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Some Features of Uncontinuable Solutions of Impulsive Dynamical Systems

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1. Introduction

The investigation of impulsive equations began with the works of V. Milman and A. Mishkis ([4], [5]). The theory of these systems develops intensively in the last few years. This development is a result of the wide range of applications of impulsive equations. The impulsive effects of these equations take place at the moments when the integral curves meet the previously given hypersurface. One particular case are the impulsive differential equations with fixed moments. The mathematical theory of the latter equations is comparatively well studied.

In the present paper a class of impulsive autonomous systems is introduced. These systems are an adequate mathematical model of finite dimensional, partially smooth and indeterminate (in the general case) evolutionary processes. Each impulsive system is defined by an ordinary system of differential equations, an impulsive set in phase space and an impulsive operator. For the system of this class it is characteristic that the impulses occur of the moments when their phase curves "reach" the boundary of the impulsive set. The evolution of the impulsive autonomous system does not differ from the evolution of the ordinary system in an arbitrary domain that does not intersect the impulsive set.

The non-reversibility is the feature differing the behaviour of the solutions of the impulsive systems from that of an ordinary system. The main goal of this paper is to study the uncontinuable solutions of impulsive autonomous systems. We shall also investigate the connections between uncontinuable solutions, non-reversibility of the system and the "beating" phenomena of the solutions.

2. Statement of the Problem

Throughout the paper will be used the following notations: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$. The n-dimensional Euclidean space (with Euclidean scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$) is denoted by $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ then $Int\, A$ (∂A) is denoted the set of all inner (limit) points of A in $\mathbb{R}^n$, $\bar{A} = A \cup \partial A$. If $a \in A, r \in \mathbb{R}_+$ then $\mathbb{B}(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$, $\mathbb{B}(A, r) = \cup\{\mathbb{B}(a, r) : a \in A\}$, $dist(A, B) = inf\{\|a - b\| : a \in A, b \in B\}, B \subset \mathbb{R}^n$. The space of all $\boldsymbol{C}^\kappa$-smooth functions from $A \subseteq \mathbb{R}^n$ in $B \subseteq \mathbb{R}^n$ is denoted by $\boldsymbol{C}^\kappa(A, B)$, $\kappa \in \mathbb{N}$. If $\Phi \in \boldsymbol{C}^0(A, \mathbb{R}^n)$ then $\Phi|A_0$ is the trace of Φ over $A_0 \subseteq A$ and $Fix\,(\Phi|A_0)$ is the set of all fixed points of the mapping $\Phi|A_0$. If $\Phi \in \boldsymbol{C}^1(A, \mathbb{R})$ then $grad_a \Phi = \left(\dfrac{\partial \Phi}{\partial x_1}(a), \dots, \dfrac{\partial \Phi}{\partial x_n}(a)\right)$.

Let $A \subset \mathbb{R}^n$. We denote the components of A by $\{A_s : s \in S\}$, where $S \subseteq \mathbb{N}$ is an index set. For every $s \in S$ we shall denote by $\{\partial_j A_s : j \in S_s\}$ the set of all components of ∂A_s, $S_s \subseteq \mathbb{N}$. Clearly,

$$\partial A = \cup_s \cup_j \{\partial_j A_s : s \in S, j \in S_s\}.$$

We shall say that the conditions **(H1)** hold if:

(H1.1) For every $s \in S$ there exists a function $h_s \in \boldsymbol{C}^\kappa(\mathbb{R}^n, \mathbb{R})$, $\kappa \in \mathbb{N}$ such that $A_s = \{x \in \mathbb{R}^n : h_s(x) \leq 0\}$ and $grad_a h_s \neq 0$ for all $a \in \partial A_s$.

(H1.2) $\Phi \in \boldsymbol{C}^\kappa(\mathbb{R}^n, \mathbb{R}^n)$.

We set $A_\delta = A \setminus \mathbb{B}(\partial A, \delta)$, $U_\delta = \mathbb{R}^n \setminus A_\delta$ and consider the autonomous system

$$\dot{y} = f(y), \quad y \in U_\delta, \tag{1}$$

where $f \in \mathbf{C}^\kappa(U_\delta, \mathbb{R}^n)$. Let $y = y(t, y_0)$ is the solution of the system (1) with initial condition

$$y(0, y_0) = y_0, \quad y_0 \in U_\delta. \tag{2}$$

Definition 1. The sets

$$E_1(f, A_s) = \{a \in \partial A_s : \langle f(a), grad_a h_s \rangle < 0\}, \quad \text{(Figure 1.a)},$$
$$E_2(f, A_s) = \{a \in \partial A_s : \langle f(a), grad_a h_s \rangle > 0\}, \quad \text{(Figure 1.b)},$$
$$E_{34}(f, A_s) = \{a \in \partial A_s : \langle f(a), grad_a h_s \rangle = 0\}, \quad \text{(Figure 1.c)},$$

are named respectively set of points of entry, exit and contact.

Clearly, the set $E_{34}(f, A_s)$ is a disjunctive union of the following sets:

$$E_3(f, A_s) = \{a \in E_{34}(f, A_s) : \{y(t, a) : t \in (-\epsilon, \epsilon)\} \setminus \{a\} \subset \mathbb{R}^n \setminus A\}$$

and

$$E_4(f, A_s) = \{a \in E_{34}(f, A_s) : \{y(t, a) : t \in (-\epsilon, \epsilon)\} \setminus \{a\} \subset Int\, A\},$$

where $\epsilon \in \mathbb{R}_+$ is a sufficiently small number.

Let $E_\sigma(f, A) = \cup\{E_\sigma(f, A_s) : s \in S\}$, $\sigma \in \{1, 2, 3, 4\}$.

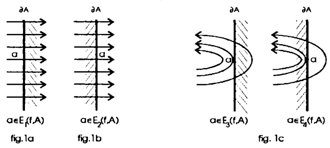

Figure 1.

DEFINITION 2. Let the Conditions **(H1)** hold. The ordered triple (f, A, Φ) is named impulsive autonomous system on $\mathbb{R}^n$ if:

1. $E_{34}(f, A)$ is smooth manifold in ∂A and

$$codim_{\partial A} E_{34}(f, A) = dim\, \partial A - dim\, E_{34}(f, A) \neq 0.$$

2. The equalities:

$$\partial A = \cup\{E_\sigma(f, A) : \sigma \in \{1, 2, 3, 4\}\},$$
$$\Phi(E_1(f, A)) = E_2(f, A), \quad \Phi(E_2(f, A)) = E_1(f, A)$$

hold.

We shall write the impulsive system (f, A, Φ) in the form

$$\dot{x} = f(x), \quad x \in U = \mathbb{R}^n \setminus A, \tag{3}$$
$$x^+ = \Phi(x), \quad x \in \partial A, \tag{4}$$

as it is taken in most works about impulsive systems.

Before giving a precise definition of the term impulsive autonomous system we shall illustrate the motion of the mapping point $x(t, x_0)$. Let us suppose that $x_0 \in U_\delta$ and $t \in \mathbb{R}_+$. Then $x(0, x_0) = x_0$ in the initial moment $t = 0$. The mapping point $x(t, x_0)$ starts from the point x_0 and moves along the orbit of the system (1) defined by x_0 in the first moment $\tau_1^+(x_0) \in \mathbb{R}_+$ such that $x(\tau_1^+(x_0), x_0) \in \partial A_s$ for any $s \in S$. The mapping point "jumps" from the point $x(\tau_1^+(x_0), x_0)$ to the point $x(\tau_1^+(x_0)+0, x_0) = \Phi\left(x(\tau_1^+(x_0), x_0)\right) \in \Phi(A_s)$ in the moment $\tau_1^+(x_0)$. After that $x(t, x_0)$ continues its motion along the orbit of the system (1) defined by the point $x(\tau_1^+(x_0) + 0, x_0)$ in the first moment $\tau_2^+(x_0) \in (\tau_1^+(x_0), \infty)$ when $x(\tau_2^+(x_0) - \tau_1^+(x_0), x(\tau_1^+(x_0) + 0, x_0)) \in \partial A$ and so on.

In the case when $t \in \mathbb{R}_-$ the dynamical picture of the process under consideration is similar.

DEFINITION 3. The interval $(\alpha, \beta) \subset \mathbb{R}$, $\alpha < 0 < \beta$ and the mapping

$$x : (\alpha, \beta) \to \mathbb{R}^n \setminus Int\, A, \quad t \to x(t, x_0)$$

will be named an existence interval of solution and solution of the impulsive systems (3), (4) with initial condition

$$x(0, x_0) = x_0, \quad x_0 \in U, \tag{5}$$

if the following conditions hold:

1. If $\tau \in \mathbb{R}_+$ $(\tau \in \mathbb{R}_-)$ is the first moment when $y(\tau, x_0) \in \partial A$, then $x(t, x_0) = y(t, x_0)$, $t \in (0, \tau]$ $(t \in [\tau, 0))$ and:
   (a) If $y(\tau, x_0) \in E_1(f, A)$ $(y(\tau, x_0) \in E_2(f, A))$ then $x(\tau + 0, x_0) = \Phi(x(\tau, x_0))$ $(x(\tau + 0, x_0) = \Phi(x(\tau, x_0)))$.
   (b) If $y(\tau, x_0) \in E_3(f, A)$ then $x(\tau + 0, x_0) = x(\tau - 0, x_0) = x(\tau, x_0)$.
2. The group equality $x(t_1 + t_2, x_0) = x\left(t_2, x(t_1, x_0)\right)$ holds for each two numbers $t_1, t_2 \in (0, \beta)$ $(t_1, t_2 \in (\alpha, 0))$ such that $t_1 + t_2 \in (\alpha, \beta)$.

The unique interval $(\omega^-(x_0), \omega^+(x_0))$ for wich the following condition holds: there exists no extension of the solution $x = x(t, x_0)$ on an interval containing $(\omega^-(x_0), \omega^+(x_0))$, will be named maximal existence interval of the solution $x = x(t, x_0)$ of the impulsive problem (3), (4), (5).

Let us denote by $\mathcal{O}(x_0) = \{x(t, x_0) : t \in (\omega^-(x_0), \omega^+(x_0))\}$ the orbit of the solution $x = x(t, x_0)$ of the impulsive problem (3), (4), (5). We put $\mathcal{O}(x_0; \alpha, \beta) = \{x(t, x_0) : t \in (\alpha, \beta) \subseteq (\omega^-(x_0), \omega^+(x_0))\}$, $\mathcal{O}^+(x_0) = \mathcal{O}(x_0; 0, \omega^+(x_0)), \mathcal{O}^-(x_0) = \mathcal{O}(x_0; \omega^-(x_0), 0)$.

3. Preliminary Notes

LEMMA 1. *Let* (f, A, Φ) *is an impulsive system in* $\mathbb{R}^n$. *Then*

$$codim_{\partial A} E_{34}(f, A) = 1.$$

Proof. It follows from the Definition of the sets $E_\sigma(f, A)$, $\sigma \in \{1, 2, 34\}$ that:

1. The sets $E_1(f, A)$ and $E_2(f, A)$ are open in ∂A, that is why $dim\ E_1(f, A) = dim\ E_2(f, A) = dim\ \partial A$.
2. If $x_0 \in E_{34}(f, A)$ and V is a neighbourhood of x_0 in ∂A, then the set $E_{34}(f, A) \cap V$ is a boundary of the disjunctive sets $E_1(f, A)$ and $E_2(f, A)$ in $\partial A \cap V$.

Hence, the inequality

$$codim_{\partial A} E_{34}(f, A) \leq 1, \tag{6}$$

follows from the known Theorem of P.S. Alecsandrov [1].

The equality $codim_{\partial A} E_{34}(f, A) = 1$ follows from the Condition 1 of Definition 2 and (6). ∎

We consider the impulsive autonomous system (3), (4). Let $x_0 \in U$. Then all local features of the solution $x = x(t, x_0)$ of the impulsive system (3), (4) do not differ from the respectively features of the solution of the ordinary system (3) in a neighbourhood $V \subset U$. For example the point x_0 determines an interval $(\alpha, \beta) \subset \mathbb{R}$, $\alpha < 0 < \beta$ of existence and uniqueness of the solution $x = x(t, x_0)$. Moreover, it is not difficult to extend the solutions $x = x(t, x_0)$ in the first moment τ_1^+ in which $x = x(t, x_0) \in \partial A$. Therefore we can see that the initial condition (5) of the impulsive system under consideration is replaced with the following less resrictive condition

$$x(0, x_0) = x_0, \quad x_0 \in \overline{U}. \tag{7}$$

Actually, if $x_0 \in \partial A = \partial U$ then there exists a point $x_1 \in U$ and a number $t_0 \in \mathbb{R}_+$ such that $\{x(t, x_0) : t \in (0, t_0)\} \cap \partial A = \emptyset$ and $x(t_0, x_1) = x_0$ or $x(t_0, x_0) = x_1$. Hence, we may accept that the point $x_1 \in U$ is the initial condition of the impulsive system (3), (4).

The next lemma proves that each point $x_0 \in \overline{U}$ determines in a unique way the maximal existence interval of the solution of the impulsive problem (3), (4), (7).

LEMMA 2. *We consider the impulsive autonomous system* (3),(4). *Then:*

1. *Each point* $x_0 \in \overline{U}$ *determines a unique maximal existence interval* $(\omega^-(x_0), \omega^+(x_0))$ *of the solution* $x = x(t, x_0)$ *of the problem* (3), (4), (7).
2. *If* (3) *is a complete system in* U_δ, *then:*

$$\omega^+(x_0) = \begin{cases} \lim_{i\to\infty} \tau_i^+(x_0), & \text{if } p^+(x_0) = \infty, \\ +\infty, & \text{if } p^+(x_0) < \infty; \end{cases} \tag{8}$$

$$\omega^-(x_0) = \begin{cases} \lim_{i\to\infty} \tau_i^-(x_0), & \text{if } p^-(x_0) = \infty, \\ -\infty, & \text{if } p^-(x_0) < \infty, \end{cases} \tag{9}$$

where $\{\tau_i^+ : i \in \{1, \dots, p^+(x_0)\}\}$ $(\{\tau_i^- : i \in \{1, \dots, p^-(x_0)\}\})$ *are all moments in which* $x(\tau_i^+, x_0) \in \partial A$ $(x(\tau_i^-, x_0) \in \partial A)$ *and* $\tau_i^+ < \tau_{i+1}^+$ $(\tau_{i+1}^- < \tau_i^-)$, $i \in \{1, \dots, p^+(x_0) - 1\}$ $(i \in \{1, \dots, p^-(x_0) - 1\})$.

Proof. We shall prove the assertion 2 of the Lemma. The proof of the statement 1 can be done in a similar way.

Let $t \in \mathbb{R}_+$, $x_0 \in U$. It follows from the closeness of the set A and the continuity of the solution of the ordinary problem (3), (5) that $\{x(t, x_0) : t \in \mathbb{R}_+\} \cap \partial A = \emptyset$ or there exists a least number $\tau_1^+(x_0) \in \mathbb{R}_+$ such that $x(\tau_1^+(x_0), x_0) \in \partial A$. In the first case, the completeness of the system (3) implies that $\omega^+(x_0) = \infty$, $(p^+(x_0) = 0)$ i.e. the formula (8) is valid.

Let us consider the second case. Let $\tau_1^+(x_0) \in \mathbb{R}_+$ is the least number in which $x(\tau_1^+(x_0), x_0) \in \partial A$ and $\{x(t, x_0) : t \in (0, \tau_1^+(x_0))\} \cap A = \emptyset$.

It follows from Definition 2 that $x(\tau_1^+(x_0), x_0) \in E_\sigma(f, A)$, $\sigma \in \{1, 3\}$. If $x(\tau_1^+(x_0), x_0) \in E_1(f, A)$ then it follows from Condition 2 of Definition 2 that $\Phi(x(\tau_1^+(x_0), x_0)) = E_2(f, A)$ and the solution $x = x(t, x_0)$ of the impulsive problem (3), (4), (7) can be extended after the moment $\tau_1^+(x_0)$. If $x(\tau_1^+(x_0), x_0) \in E_3(f, A)$ then the Condition 1.(b) of Definition 3 implies that the solution $x = x(t, x_0)$ is continuous in the moment $\tau_1^+(x_0)$ i.e. it can be extended after this moment.

It is not difficult to prove that the number $p^+(x_0)$ of all moments of impulsive effect of the solution of the impulsive problem (3), (4), (7) is either finite (in the case when $\omega^+(x_0) = \infty$) or $p^+(x_0) = \infty$ and $\omega^+(x_0) = \lim_{i\to\infty} \tau_i^+(x_0)$. This assertion is proved by means of mathematical induction.

If $t \in \mathbb{R}_-$ then the proof is similar. ∎

4. Main Results

4.1. Determined impulsive dynamical systems. The phenomenon "fusion" of the solutions of the impulsive system (3), (4), (7) is illustrated in the following example. We must note that the phenomenon "fusion" of the orbit of a dynamical system is possible for non-reversible system only.

Example 1. Let $A = A_1 \cup A_2 \cup A_3 \subset \mathbb{R}^2$, $A_j = \{(x_1, x_2) : x_1 \in [2j - 1, 2j],\ x_2 \in \mathbb{R}\}$, $j \in \{1, 2, 3\}$;

$$\Phi : A \to A, \quad \Phi(x_1, x_2) = \begin{cases} (7 - x_1, x_2), & (x_1, x_2) \in A_1, \\ (9 - x_1, x_2), & (x_1, x_2) \in A_2, \\ (7 - x_1, x_2), & (x_1, x_2) \in A_3 \end{cases}$$

and $f(x_1, x_2) = (1, 0)$.

We consider the impulsive system (3), (4). Obviously

$$\mathcal{O}^+(0,0) = \{(x_1, x_2) : x_1 \in (0, 1], x_2 = 0\} \cup \{(x_1, x_2) : x_1 \in (6, \infty], x_2 = 0\},$$

$$\mathcal{O}^+(2.5,0) = \{(x_1, x_2) : x_1 \in [2.5, 3], x_2 = 0\} \cup \{(x_1, x_2) : x_1 \in (6, \infty], x_2 = 1\}.$$

Hence

$$\mathcal{O}^+((0,0); 1, \infty) = \mathcal{O}^+((2.5, 0); 0.5, \infty).$$

We shall note that the system (3), (4) represents a non-reversible evolutionary process: If the mapping point "starts" from the initial point $(2.5, 0)$ it can not be "returned" in the point $(2.5, 0)$ after the time $t > 0.5$.

Definition 4. The impulsive system (3), (4) will be called determinable if for each point $x_0 \in \overline{U}$ and for each two numbers $t_1, t_2 \in (\omega^-(x_0), \omega^+(x_0))$ for which $t_1 + t_2 \in (\omega^-(x_0), \omega^+(x_0))$, the group equality $x(t_1 + t_2, x_0) = x(t_2, x(t_1, x_0))$ is valid.

Lemma 3. *Let* (3), (4) *is a determinable impulsive system in* $\mathbb{R}^n$, $x_1, x_2 \in \overline{U}$ *and* $\mathcal{O}(x_1) \cap \mathcal{O}(x_2) \neq \emptyset$. *Then* $\mathcal{O}(x_1) = \mathcal{O}(x_2)$.

The proof of Lemma 3 is similar to the proof of the corresponding statement from the theory of smooth dynamicall systems.

Theorem 1. *Let* (3), (4) *is an impulsive autonomous system in* $\mathbb{R}^n$. *Then* (3), (4) *is a determinable impulsive system if and only if the impulsive operator* $\Phi : \partial A \to \partial A$ *is an involution.*

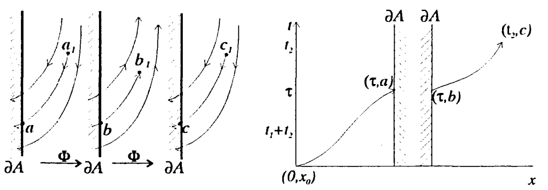

Figure 2.

Proof. **1.** Let (3), (4) is a determinable autonomous system.

Let us suppose that the operator $\Phi|E_1(f,A)$ is not involutive. Then there exists a point $a \in E_1(f,A)$ such that $c = \Phi^2(a) \neq a$ where $b = \Phi(a) \in E_2(f,A)$, $c = \Phi(b) \in E_1(f,A)$. It follows from the Definition of the sets $E_1(f,A)$ and $E_2(f,A)$, the inclusions $\mathcal{O}(a;-t_1,0) \subset U$, $\mathcal{O}(b;0,t_1) \subset U$ and $\mathcal{O}(c;-t_1,0) \subset U$ for sufficiently small $t_1 \in \mathbb{R}_+$.

We put $a_1 = x(-t_1,a)$, $b_1 = x(t_1,b)$, $c_1 = x(-t_1,c)$ (Figure 2). It follows from the condition of determinative that $a_1 = x(0,a_1) = x(-2t_1, x(2t_1,a_1)) = x(-2t_1,b_1) = c_1$. That is why (see Lemma 3) $\mathcal{O}(a_1) = \mathcal{O}(c_1)$ and in particular $a = c$. The last equality contradicts to the above supposing. Hence the operator $\Phi|E_1(f,A)$ is an involution. It proves that $\Phi|E_2(f,A)$ is an also involution by similar contentions. Hence the operator $\Phi|\partial A$ is an involution since the set $E_1(f,A) \cup E_2(f,A)$ is everywhere dense in ∂A (see the Condition 1 of the Definition 2).

2. Let $\Phi : \partial A \to \partial A$ is an involution, $x_0 \in U$, $t_1, t_2 \in (\omega^-(x_0), \omega^+(x_0))$ and $t_1 + t_2 \in (\omega^-(x_0), \omega^+(x_0))$.

It follows from Definition 3 that (3), (4), (7) is a local determinative problem in the interval (τ_1^-, τ_1^+), i.e. if $t_1, t_2 \in (\tau_1^-, \tau_1^+)$ and $t_1 + t_2 \in (\tau_1^-, \tau_1^+]$ then $x(t_1 + t_2, x_0) = x(t_2, x(t_1, x_0))$. Analogous, if $x(\tau_1^+, x_0) \in E_3(f,A)$ then (3), (4), (7) is a local determinative problem in the interval $(\tau_1^-, \tau_2^+) \supset (\tau_1^-, \tau_1^+)$ and so on.

Let $t_1 < 0 < t_2$, $0 < t_1 + t_2$ and let the interval $(t_1 + t_2, t_2)$ contains a unique moment τ, in which $x(\tau, x_0) \in E_1(f,A)$. It follows from the above said that $x(t_1, x(t_2, x_0)) = x(t_1, x(t_2 - \tau, b)) = x(t_1, c)$ where $a = x(\tau, x_0)$, $b = \Phi(a)$, and $c = x(t_2 - \tau, b)$. On the other hand the inequality $\tau - (t_1 + t_2) > 0$

implies that $x(\tau - t_2 - (\tau - t_2 - t_1), c) = x(-\tau + t_2 + t_1, x(\tau - t_2, c))$. But $x(-\tau + t_2 + t_1, x(\tau - t_2, c)) = x(-\tau + t_2 + t_1, a) = x(-\tau + t_2 + t_1, x(\tau, x_0)) = x(t_1 + t_2, x_0)$. That is why the equations and $x(t_1 + t_2, x_0) = x(t_2, x(t_1, x_0))$ come from the involutionary of the operator Φ. ∎

Another feature of the solutions of the impulsive autonomous system is illustrated in the following example.

EXAMPLE 2. Let $A = A_1 \cup A_2 \cup A_3$, $A_j = \{(x_1, x_2) : x_1 \in (j - 0.1, j + 0.1),\ x_2 \in \mathbb{R}\}$, $j \in \{1, 2, 3\}$;

$$\Phi : A \to A, \quad \Phi(x_1, x_2) = \begin{cases} (4 - x_1, x_2), & (x_1, x_2) \in A_1, \\ (3 - x_1, x_2), & (x_1, x_2) \in A_2, \\ (5 - x_1, x_2), & (x_1, x_2) \in A_3 \end{cases}$$

and let $f(x_1, x_2) = (1, 0)$.

We consider the indeterminate impulsive autonomous problem

$$\dot{x} = f(x), \ x \in U = \mathbb{R}^2 \setminus A, \tag{10}$$

$$x^+ = \Phi(x), \ x \in \partial A, \tag{11}$$

$$x(0, (2.5, 1)) = (2.5, 1). \tag{12}$$

Clearly,

$$\mathcal{O}^-(2.5, 1) = (-\infty, 0.9] \cup \{(x_1, x_2) : x_1 \in (2.1, 2.5], x_2 = 1\},$$

$$\mathcal{O}^+(2.5, 1) = \\ \{(x_1, x_2) : x_1 \in [2.5, 2.9], x_2 = 1\} \cup \{(x_1, x_2) : x_1 \in (2.1, 2.9], x_2 = 1\}.$$

That is why $x = x(t, (2.5, 1))$ is a periodical solution with period 1, at $t \in \mathbb{R}_+$, i.e. $x(t + 1, (2.5, 1)) = x(t, (2.5, 1))$, $t \in \mathbb{R}_+$. The solution $x = x(t, (2.5, 1))$ is not periodical, at $t \in \mathbb{R}_-$.

4.2. THE PHENOMENA "BEATING" FOR IMPULSIVE AUTONOMOUS SYSTEMS. We shall consider some features of the phenomenon "beating" for impulsive autonomous systems. Note that this phenomenon is typical for discontinuous systems only.

DEFINITION 5. Let (3), (4) is an impulsive autonomous system.

1. We shall say that the phenomenon "beating" of the solution $x = x(t, x_0)$ is absent at $t \in \mathbb{R}_+$ $(t \in \mathbb{R}_-)$, if there exists $\epsilon \in \mathbb{R}_+$ such that for every $i \in \{1, \dots, p^+(x_0)\}$ the following inequalities

$$\tau_{i+1}^+(x_0) - \tau_i^+(x_0) \geq \epsilon \quad (\tau_i^-(x_0) - \tau_{i+1}^-(x_0) \geq \epsilon)$$

are valid.

2. We shall say that the phenomenon "beating" is absent for the impulsive system (3), (4) if the phenomenon "beating" is absent for all solutions $x = x(t, x_0), x_0 \in U, t \in \mathbb{R}$ of the impulsive system.

The main aim of the present subsection is to study the behaviour of the solutions of impulsive system for which the phenomenon "beating" is not absent.

DEFINITION 6. We shall name the point $x_1 \in \overline{U}$ β_+-limit (β_--limit) point of the solution $x = x(t, x_0)$ of the impulsive problem (3), (4), (7) if $p^+(x_0) = \infty$ and there exists a sequence $\{t_i : i \in \mathbb{N}\}$ such that $\lim_{i\to\infty} t_i = \omega^+(x_0)$ and $\lim_{i\to\infty} x(t_i, x_0) = x_1$ ($p^-(x_0) = \infty, \omega^-(x_0) > -\infty$ and there exists a sequence $\{t_i : i \in \mathbb{N}\}$ such that $\lim_{i\to\infty} t_i = \omega^-(x_0)$ and $\lim_{i\to\infty} x(t_i, x_0) = x_1$).

We shall denote the set of all β_+-limit (β_--limit) points of the problem (3), (4), (7) by $\beta_+(x_0)$ $(\beta_-(x_0))$; $\beta(x_0) = \beta_+(x_0) \cup \beta_-(x_0)$.

LEMMA 4. *Let $x_0 \in U$, $a \in \beta(x_0) \cap E_4(f, A)$ and $\Phi^{-1}(a) \in E_3(f, A)$. Then $\omega^+(x_0) = \infty$.*

Proof. We shall accept that $a \in \beta_+(x_0)$. The case $a \in \beta_-(x_0)$ is considered similarly.

Let V_a is a neighbourhood of the point a such that $V_a \subset U_\delta$, $V_a^0 = V_a \cap \partial A$, $W_a^1 = E_1(f, A) \cap V_a^0$, $W_a^2 = E_2(f, A) \cap V_a^0$.

We shall accept that

$$V_a^0 = W_a^1 \cup (E_4(f, A) \cap V_a) \cup W_a^2,$$
$$\Phi^{-1}(V_a^0) = \Phi^{-1}(W_a^1) \cup (E_3(f, A) \cap \Phi^{-1}(V_a)) \cup \Phi^{-1}(W_a^2),$$
$$\Phi^{-1}(W_a^1) = E_2(f, A) \cap \Phi^{-1}(V_a^0), \quad \Phi^{-1}(W_a^2) = E_1(f, A) \cap \Phi^{-1}(V_a^0),$$

without loss of the community.

Let $b = \Phi^{-1}(a)$, $V_b^0 = \Phi^{-1}(V_a^0)$, $W_b^1 = \Phi^{-1}(W_a^2)$, $W_b^2 = \Phi^{-1}(W_a^1)$, V_b is a neighbourhood of the point b in U_δ such that $V_b \cap \partial A = V_b^0$ and $inf\{\|f(x)\| :$

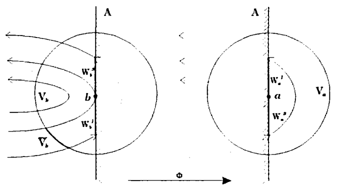

Figure 3.

$x \in V_b\} > 0$ (Figure 3), and let $\widetilde{V}_b$ is an open set in ∂V_b that contains all points $x_1 \in \widetilde{V}_b$ for which there exists a number $\theta = \theta(x_1) \in \mathbb{R}_+$ such that $\{x(t, x_1) : t \in (0, \theta(x_1))\} \subset Int(U \cap V_b)$ and $x(\theta(x_1), x_1) \in W_b^1$.
We denote by b_0 the unique point in ∂V_b for which there exists $\theta_0 \in \mathbb{R}_+$ such that $\{x(t, b_0) : t \in (0, \theta_0)\} \subset (U \cap V_b)$ and $x(\theta_0, b_0) = b$. Then, it follows from the definition of $\theta : \widetilde{V}_b \to \mathbb{R}_+$ that

$$(13) \qquad \lim_{x \to \partial \widetilde{V}_b \cap \partial A} \theta(x) = 0, \; x \in \widetilde{V}_b, \quad \lim_{x \to b_0} \theta(x) = \theta_0.$$

Let us suppose that

$$(14) \qquad \omega^+(x_0) < \infty.$$

It follows from Definition 6 that there exists a sequence $\{t_i : i \in \mathbb{N}\} \subset (0, \omega^+(x_0))$ so that $\lim_{i\to\infty} t_i = \omega^+(x_0)$, $\lim_{i\to\infty} x(t_i, x_0) = a$ and $x(t_i, x_0) \in V_a$. We denote by $\{k_i : i \in \mathbb{N}\}$ the sequence of integer for which $t_i \in (\tau^+_{k_i-1}(x_0), \tau^+_{k_i}(x_0))$, $i \in \mathbb{N}$. We shall accept that the sequence $\{t_i : i \in \mathbb{N}\}$ is chosen so that $k_i < k_{i+1}$, $x(\tau^+_{2k_i-1}(x_0), x_0) \in W_b^1$ and $x(\tau^+_{2k_i}(x_0), x_0) \in W_a^1$ for all $i \in \mathbb{N}$. It follows from the inequalities $\tau^+_{k_i-1}(x_0) < t_i < \tau^+_{k_i}(x_0)$ and from the equalities $\lim_{i\to\infty} x(t_i, x_0) = a$, $\lim_{i\to\infty}(\tau^+_{k_i}(x_0), \tau^+_{k_i-1}(x_0)) = 0$ that

$$(15) \qquad \lim_{i\to\infty} x(\tau^+_{2k_i-1}(x_0), x_0) = b.$$

Let us choose the sequence of points

$$\{x_{2k_i-1} : i \in \mathbb{N}\} \subset \widetilde{V}_b$$

so that $x(\theta(x_{2k_i-1}), x_{2k_i-1}) = x(\tau^+_{2k_i-1}(x_0), x_0)$. Then

$$\lim_{i\to\infty} x_{2k_i-1} = b_0 \text{ and } \lim_{i\to\infty} \theta(x_{2k_i-1}) = \theta_0. \tag{16}$$

follow from equalities (13), (15) and from the definition of the map θ. That is why $\theta(x_{2k_i-1}) > \theta(x_{2k_1-1})$ for all $i \in \mathbb{N}$. Therefore

$$\omega^+(x_0) > \sum_{i=1}^{\infty} \theta(x_{2k_i-1}) > \sum_{i=1}^{\infty} \theta(x_{2k_1-1}) = \infty. \tag{17}$$

The inequalities (17) contradict to (14). ∎

THEOREM 2. *Let $x = x(t, x_0)$ is a boundary solution of the impulsive autonomous system* (3), (4), (7) *for which $\omega^+(x_0) < \infty$. Then*

1. $p^+(x_0) = \infty$.
2. *The phenomenon "beating" is not absent for the solution $x = x(t, x_0)$.*
3. *If x^* is a limit point of the sequence $\nu = \{x(\tau_i^+(x_0), x_0) : i \in \mathbb{N}\}$ then $x^* \in \beta(x_0) \cap E_4(f, A)$.*
4. *$\Phi^{-1}(x^*) \in E_4(f, A)$ and $\Phi(x^*) \in E_4(f, A)$.*
5. *The set of all limit points of the sequence ν is a boundary and it has codimension in ∂A higher or equal to one.*

Proof. 1. Let us suppose that $p^+(x_0) < \infty$. There exists a compact $K \subset U_\delta$ such that $\overline{\mathcal{O}(x_0)} \subset K$ because the solution $x = x(t, x_0)$ is limited. That is why it follows from the [3], Chapter II, Theorem 3.1 that $\omega(x_0) = \infty$. The obtained contradiction proves the statement 1 of the Theorem 2.

2. The statement 2 of the Theorem follows from Definition 5 and the equality $\lim_{i\to\infty} \tau_i^+(x_0) = \omega^+(x_0) < \infty$.

3. It follows immediately from Definitions 5 and 6 that each limit point x^* of the sequence ν is β_+-limit point of the impulsive system (3), (4). We shall prove that $x^* \in E_4(f, A)$. Let K is a compact in U_δ and V is an open set in U_δ such that:

$$\overline{\mathcal{O}^+(x_0)} \subset K, \quad \overline{\mathcal{O}^+(x_0)} \cap \partial A \subset V \subset K, \\ \text{and } 0 < m = \mathit{inf}\{\|f(x)\| : x \in V\}. \tag{18}$$

Let $\{t_j : j \in \mathbb{N}\}$ is a sequence for which $0 < t_j < t_{j+1} < \omega^+(x_0)$, $\lim_{j\to\infty} t_j = \omega^+(x_0)$, $\lim_{j\to\infty} x(t_j, x_0) = x^*$ and $x(t_j, x_0) \notin \partial A$, $j \in \mathbb{N}$. We denote by $\{k_j : j \in \mathbb{N}\}$ the strictly monotone sequence of integer such that $t_j \in (\tau^+_{k_j-1}(x_0), \tau^+_{k_j}(x_0))$, $j \in \mathbb{N}$. It follows from the inequalities $\tau^+_{k_j-1}(x_0) < t_j < \tau^+_{k_j}(x_0)$ and from $\lim_{j\to\infty}(\tau^+_{k_j}(x_0) - \tau^+_{k_j-1}(x_0)) = 0$ that $\lim_{j\to\infty} x(\tau^+_{k_j}(x_0), x_0) = x^*$. Moreover from

$$\begin{aligned}\|x(t,x_0) - x(\tau^+_{k_j-1}(x_0), x_0)\| &\le M|t - \tau^+_{k_j-1}(x_0)| \\ &\le M|\tau^+_{k_j}(x_0) - \tau^+_{k_j-1}(x_0)| \xrightarrow[j\to\infty]{} 0,\end{aligned}$$

where $t \in (\tau^+_{k_j-1}(x_0), \tau^+_{k_j}(x_0))$ and $M = sup\{\|f(x)\| : x \in V\}$, follows that there exists $j_0 \in \mathbb{N}$ such that for each $j \ge j_0$ the following implications

$$\begin{aligned}&\mathcal{O}(x_0; \tau^+_{k_j-1}(x_0), \tau^+_{k_j}(x_0)) \subset V \setminus A, \\ &\lim_{j\to\infty} \|x(\tau^+_{k_j}(x_0), x_0) - x(\tau^+_{k_j-1}(x_0), x_0)\| = 0,\end{aligned} \tag{19}$$

are valid.

That is why the implications (19) are valid for every open set V in U_δ for which (18) is valid and $x^* \in V$. Therefore $x^* \in E_4(f, A)$.

4. The statement 4 follows from Lemma 4 and the condition of the boundary of maximal existence interval.

5. The statement 5 follows from the Condition 1 of Definition 2 and the boundary of solution $x = x(t, x_0)$. ∎

REMARK 1. Let $x = x(t, x_0)$ is a boundary solution of the impulsive system (3), (4) for which $\omega^+(x_0) < \infty$. Then it follows from Theorem 2 that the motion of the mapping point "concentrates" in sufficiently small neighbourhoods of finite number of points from the set $E_4(f, A)$ when t is near to $\omega^+(x_0)$.

COROLLARY 1. *Let the impulsive autonomous system* (3), (4) *is defined in the boundary domain U and $E_4(f, A) = \emptyset$. Then*

1. *The maximal existence interval of every solution of the impulsive problem* (3), (4), (7) *is unlimited.*
2. *The phenomenon "beating" is absent for the impulsive system* (3), (4).

The proof of the Corollary follows from the condition for the boundary of the set U and Theorem 2.

COROLLARY 2. *Let* (3), (4) *is an impulsive autonomous system in the boundary domain* U *and let*

$$\Phi(E_3(f, A)) = E_4(f, A). \tag{20}$$

Then

1. *The maximal existence interval of every solution of the impulsive problems* (3), (4), (7) *is unlimited.*
2. *The phenomenon "beating" is absent for the impulsive system* (3), (4).

Proof. Let $x_0 \in U$. If $p^+(x_0) \in \mathbb{N}$ then it follows from [3], Ch.II, Theorem 3.1 (applied to the solution $x = x(t, x(\tau^+_{p^+(x_0)}, x_0))$ of the ordinary differential equation (3) that $\omega(x_0) = \infty$. Let $p^+(x_0) = \infty$. Let us suppose that $\omega^+(x_0) < \infty$. Then from the boundary of the set U and Theorem 2 follows that if x^* is a limit point of the sequence $\{x(\tau_i^+(x_0), x_0) : i \in \mathbb{N}\}$ then $x^* \in E_4(f, A)$ and $\Phi^{-1}(x^*) \in E_4(f, A)$, $\Phi(x^*) \in E_4(f, A)$. The obtained equalities contradict to (20). ∎

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