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Rolle's Theorem and Negligibility of Points in Infinite Dimensional Banach Spaces†

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Rolle's theorem in finite dimensional spaces states that for every open connected and bounded subset $\mathcal{U}$ in $\mathbb{R}^n$ and every continuous function $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ such that f is differentiable in $\mathcal{U}$ and constant on $\partial\mathcal{U}$, there exists an x in $\mathcal{U}$ such that $df(x) = 0$. In a paper published in 1992 [7], S.A. Shkarin proved that Rolle's theorem fails in a large class of infinite dimensional Banach spaces, including all super-reflexive and all non-reflexive Banach spaces having a Fréchet differentiable norm —although he did not study the reflexive but non-super-reflexive case. Other explicit examples were found in c_0 and ℓ_2 by J. Ferrera and J. Bés [5] and independently by J. Ferrer [6]. On the other hand it is clear that Rolle's theorem trivially holds in all non-Asplund Banach spaces because of the harmonic behaviour of differentiable maps in such spaces (see [3], chapter III). It is natural to conjecture that a reasonable version of Rolle's theorem in infinite dimensional Banach spaces holds if and only if our space does not have a C^1 bump function and we prove this conjecture to be true within the class of those Banach spaces X which can be linearly injected into a Banach space Y with an equivalent norm whose dual norm is locally uniformly rotund (LUR) in Y^*. This geometrical condition, which we shall call $(*)$ for short, is satisfied by every WCG Banach space, every space which can be injected into some $c_0(\Gamma)$, and even by every space injectable into some $C(K)$ being K a scattered compact with $K^{(\omega_1)} = \emptyset$. This conjecture is closely related to the question posed in [4] whether for every Banach space X having a C^1 bump function there exists a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such

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that φ is the identity out of a ball, and we give an affirmative answer to this question within the class of all Banach spaces X verifying $(*)$.

An interesting approximate version of Rolle's theorem remains nevertheless true in all Banach spaces. By an approximate Rolle's theorem we mean that if a differentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point in the interior of the ball in which the differential of the function has norm equal or less than ε. Namely, we have found the following

THEOREM 1. (Approximate Rolle's theorem) *Let X be a Banach space and $\mathcal{U}$ be an open bounded connected subset of X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be a continuous bounded function. Suppose that f is Gâteaux differentiable in $\mathcal{U}$ and $\overline{f(\partial\mathcal{U})} \subseteq [a,b]$, with $a < b$. Then, for every $x_0 \in \mathcal{U}$ and $R > 0$ such that $B(x_0,R) \subseteq \mathcal{U}$, there exists $x_1 \in \mathcal{U}$ such that*

$$\|df(x_1)\| \leq \frac{b-a}{2R}.$$

This result is an immediate consequence of the next two lemmas, which are themselves interesting.

LEMMA 2. *Let X be a Banach space and $\mathcal{U}$ be an open bounded connected subset of X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be a continuous bounded function such that:*

1. *f is Gâteaux differentiable in $\mathcal{U}$*
2. *$\inf f(\overline{\mathcal{U}}) < \inf f(\partial\mathcal{U})$ or $\sup f(\overline{\mathcal{U}}) > \sup f(\partial\mathcal{U})$.*

Then, for every $\alpha > 0$ there exists $x \in \mathcal{U}$ such that $\|df(x)\| \leq \alpha$.

LEMMA 3. *Let X be a Banach space and $\mathcal{U}$ be an open bounded connected subset of X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be a continuous bounded function such that:*

1. *f is Gâteaux differentiable in $\mathcal{U}$*
2. *$f(\overline{\mathcal{U}}) \subseteq [a,b]$, where $a < b$.*

Then, for every $x_0 \in \mathcal{U}$ and $R > 0$ such that $B(x_0,R) \subseteq \mathcal{U}$, there exists $x_1 \in B(x_0,R)$ such that$\|df(x_1)\| \leq (b-a)/2R$.

From theorem 1 we can immediately deduce the following

Corollary 4. *Let $\mathcal{U}$ be an open connected bounded subset of a Banach space X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be continuous, bounded, and Gâteaux differentiable in $\mathcal{U}$. Suppose that f is constant on $\partial\mathcal{U}$. Then,*

$$\inf_{x \in \mathcal{U}} \|f'(x)\| = 0.$$

And also

Corollary 5. *Let X be a Banach space and $f : X \longrightarrow \mathbb{R}$ be continuous, Gâteaux differentiable and bounded on X. Then,*

$$\inf_{x \in X} \|f'(x)\| = 0.$$

On the other hand we have used Bessaga's non-complete norm technique to prove that every Banach space X verifying the condition

> $(*)$ There exists a Banach space Y with an equivalent norm $\|.\|$ whose dual norm $\|.\|^*$ is locally uniformly rotund (LUR) in Y^* and a continuous linear injection $T : X \longrightarrow Y$

is C^1 diffeomorphic to $X \setminus \{0\}$. If moreover X has a differentiable bump function then there exists a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that φ is the identity out of a ball centered at 0. Condition $(*)$ is equivalent to say that X admits a continuous (not necessarily equivalent) norm whose dual norm is (LUR).

The main result here is

Theorem 6. *Let X be an infinite dimensional Banach space that verifies condition $(*)$. Then*

1. *X admits a $C^1(X \setminus \{0\})$ non-complete norm ω.*
2. *there exists a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that $\varphi(x) = x$ if $\omega(x) \geq 1$.*

from which it is easily deduced

Theorem 7. *For a Banach space X satisfying condition $(*)$, the following are equivalent.*

1. *X has a C^1 bump function.*

2. *There exists a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that φ is the identity out of a ball centered at* 0.

Theorem 7 gives an affirmative answer, within the class of all Banach spaces verifying $(*)$, to the question posed by T. Dobrowolski [4] whether for every C^1 smooth Banach space X there exists a diffeomorphism between X and $X \setminus \{0\}$ which is the identity out of a ball centered at 0.

Using this one can prove as in [1] the following

COROLLARY 8. *If a Banach space X verifies condition $(*)$ and has a Fréchet smooth equivalent norm then the sphere S_X is C^1 diffeomorphic to each hyperplane in X. If moreover X is isomorphic to one of its hyperplanes, then X is C^1 diffeomorphic to its sphere.*

One can use the preceding results and some of the ideas in [7] to prove that an exact Rolle's theorem either fails or trivially holds in infinite dimensional Banach spaces verifying $(*)$. The following result provides a characterization of spaces that do not verify Rolle's theorem within the class of those spaces.

THEOREM 9. *If a Banach space X verifies condition $(*)$, the following are equivalent:*

1. *X has a C^1 bump function.*

2. *There exists an open connected bounded subset $\mathcal{U}$ and a continuous bounded function $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ such that f is $C^1(\mathcal{U})$, $f \equiv 0$ on $\partial\mathcal{U}$ and yet $df(x) \neq 0$ for all $x \in \mathcal{U}$; that is, Rolle's theorem fails in X.*

3. *There exists a $C^1(X)$ bounded function $f : X \longrightarrow \mathbb{R}$ and an open connected bounded subset $\mathcal{U}$ in X such that $f \equiv 0$ on $X \setminus \mathcal{U}$ and yet $df(x) \neq 0$ for all $x \in \mathcal{U}$.*

In closing we observe that all these results remain true if we replace condition $(*)$ by the following one

$(**)$ There exist a Banach space Y with an equivalent differentiable norm $\|.\|$, a reflexive closed subspace $Z \subseteq Y$ and a continuous linear injection $T : X \longrightarrow Y$ such that $Z \subseteq T(X) \subseteq Y$.

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