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Isomorphisms of Jordan-Banach Algebras

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Introduction

In [5], Kaplansky proved that every ring isomorphism Φ from a semi-simple complex Banach-algebra A onto another provides a decomposition of A into a direct sum of closed two-sided ideals $A = A_1 \oplus A_2 \oplus A_3$ such that A_1 has finite-dimension, Φ is complex-linear on A_2, and Φ is conjugate-linear on A_3. The aim of this paper is to translate to noncommutative Jordan-Banach algebras the above-mentioned result of Kaplansky. To this end we follow the pattern established in [5] although some difficulties have to be overcome.

We recall that a nonassociative algebra A, over a field K, of characteristic not two is said to be a noncommutative (in short: n.c) Jordan algebra if it satisfies:

$$(ab)\,a = a\,(ba)\,; \quad (a^2 b)\,a = a^2\,(ba)$$

for all $a, b \in A$. It is worth pointing out that n.c. Jordan algebras contain all associative algebras and (commutative) Jordan algebras. We note that for any n.c Jordan algebra A, the algebra A^+ obtained by symmetrisation of the product of A is a Jordan-algebra. A n.c Jordan-Banach algebra is a real or complex n.c Jordan-algebra whose underlying linear space is endowed with a complete norm $\|\cdot\|$ satisfying

$$\|ab\| \leq \|a\| \ \|b\|$$

for all $a,\, b \in A$.

It should be pointed out that there is a well settled spectral theory and a holomorphic functional calculus for n.c. Jordan-Banach algebras (see [2]). Furthermore, K. McCrimmon introduced in [6] a judicious notion of radical for n.c Jordan algebras and later, L. Hogben and K. McCrimmon [4] showed that this radical, for a Jordan algebra, may be obtained as the intersection of all "primitive" ideals in the algebra. Finally A. Fernández and A. Rodríguez [3] extend this result to the noncommutative framework.

In the sequel, A and B denote complex n.c. Jordan-Banach algebras and Φ stands for a ring isomorphism from A onto B.

THEOREM 1. *If A and B are infinite-dimensional and primitive, then Φ is automatically real-linear (and accordingly either complex-linear or conjugate-linear).*

Proof. It is well known that Φ provides a ring isomorphism, say σ, between the centroids of A and B which, on account of [7; (Remark 5.i)], equal $\mathbb{C}$. Accordingly Φ becomes a σ-linear map and therefore we only need to show that σ is continuous. To obtain a contradiction, suppose that σ were discontinuous.

If there exists $a \in A$ with an infinite spectrum, then it may be argued as in [5; Lemma 11] to yield a contradiction.

On the other hand according to [1], if every element a in A has finite spectrum, then A^+ and consequently B^+ are the Jordan-Banach algebras associated to continuous nondegenerate symmetric bilinear forms f and g on complex infinite-dimensional Banach spaces X and Y, respectively. From this we have

$$g(\Phi(x), \Phi(y)) = \sigma(f(x,y)) \quad \forall x, y \in X.$$

Let $\{e_n\}$ be an infinite orthonormal sequence for f and, for each $n \in \mathbb{N}$, choose $\lambda_n \in \mathbb{C}$ satisfying

$$\|\lambda_n e_n\| < 2^{-n} \quad \text{and} \quad 2^n \|\Phi(e_n)\| < |\sigma(\lambda_n)|.$$

The elements $a \in A$ and $b \in B$ defined by $a = \sum_{n=1}^{\infty} \lambda_n e_n$ and $b = \sum_{n=1}^{\infty} 2^{-n} \|\Phi(e_n)\|^{-1} \Phi(e_n)$ satisfy

$$\begin{aligned} g(\Phi(a), b) &= \sum_{n=1}^{\infty} \frac{1}{2^n \|\Phi(e_n)\|} g(\Phi(a), \Phi(e_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n \|\Phi(e_n)\|} \sigma(f(a, e_n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n \|\Phi(e_n)\|} \sigma(\lambda_n), \end{aligned}$$

which is impossible, since the last series diverges. ∎

It is rather obvious that every primitive $\mathbb{Q}$-linear ideal of A is a primitive ideal of A. Consequently, for every primitive ideal P of A, $Q = \Phi(P)$ is a primitive ideal of B and Φ drops into a ring isomorphism Φ_P from A/P onto B/Q. Since A/P and B/Q are primitive, the preceding results shows that Φ_P is real-linear whenever P has infinite codimension.

It is a simple matter to adapt Lemmas 12 and 13 in [5] to n.c. Jordan-Banach algebras and this gives the following result.

THEOREM 2. *There exists only a finite number of primitive ideals P of A for which Φ_P is not real-linear. Furthermore, these exceptional ideals have finite codimension.*

Finally, our main result may be proved in much the same way as [5; Theorem].

THEOREM 3. *If A and B are semisimple, then there are closed two-sided ideals A_1, A_2, A_3 in A such that $A = A_1 \oplus A_2 \oplus A_3$, A_1 has finite dimension, Φ is complex-linear on A_2 and Φ is conjugate-linear on A_3.*

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