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Uncomplemented Copies of $C(K)$ Inside $C(K)$

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Throughout this note, whenever K is a compact space $C(K)$ denotes the Banach space of continuous functions on K endowed with the sup norm. Though it is well known that every infinite dimensional Banach space contains uncomplemented subspaces, things may be different when only $C(K)$ spaces are considered. For instance, every copy of $\ell_\infty = C(\beta\mathbb{N})$ is complemented wherever it is found.

In [5] Pelczynski proved

Theorem 1. *Let K be a compact metric space. If a separable Banach space X contains a subspace Y isomorphic to $C(K)$ then Y contains a new subspace Z isomorphic to $C(K)$ and complemented in X.*

Our aim is to obtain the "uncomplemented" version of Pelczynski's Theorem 1. Some restrictions are needed: if X is separable and $K = \mathbb{N}^*$, the one point compactification of $\mathbb{N}$, then $C(K) = c$ (the space of convergent sequences) and Sobczyk's theorem [6] yields that c is complemented in X. So, the hypothesis $K^{(\omega)} \neq \emptyset$ is necessary. Recall that for a given topological space K, the set of accumulation points of K is denoted K'; and if λ is an ordinal, the topological derived of order λ, denoted $K^{(\lambda)}$ is defined by transfinite induction as follows:

$$K^{(0)} = K, \quad K^{(\lambda+1)} = \left(K^{(\lambda)}\right)', \quad \text{and} \quad K^{(\lambda)} = \bigcap_{\beta<\lambda} K^{(\beta)}$$

when λ is a limit ordinal. The first limit ordinal is denoted ω.

Theorem 2. *Let K be a compact metric space with $K^{(\omega)} \neq \emptyset$. If a Banach space X contains a subspace Y isomorphic to $C(K)$ then Y contains a new subspace Z isomorphic to $C(K)$ uncomplemented in Y (and a fortiori, in X).*

The proof is easy since the basic tools were provided by Amir [1] and Baker [2]. Amir proved that $C[0,1]$ contains an uncomplemented subspace isomorphic to $C[0,1]$; Baker proved that $C(\omega^\omega)$ contains an uncomplemented subspace isomorphic to $C(\omega^\omega)$.

Proof of Theorem 2. If K contains a compact perfect subset P, i.e. a subset for which $P' = P$, then [3] there is a quotient map $q : K \longrightarrow [0,1]$; therefore, the map $T : C[0,1] \longrightarrow C(K)$ given by $Tf = f \circ q$ is an into isomorphism. The uncomplemented copy of $C[0,1]$ inside of $C[0,1]$ provides an uncomplemented copy of $C(K)$ inside of $C(K)$ since $C[0,1] = C(K)$ by Milutin's theorem (see [4, theorem 8.5]).

If K is dispersed, i.e. does not contain perfect subsets, and metric then K is homeomorphic to an ordinal compact α; and since $K^{(\omega)} \neq \emptyset$, $\alpha \geq \omega^\omega$. Now, a transfinite induction and Baker's uncomplemented copy of $C(\omega^\omega)$ inside of $C(\omega^\omega)$ yield an uncomplemented copy of $C(\alpha)$ inside of $C(\alpha)$ for all countable ordinals α: assume that C_α is an uncomplemented copy of $C(\alpha)$ inside of $C(\alpha)$ for $\alpha < \beta$. If β is not a limit ordinal, $\beta = \alpha + 1$ then $C(\beta) = C(\alpha) \oplus \mathbb{K}$ and $C_\alpha \oplus \mathbb{K}$ is an uncomplemented copy of $C(\beta)$ inside $C(\beta)$; and if β is a limit ordinal then for some sequence of ordinals $\alpha_n \uparrow \beta$ one has $C(\beta) = c_0(C(\alpha_n))$. Therefore,

$$C(\beta) = c_0(C_{\alpha_n}) \hookrightarrow c_0(C_{\alpha_n}) = C(\beta)$$

is the uncomplemented copy of $C(\beta) = c_0(C(\alpha_n))$ one is looking for. ∎

The hypothesis K metric is necessary as the example of ℓ_∞ shows.

REFERENCES

[1] AMIR, D., Continuous function spaces with the bounded extension property, *Bull. Res. Concil Israel Sect. F,* **101** (1962), 133–138.

[2] BAKER, J.W., Some uncomplemented subspaces of $C(X)$ of the type $C(Y)$, *Studia Math.,* **36** (1970), 85–103.

[3] LACEY, H.E., "The isometric theory of classical Banach spaces", Springer-Verlag, 1974.

[4] PELCZYNSKI, A., Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions, *Dissertationes Math.,* **58** (1968).

[5] PELCZYNSKI, A., On $C(S)$-subspaces of separable spaces, *Studia Math.,* **31** (1968), 513–522.

[6] SOBCZYK, A., Projection of the space (m) on its subspace (c_0), *Bull. Amer. Math. Soc.,* **7** (1941), 938–947.