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Polynomials and Multilinear Forms on Spaces Isomorphic to Their Cartesian Square

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Along this talk we plan to show examples and counterexamples related to the coincidence of the space of n—linear mappings and the space of n—homogeneous polynomials constructed on a Banach or a Fréchet space. We first recall, from a joint work with S. Dineen, the positive results that fix the background for the examples.

Let E be an infinite dimensional Fréchet space. Denote by $L(^nE)$ the space of n—linear forms defined on E and by $L_s(^nE)$ the subspace of symmetric forms. $L_s(^nE)$ can be identified, via the Polarization Formula, to the space of n—homogeneus polynomials on E. An n—homogeneous polynomial is the restriction, to the diagonal of E^n, of a symmetric n—linear form.

We want to clarify the relationship between $L(^nE)$ and $L_s(^nE)$. On one hand $L_s(^nE)$ is a canonically complemented subspace of $L(^nE)$. On the other hand $L(^nE)$ is a complemented subspace of $L_s(^n(E^n))$; the latter result is far from being trivial, it is due to Bonet and Peris for $n = 2$ [1] and to Defant and Maestre for $n > 2$ [2]. In particular, if E is isomorphic to its (cartesian) square the spaces $L(^nE)$ and $L_s(^nE)$ contain each other as complemented subspaces. This property implies that both spaces are very 'alike' and share many topological properties but it is not enough to assure that they are isomorphic.

Concerning this point let us comment for a while the so called Schroeder-Bernstein problem: Let X and Y be Banach spaces which contain each other as complemented subspaces. Are they isomorphic? The answer is yes at least in the following cases:

(S-B.1) One of the spaces, say X, is isomorphic to $\ell_p(X)$ for $1 \leq p \leq \infty$, or $p = 0$. The proof is the celebrated Pelcynski's decomposition method.

(S-B.2) Both spaces are isomorphic to their cartesian square. The proof is a weak form of Pelcynski's decomposition method.

(S-B.3) The space X is isomorphic to its square and Y is isomorphic to $Y^2 \times X$. It is proved in a similar way as (S-B.2). This condition arose in [4].

There is a recent and very important counterexample, due to Gowers [5], for the Shroeder-Bernstein problem. Gowers constructs Banach spaces X and Y such that X is isomorphic to X^2, Y is isomorphic to Y^3, and they contain complementably each other, but they are not isomorphic.

In our particular case we proved [4] that $L(^nE)$ is isomorphic to $L_s(^nE)$ provided that E is isomorphic to its square. The main technical result is.

LEMMA 1. *If E is isomorphic to its square then $L(^nE)$ is isomorphic to its square and $L_s(^nE)$ is isomorphic to $L_s(^nE)^2 \times L(^nE)$.*

As a consequence we obtain the following theorem.

THEOREM 1. *If E is isomorphic to its square then $L(^nE)$ is algebraically isomorphic to $L_s(^nE)$, for every n. When both spaces are endowed, respectively, with topologies τ_1 and τ_2, the isomorphism is topological in each of the following cases:*

(i) *τ_1 and τ_2 are compact open topologies.*

(ii) *τ_1 and τ_2 are the topologies of the uniform convergence on bounded sets.*

(iii) *τ_1 is the inductive dual topology (where $L(^nE)$ is considered as the dual of the projective tensor product $\hat{\otimes}_{n,\pi}E$) and τ_2 is the topology induced by the Nachbin ported topology, τ_ω, which is defined on the space of polynomials on E.*

Moreover the projective tensor product $\hat{\otimes}_{n,\pi}E$ is isomorphic to the subspace of symmetric tensors.

The first example arises to show that the hypothesis that E is isomorphic to its square cannot be removed from the statement before.

EXAMPLE 1. Given $1 < p < \infty$ we consider the following James space

$$J_p := \left\{ (x_i) \colon \sup_k \sup_{0=n_0<\cdots<n_k} \left(\sum_{j=0}^{k-1} \left(\sum_{i=n_j+1}^{n_{j+1}} x_i \right)^p \right)^{1/p} < \infty \right\}.$$

The classical James space is J_2. It was the first known Banach space non isomorphic to its cartesian square. This is also true for every J_p, $1 < p < \infty$, and it is deduced since the canonical embedding of J_p in $(J_p)''$ has codimension one. It is known that J_p is hereditarily ℓ_p, i.e. every closed subspace of J_p contains a copy of ℓ_p. Moreover it has the following property.

LEMMA 2. *Every weakly null and normalized sequence of J_p has a subsequence equivalent to the canonical basis of ℓ_p.*

As a consequence of lemma 2 and Pitt's Theorem, every continuous linear operator from J_p into $(J_p)'$ is compact if $p > 2$. It is then deduced that the bidual of $J_p\hat{\otimes}_\pi J_p$ is $(J_p)''\hat{\otimes}_\pi(J_p)''$. Analogously the bidual of the subspace $J_p\hat{\otimes}_{s,\pi}J_p$ of symmetric tensors is the space $(J_p)''\hat{\otimes}_{s,\pi}(J_p)''$ of symmetric tensors of the biduals. Studying the canonical embeddings of the spaces into the biduals yields the following conclusion.

THEOREM 2. *The space $L(^2J_p)$ is not isomorphic to any complemented subspace of $L_s(^2J_p)$ for $p > 2$.*

The next example is a Fréchet space. It is also given to show the necessity of the hypothesis of theorem 1.

EXAMPLE 2. Let us introduce some notation. If $\alpha = (\alpha_n)$ is an increasing sequence of positive real numbers such that $\sup_n \log(n)/\alpha_n < \infty$ (respectively $\lim_n \log(n)/\alpha_n=0$), we define the power series space of infinite type (respectively of finite type) $\Lambda_\infty(\alpha)$ (respectively $\Lambda_1(\alpha)$) as the following Köthe space:

$$\Lambda_\infty(\alpha) := \{(x_i);\ \|(x_i)\|_k = \sum_{i=1}^{\infty} |x_i| \exp(k\alpha_i) < \infty,\ k \in \mathbb{N}\},$$

$$\text{respectively}\quad \Lambda_1(\alpha) := \{(x_i);\ \|(x_i)\|_k = \sum_{i=1}^{\infty} |x_i| \exp(-\alpha_i/k) < \infty,\ k \in \mathbb{N}\}.$$

Power series spaces were introduced by Grothendieck and have been thoroughly studied by several authors. The conditions relating (α_n) with $(\log n)$ are equivalent to the nuclearity of $\Lambda_\tau(\alpha)$, $\tau = 1$ or $\tau = \infty$. For these spaces we obtain the following result.

THEOREM 3. *A power series space $E = \Lambda_\tau(\alpha)$, $\tau = 1$ or ∞, is isomorphic to its square if and only if $L(^2E)$ is isomorphic to a subspace of $L_s(^2E)$.*

The last example shows that theorem 3 cannot be obtained for the class of Banach spaces. We provide a Banach space $X = C(\Omega)$ such that $L(^2X)$ is isomorphic to $L_s(^2X)$ but X is not isomorphic to its square.

EXAMPLE 3. (Semadeni) Let Ω be the first uncountable ordinal. The space of continuous functions $C(\Omega)$ is not isomorphic to its cartesian square [7]. Howewer its topological dual $\ell_1(\Omega)$ is isomorphic to its square, moreover every continuous linear mapping from $C(\Omega)$ into $\ell_1(\Omega)$ is compact; from these properties it can be deduced that $L(^2C(\Omega))$ is isomorphic to $L_s(^2C(\Omega))$.

We finish this talk with some open problems which rise from the results obtained so far.

QUESTION 1. Is $L(^2J_2)$ isomorphic to $L_s(^2J_2)$? (see Theorem 2).

QUESTION 2. Let E' be isomorphic to its square. Is $L(^nE)$ isomorphic to $L_s(^nE)$ for every n? (see Example 3).

QUESTION 3. (Jarchow) It follows from Theorem 1 that the space of continuous linear operators from ℓ_2 into ℓ_2, considered as a space of infinite matrices, is isomorphic to the subspace of symmetric matrices. Which is the Banach-Mazur distance between both spaces?

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