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Holomorphic Functions on Fréchet Spaces with Schauder Basis

Seán Dineen

Department of Mathematics, Univ. College Dublin, Belfield, Dublin 4, *Ireland*

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In this article we discuss when the ported (or Nachbin) topology, τ_ω, coincides with the topology generated by the countable open covers, τ_δ, on balanced open subsets of certain Fréchet spaces.

Our main result is the following theorem.

Theorem 1. *Let $\{E_n\}_n$ denote a sequence of Banach spaces each of which has a finite dimensional Schauder decomposition, let $\lambda(A)$ denote a Fréchet nuclear space with DN, let $E := \lambda(\{E_n\}) = \{(x_n)_n;\ x_n \in E_n$ all n and $(\|x_n\|_n)_n \in \lambda(A)\}$ and let U denote a balanced open subset of polydisc type in a complemented subspace F of E. If $H(U)$ is the space of all C-valued holomorphic functions on U then $\tau_\omega = \tau_\delta$ on $H(U)$.*

The space E is a Fréchet space when endowed with the topology generated by the norms

$$\|(x_n)_n\|_k := \sum_{k=1}^{\infty} a_{n,k}\|x_n\|$$

where $A = (a_{n,k})_{1\le n,k<\infty}$.

Corollary 1. *If U is a balanced open subset of a separable Banach space with the bounded approximation property (BAP) then $\tau_\omega = \tau_\delta$ on $H(U)$.*

Corollary 1 was obtained for Banach spaces with an unconditional basis in [4] and generalised to other Banach spaces in [3]. The method used in [4] was modified in [5] to obtain $\tau_\omega = \tau_\delta$ on $H(U)$ when U is an open polydisc in a Fréchet nuclear space with basis and DN and refined further in [6] to obtain theorem 1 when each E_n has an unconditional finite dimensional Schauder decomposition. The fact that unconditionality was not required in theorem 1 was noticed by the author in preparing lectures which were given at Universidade Federal do Rio de Janeiro during July 1995. At the same time and independently J. Mujica discovered that unconditionality was not required in the Banach space case and also obtained corollary 1. This is the main result in J. Mujica "Spaces of Holomorphic mappings on Banach spaces with a Schauder basis" Preprint RP 48/95, Universidade Estadual de Campinas.

Our method is a further refinement of the technique in [6] while Mujica extends the methods of [4]. In both cases the results are very technical. In this research announcement I include the crucial step which differs from the proof in [6] and which allows the removal of the unconditionality hypothesis. The full details of my proof will be given in [7] while Jorge Mujica will publish his proof separately.

We now describe the modification needed to prove the key technical result required for the proof of theorem 1.

Let $\{F_n\}_n$ denote a Schauder decomposition of a Fréchet space F which is unconditional in the even variables. This means, in particular, that the topology of F is generated by semi-norms having the form

$$\left\|\sum_{n=1}^{\infty} x_n\right\| = \sup_{\substack{|\lambda_n|\leq 1 \\ m=1,2,\ldots}} \left\|\sum_{n=1}^{m} x_{2n-1} + \sum_{n=1}^{\infty} \lambda_n x_{2n}\right\| \tag{$*$}$$

Let $(\beta_n)_n$ denote a sequence of positive real numbers, $\beta_n \geq 1$, $\beta_{2n-1} = 2$ all n and suppose $\sum_{n=1}^{\infty} \beta_n^p x_n \in F$ whenever $\sum_{n=1}^{\infty} x_n \in F$ and $p \geq 1$.

Let $\mathcal{P}_e(F)$ denote the set of all continuous homogeneous polynomials on F which are, in addition, homogeneous in each even variable.

LEMMA 1. *Let U be a balanced open subset of $F := \{F_n\}_n$, G a Banach space and $T : (H(U), \tau_\delta) \to G$ a continuous linear function. If $\delta > 1$ and there exists $c > 0$ such that*

$$\|T(P)\| \leq c\|P\|$$

for all $P \in \mathcal{P}_e(F)$ then there exists a positive integer j and $c' > 0$ such that

$$\|T(P)\| \leq c'\|P\|_{j,\delta}$$

for all $P \in \mathcal{P}_e(F)$.

We have let $\|P\| = \sup\{|P(x)| : \|x\| \leq 1\}$, where $\| \cdot \|$ satisfies $(*)$, and

$$\|P\|_{j,\delta} = \sup\left\{ |P(x)|; x = \sum_{n=1}^{\infty} x_n \text{ and } \left\| \sum_{n=1}^{j} x_n + \sum_{n=j+1}^{\infty} \beta_n x_n \right\| \leq \delta \right\}.$$

The statement and proof of this lemma are modelled on Proposition 3 of [6]. The unconditionality hypothesis however, have been removed and the two cases considered are modified. We consider a sequence of polynomials $(P_j)_j$ in $\mathcal{P}_e(F)$, P_j homogeneous of degree $|m_j|$, such that

$$P_j\left(\sum_{n=1}^{j} x_{2n-1} + \sum_{n=j}^{\infty} \lambda x_{2n+1} + \sum_{n=1}^{\infty} \lambda_n x_{2n}\right) = \lambda^{l_j} \prod_{n=1}^{\infty} \lambda_n^{m_{2n}^j} P_j\left(\sum_{n=1}^{\infty} x_n\right)$$

for all $\lambda \in C$, $\lambda_n \in C$ all n, and $\sum_{n=1}^{\infty} x_n \in F$. Let $\beta^{s_j} = \prod_{i \geq j} \beta_{2i}^{m_{2i}^j}$. The two cases considered in the new proof (and these cover all possibilities) are

Case (1) $\qquad \lim_{j\to\infty} (\alpha^{l_j} \beta^{s_j})^{\frac{1}{|m_j|}} = 1 \qquad$ for all $\alpha \geq 1$

Case (2) $\qquad$ there exists $\alpha > 1$ such that $\limsup_{j\to\infty} (\alpha^{l_j} \beta^{s_j})^{\frac{1}{|m_j|}} = w > 1.$

This gives lemma 3 and afterwards theorem 1 when $F = \lambda(\{E_n\}_n)$. The proof is extended to complemented subspaces in a standard fashion and corollary 1 is obtained from the fact that separable Banach spaces with the BAP are complemented subspaces of Banach spaces with a Schauder basis ([8]).

Since a separable reflexive Banach space has BAP if and only if it has the approximation property [8] we also have the following corollary (see [1,2,9]).

COROLLARY 2. *If E is a separable reflexive Banach space with the approximation property then the following are equivalent;*

(a) $(H(U), \tau_\omega)$ *is reflexive for any balanced open subset U of E,*
(b) $(\mathcal{P}(^nE), \| \cdot \|)$ *is reflexive for all n,*
(c) *each norm continuous polynomial on E is weakly continuous on bounded sets,*
(d) *each norm continuous polynomial on E is weakly sequentially continuous.*

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