space $\mathcal{X}$ containing every separable Banach space which admits a separating polynomial of degree at most $k$?

We finish this Section with a result about smoothness and saturation of Banach spaces due to Deville [4] (see also [6 V.5.1]).

THEOREM 2.9. *Suppose that $X$ is $C^{2k}$-smooth for some integer $k$, and it is saturated with subspaces of cotype $2k$. Then, $X$ has a separating polynomial and has exact cotype $2k$.*

From this it is possible to obtain Makarov's theorem [23], where the Hilbert space case was established: that is, if $X$ is $C^2$-smooth and it is saturated with $\ell_2$ subspaces then $X$ is in fact isomorphic to a Hilbert space.

## 3. SEPARATING POLYNOMIALS ON SPACES WITH SYMMETRIC BASIS AND R.I. FUNCTION SPACES

In this Section we charaterize those spaces with symmetric structrure that admit a separating polynomial. We begin with spaces with a symmetric basis (see [21.I] for an extensive treatment). Recall that a Schauder basis $\{e_j\}$ on a Banach space is said to be symmetric if it is equivalent to all of its permutations. If $X$ is a Banach space with symmetric basis $\{e_j\}$, then it is possible to find an equivalent norm verifying that for every $n$ and $a_1, \dots, a_n \in \mathbb{R}$,

$$\| \sum_{j=1}^{n} a_j e_j \| = \| \sum_{j=1}^{n} a_j e_{\sigma(j)} \|$$

whenever $\sigma$ is a permutation of the integers. We always consider this norm on $X$.

A well known class of spaces with symmetric basis are $\ell_p$ spaces. As we have seen, such spaces admit a separating polynomial only in the case that $p$ is an even integer.

Another remarkable class of spaces with symmetric basis are Orlicz sequence spaces $\ell_M$ associated to an Orlicz function $M$. We also refer to [21.I] for an account on this topic. Maleev and Troyanski [24,25] studied the highest order of smoothness of bump functions in these spaces and from their results it follows that $\ell_M$ admits a separating polynomial only in the case that it is isomorphic to $\ell_p$ for some even integer $p$.

These results can be generalized to the context of all symmetric sequence spaces as it was proved by Gonzalo and Jaramillo [13]. They gave the following characterization of symmetric sequence spaces with separating polynomial:

THEOREM 3.1. *Let $X$ be a Banach space with a symmetric basis. The following are equivalent:*

(i) *$X$ admits a separating polynomial.*

(ii) *$X$ is isomorphic to $\ell_{2k}$ for some integer $k$.*

The continuous analog and the generalization of symmetric sequence spaces are rearrangment invariant function spaces, or r.i. spaces, in short. We will be concerned here only with separable r.i. spaces. Every space with a symmetric basis is, in a natural way, an r.i. space on $\mathbb{N}$ (see [21.II] for details; we refer to [21.II] or [3] for further information about r.i. spaces). In this context, the analog to Theorem 3.1 for function spaces on $[0,1]$ is the following.

THEOREM 3.2. *Let $X[0,1]$ be a separable rearrangement invariant function space on $[0,1]$. The following are equivalent:*

(i) *$X[0,1]$ admits a separating polynomial.*

(ii) *$X[0,1]$ coincides with $L_{2k}[0,1]$ for some integer $k$, up to an equivalent renorming.*

In order to prove both Theorem 3.1 and 3.2 only techniques of polynomials are used. Of course, the symmetric structure of the space is essential in these results; indeed, the proof makes use in an strong way of symmetric polynomials.

Recall that for a Banach space $X$ with symmetric basis $\{e_j\}$, a polynomial $P$ on $X$ is said to be symmetric if:

$$P(\textstyle\sum_{j=1}^n a_j e_j) = P(\sum_{j=1}^n a_j e_{\sigma(j)})$$

for every $a_1, \dots, a_n \in \mathbb{R}$, every $n$ and every permutation $\sigma$ of integers.

This notion can be extended to rearrangement invariant function spaces on $[0,1]$. Note that the main feature of a r.i. function space on $[0,1]$ is that for every automorphism $\sigma$ of $[0,1]$ we have that $f \circ \sigma \in X[0,1]$ whenever $f \in X[0,1]$. Thus, a polynomial on $X[0,1]$ is said to be symmetric if it is invariant under automorphisms; more precisely if for all automorphism $\sigma$ on $[0,1]$ it holds:

$$P(f) = P(f \circ \sigma).$$

The nicest example of an $N$-homogeneous symmetric polynomial, for an integer $N$, on spaces with symmetric basis is

$$P_N(\textstyle\sum_{j=1}^{\infty} x_j e_j) = \sum_{j=1}^{\infty} x_j^N$$

whenever it is well defined, i.e., if $\sum_j |x_j|^N < \infty$ for all $x = \sum_j x_j e_j \in X$. These are called the elementary symmetric polynomials. In the continuous case, the elementary symmetric polynomials are defined by

$$P_N(f) = \int_0^1 f^N \qquad f \in X[0,1];$$

provided $f \in L_N[0,1]$ for all $f \in X[0,1]$.

It is well known that in finite dimensional spaces every symmetric polynomial can be obtained as an algebraic combination of elementary symmetric polynomials (see for instance [17]). In [28] it is showed that the situation is exactly the same for Hilbert and $\ell_p$-spaces, and also for $L_p[0,1]$ whenever $1 < p < \infty$. This was generalized to the context of all symmetric sequence spaces and separable r.i. function spaces in [11]. Namely, it was proved there that every symmetric polynomial on such spaces may be written as an algebraic combination of the elementary symmetric polynomials.

There is a kind of symmetrization procedure for polynomials [28, 11] that allows us to obtain from a given polynomial a new symmetric polynomial preserving some properties of the original one. In the case of symmetric sequence spaces the procedure is as follows:

For a fixed polynomial $P$ consider the sequence of polynomials $\{P_n\}$ defined by

$$P_n(\textstyle\sum_{j=1}^{\infty} a_j e_j) = \frac{1}{n!} \sum_{\sigma \in \Pi_n} P(\sum_{j=1}^{n} a_j e_{\sigma(j)}),$$

where $\Pi_n$ denotes the permutations of $\{1, \ldots, n\}$. Since $\{P_n\}$ is a bounded sequence of polynomials with fixed degree, by [28, Th.4] it is possible to extract a subsequence pointwise convergent to a polynomial $P^*$, which is now a symmetric polynomial.

For r.i. function spaces there is an analogous procedure (see [11] for details). Using this method, the following result is proved.

THEOREM 3.3. *Let $X$ be either a Banach space with symmetric basis or a separable rearrangement invariant function space on $[0,1]$. If there is a separating polynomial on $X$ then there is a symmetric separating polynomial.*

Now Theorems 3.1 and 3.2 are obtained by combining the above Theorem with the description of symmetric polynomials given in [11].

The situation for r.i. function spaces on $[0,\infty)$ is not exactly the same, in the sense that spaces $L_{2k}[0,\infty)$ are not the only ones that admit a separating polynomial. Indeed, consider the following interpolation space: for $p,q>1$ the space $L_p[0,\infty)\cap L_q[0,\infty)$ is defined as the set theoretic intersection endowed with the norm

$$\|f\|_{L_p\cap L_q} = \max\{\|f\|_p, \|f\|_q\}.$$

Then for integers $n,m$, the space $L_{2n}[0,\infty)\cap L_{2m}[0,\infty)$ admits a separating polynomial, namely

$$P(f) = \int_0^\infty f^{2n} + \int_0^\infty f^{2m}.$$

Nevertheless, as it is shown in [11], the above examples are the only separable r.i. function spaces on $[0,\infty)$ that admit a separating polynomial. More precisely, we have:

THEOREM 3.4. *Let $X[0,\infty)$ be a separable rearrangement invariant function space on $[0,\infty)$. The following are equivalent:*

(i) *$X[0,\infty)$ admits a separating polynomial.*
(ii) *$X[0,\infty)$ coincides with $L_{2n}[0,\infty)\cap L_{2m}[0,\infty)$ for some integers $n,m$, up to an equivalent renorming.*

An application of the results in this Section is the characterization of Orlicz sequence spaces $\ell_M$ (also obtained in [24,25]), and Orlicz function spaces $L_M[0,1]$ and $L_M[0,\infty)$ which admit a separating polynomial.

COROLLARY 3.5. *Let $M$ be an Orlicz function satisfying the $\Delta_2$-condition at 0 and $\infty$. Then:*

(i) *$\ell_M$ admits a separating polynomial if and only if $M\sim t^{2k}$ at 0 for some integer $k$.*

(ii) $L_M[0,1]$ *admits a separating polynomial if and only if* $M \sim t^{2k}$ *at* $\infty$ *for some integer* $k$.

(iii) $L_M[0,\infty)$ *admits a separating polynomial if and only if* $M \sim \max\{t^{2n}, t^{2m}\}$ *at* 0 *and at* $\infty$, *for some integers* $m, n$.

The last part in the above Corollary is a consequence of the fact that the space $L_{2n}[0,\infty) \cap L_{2m}[0,\infty)$ coincides with the Orlicz space $L_M[0,\infty)$ associated to the Orlicz function $M(t) = \max\{t^{2n}, t^{2m}\}$.

We consider now spaces with subsymmetric basis, a notion which is weaker than symmetric basis. Recall that a basis $\{e_n\}$ in a Banach space is said to be subsymmetric if it is unconditional and, for every increasing sequence $\{n_i\}$ of integers, $\{e_{n_i}\}$ is equivalent to $\{e_n\}$. Every symmetric basis is subsymmetric, and the converse is not true in general (see [21.I]). The characterization obtained in Theorem 3.1 holds, in fact, for spaces with subsymmetric basis. Thus we have:

THEOREM 3.6. *Let* $X$ *be a Banach space with subsymmetric basis. Then* $X$ *admits a separating polynomial if, and only if,* $X$ *is isomorphic to* $\ell_{2k}$ *for some integer* $k$.

This was proved in [13] using smoothness techniques. An alternative proof is given in [12], which is somewhat analogous to the symmetric case but using subsymmetric polynomials instead of symmetric polynomials. We say that a polynomial $P$ is subsymmetric if it is invariant under spreading of the basis, that is, if

$$P(\textstyle\sum_{j=1}^k a_j e_j) = P(\textstyle\sum_{j=1}^k a_j e_{n_j})$$

for all $a_1, \dots, a_k \in \mathbb{R}$ and $n_1 < \cdots < n_k$. The proof of [12] is based on an explicit description of all subsymmetric polynomials on a Banach space with subsymmetric basis.

In [13] (see also [12]) a kind of subsymmetrization procedure is described that allows us to obtain, from a given polynomial on a Banach space $X$, an associated subsymmetric polynomial on a spreading model of $X$. The method uses Ramsey Theorem much in the same way as in the construction of the spreading norm itself (we refer to [1] for an account on the theory of spreading models). More precisely, we have:

THEOREM 3.7. *Let* $\{y_n\}$ *be a normalized weakly null sequence in a Banach space* $X$, *which admits an spreading model* $E$ *with unconditional basis* $\{e_n\}$.

*Let $P$ a polynomial on $X$. Then there exists a subsymmetric polynomial $\mathbb{P}$ on $E$ and a subsequence $\{x_n\}$ of $\{y_n\}$ such that for all $a_1, \dots, a_k \in \mathbb{R}$:*

$$\mathbb{P}(\textstyle\sum_{j=1}^k a_j e_j) = \lim_{n_1 < \cdots < n_k} P(\sum_{j=1}^k a_j x_{n_j}).$$

*Moreover, if $P$ is a separating polynomial then so is $\mathbb{P}$.*

Note that if, in addition, $\{y_n\}$ is a subsymmetric basis of $X$, then the spreading model $E$ is isomorphic to $X$. Therefore, in this case, the associated polynomial $\mathbb{P}$ is also defined on $X$.

As a consequence of the above, we obtain the following result.

THEOREM 3.8. *Supose that $X$ admits a separating polynomial. Then, every spreading model built over a normalized weakly null sequence in $X$ is isomorphic to $\ell_{2k}$ for some integer $k$.*

## 4. SEPARATING POLYNOMIALS ON $L^p(L^q)$

The nice characterization of spaces with symmetric or subsymmetric basis admiting separating polynomial, given in Theorems 3.1 and 3.6, cannot be generalized to the framework of spaces with unconditional basis, as the example $\ell_2 \oplus \ell_4$ easily shows. One can ask if, under the assumption of having a separating polynomial, a space with unconditional basis and $\ell_{2k}$-saturated for a fixed $k$ must be isomorphic to $\ell_{2k}$. This is not true, as we can see with the space

$$X = \left( \bigoplus_{n=1}^{\infty} \ell_4^{(n)} \right)_{\ell_8}$$

that consists of all sequences $x = (x_n)$, with $x_n \in \ell_4^{(n)}$ and $\sum_n \|x_n\|_4^8 < \infty$. It is easy to see that $X$ has a separating polynomial (in fact, a polynomial norm). Indeed, the following expression defines a 8-homogeneous polynomial on $X$:

$$P(x) = \|x\|^8 = \sum_{n=1}^{\infty} \|x_n\|_4^8 = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (x_n^k)^4 \right)^2$$

where $x_n = x_n^1 e_n^1 + \cdots + x_n^n e_n^n$ and $\{e_n^1, \dots, e_n^n\}$ is the canonical basis in $\ell_4^{(n)}$ for each $n$. Now $X$ has unconditional basis and is $\ell_8$-saturated. In fact, every weakly null normalized sequence in $X$ has a subsequence equivalent to the usual basis of $\ell_8$. On the other hand, $X$ is not isomorphic to $\ell_8$.

Actually, this kind of Banach spaces provide interesting examples of spaces with or without a separating polynomial. For instance, the space $\left(\bigoplus_{n=1}^{\infty} \ell_4^{(n)}\right)_{\ell_2}$ is saturated with Hilbert spaces but fails to have a separating polynomial. This follows by the fact that every $\ell_2$-saturated space with separating polynomial is indeed a Hilbert space (see Theorem 2.9).

In [8] a systematic study about smoothness and separating polynomials of the spaces

$$X = \left(\bigoplus_{n=1}^{\infty} \ell_q^{(n)}\right)_{\ell_p}$$

for $1 < p, q < \infty$, is made. A main result in this line is the following.

THEOREM 4.1. *Let $1 < p, q < \infty$. Then, the following are equivalent:*

(i) *The space $X = \left(\bigoplus_{n=1}^{\infty} \ell_q^{(n)}\right)_{\ell_p}$ admits a separating polynomial.*

(ii) *Both $p$ and $q$ are even integers, and $p$ is a multiple of $q$.*

The main idea to prove this result is that, although $\ell_q$ is not contained in $X$, from a polynomial $P$ on $X$ it is possible to construct an associated polynomial $\tilde{P}$ on $\ell_q$ in the following way:

Let $\{e_j\}$ be the usual basis of $\ell_q$ and for each $n$ consider $\{e_1, \dots, e_n\}$ as the basis of the subspace $\ell_q^{(n)}$ of $X$. For each $n$ consider the polynomial $P_n$ on $\ell_q$ defined by

$$P_n(\textstyle\sum_{j=1}^{\infty} x_j e_j) = P(x_1 e_1 + \cdots + x_n e_n).$$

Then $\{P_n\}$ is a bounded sequence of polynomials with fixed degree, and by [28, Th.4] there exists a subsequence of $\{P_n\}$ which is pointwise convergent to a polynomial $\tilde{P}$ on $\ell_q$.

It is easily seen that if $P$ is a separating polynomial then so is $\tilde{P}$. It can be shown that, in this case, both $P$ and $\tilde{P}$ can be chosen to be $p$-homogeneous. Now we only have to use the fact (see [15]) that if there is a $p$-homogeneous separating polynomial on $\ell_q$, then $p$ must be a multiple of $q$ .

The following result used in the proof of Theorem 4.1 may be of independent interest.

PROPOSITION 4.2. *Suppose that $X$ admits a separating polynomial. If every weakly null sequence in $X$ has a subsequence equivalent to the usual basis of $\ell_{2k}$, then there is a $2k$-homogeneous separating polynomial on $X$.*

Now let $1 < p < \infty$ and let $(\Omega, \mu)$ be a measure space such that the corresponding $L^p(\mu)$ is infinite-dimensional. Recall that if $X$ is a Banach space, the Bochner space $L^p(\mu, X) = L^p(X)$ is defined as the space of all measurable functions $u : \Omega \to X$ such that

$$\int_\Omega \|u(s)\|^p d\mu(s) < \infty.$$

First note that, if $X$ admits a $m$-homogeneous separating polynomial $P$, and $p = km$ is a multiple of $m$, then $L^p(X)$ admits a separating polynomial $\hat{P}$ defined by

$$\hat{P}(u) = \int_\Omega P(u(s))^k d\mu(s).$$

The smoothness of spaces $L^p(X)$ were studied by Leonard and Sundaresan [20].

Now consider the case that $X = L^q$ for some (probably different) measure and $1 < q < \infty$, where we also suppose that $L^q$ is infinite-dimensional. Thus we obtain the space $L^p(L^q)$, whose properties regarding smoothness and separating polynomials were studied in [8]. In closing, we present the following result from [8].

THEOREM 4.3. *Let $1 < p, q < \infty$ and consider the space $L^p(L^q)$. Then the following are equivalent:*

(i) *$L^p(L^q)$ admits a separating polynomial.*

(ii) *Both $p$ and $q$ are even integers and $p$ is a multiple of $q$.*

Note that $\left(\bigoplus_{n=1}^\infty \ell_q^{(n)}\right)_{\ell_p}$ is a subspace of $L^p(L^q)$. Thus, Theorem 4.3 is in fact a consequence of Theorem 4.1.

## Acknowledgements

It is a pleasure to thank Professor Angeles Prieto for several helpful conversations concerning this paper. Thanks are also due to Professor Manuel González for his careful reading of the manuscript.

## References

[1] Beauzamy, B., Lapresté, J.T., "Modèles Etalés des Espaces de Banach", Hermann, Paris, 1984.

[2] Bonic, R., Frampton, J., Smooth functions on Banach manifolds, *J. Math. Mech.,* **15** (1996), 877–898.

[3] BENNET, C., SHARPLEY, R., "Interpolation of Operators", Academic Press, Boston, 1988.

[4] DEVILLE, R., Geometrical implications of the existence of very smooth bump functions in Banach spaces, *Israel J. Math.,* **67** (1989), 1-22.

[5] DEVILLE, R., A characterization of $C^\infty$-smooth Banach spaces, *Bull. London Math. Soc.,* **22** (1990), 13-17.

[6] DEVILLE, R., GODEFROY, G., ZIZLER, V., "Smoothness and Renormings in Banach Spaces", Longman, Harlow, 1993.

[7] DEVILLE, R., GODEFROY, G. AND ZIZLER, V., Smooth bump functions and geometry of Banach spaces, *Mathematika,* **40** (1993), 305-321.

[8] DEVILLE, R., GONZALO, R., JARAMILLO, J.A., Renormings of $L^p(L^q)$, preprint.

[9] DINEEN, S., "Complex Analysis in Locally Convex Spaces", North Holland, Amsterdam, 1981.

[10] FABIAN, M., PREISS, D., WHITFIELD, J.H.M., ZIZLER, V., Separating polynomials on Banach spaces, *Quart. J. Math. Oxford,* **40** (1989), 409-422.

[11] GONZÁLEZ, M., GONZALO, R., JARAMILLO, J.A., Symmetric polynomials on rearrangement invariant function spaces, to appear in *Jour. London Math. Soc.*

[12] GONZALO, R., Multilinear forms, subsymmetric polynomials, and spreading models on Banach spaces, *J. Math. Anal. Appl.,* **202** (1996), 379-397.

[13] GONZALO, R., JARAMILLO, J.A., Smoothness and estimates of sequences in Banach spaces, *Israel J. Math.,* **89** (1995), 321-341.

[14] GONZALO, R., JARAMILLO, J.A., Compact polynomials on Banach spaces, *Proc. Royal Irish Acad.,* **95A** (1995), 213-226.

[15] HABALA, P., HÁJEK, P., Stabilization of polynomials, *C.R. Acad. Sci. Paris,* **320** (1995), 821-825.

[16] HEINRICH, S., Ultraproducts in Banach space theory, *J. Reine Angew. Math.,* **313** (1980), 72-104.

[17] JACOBSON, N., "Lectures on Abstract Algebra", Vol. I, Springer-Verlag, New York, 1975.

[18] KURZWEIL, J., On approximation in real Banach spaces, *Studia Math.,* **14** (1954), 214-231.

[19] KURZWEIL, J., On approximation in real Banach spaces by analytic operations, *Studia Math.,* **16** (1957), 124-129.

[20] LEONARD, I.E., SUNDARESAN, K., Geometry of Lebesgue-Bochner function spaces. Smoothness, *Trans. A.M.S.,* **198** (1974), 229-251.

[21] LINDENSTRAUSS, J. AND TZAFRIRI, L., "Classical Banach spaces I and II", Springer, Berlin, 1977 and 1979.

[22] LINDSTRÖM, M. AND RYAN, R., Applications of ultraproducts to infinite dimensional holomorphy, *Math. Scand.,* **71** (1992), 229-242.

[23] MAKAROV, B.M., One characterization of Hilbert space, *Mat. Zametki,* **26** (1979), 739-746.

[24] MALEEV, R. AND TROYANSKI, S., Smooth functions in Orlicz spaces, *Contemporary Math.,* **85** (1989), 355-369.

[25] MALEEV, R. AND TROYANSKI, S., Smooth norms in Orlicz spaces, *Cana-*

*dian Math. Bull.,* **34** (1991), 74–82.

[26] MESHKOV, V.Z., Smoothness properties of Banach spaces, *Studia Math.,* **63** (1978), 111–123.

[27] MUJICA, J., "Complex Analysis in Banach spaces", North-Holland, Amsterdam, 1986.

[28] NEMIROVSKI, A.S., SEMENOV, S.M., On polynomial approximation of functions on Hilbert space, *Math. Sb.,* **21** (1973), 255–277.

[29] PELCZYŃSI, A., A property of multilinear operations, *Studia Math.,* **16** (1957), 173–182.

[30] PISIER, G., Martingales with values in uniformly convex spaces, *Israel J. Math.,* **20** (1975), 326–350.

[31] SUNDARESAN, K., SWAMINATHAN, S., "Geometry and Nonlinear Analysis in Banach Spaces", Lecture Notes in Math., Vol. 1131, Springer, Berlin, 1985.