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Semi-Simplicity of some Semi-Prime Banach Algebras

DUMITRU D. DRĂGHIA

National Communications Research Center, P.O. Box 1-816, 70101-*Bucureşti, România*

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1. INTRODUCTION

The notions of radical and socle are of fundamental importance in the theory of Banach algebras. If A is a complex Banach algebra, the socle of A and the Jacobson radical of A are denoted by $\mathrm{Socl}(A)$ and $\mathrm{Rad}(A)$, respectively. Recall that $\mathrm{Socl}(A)$ exists when A is a semi-prime algebra with minimal one-sided ideals. For details see [9, Chapter II], and [2, Chapter III and Chapter IV].

Recently, in [3], and [7], the connection between the socle and the radical has been investigated. Thus, in [3] it is established that if A is a semi-prime Banach algebra, then $\mathrm{Socl}(A) \cap \mathrm{Rad}(A) = \{0\}$, using the fact that "the socle consists of all finite sums of compactly acting single elements that are not in radical". Also, in [7] it was proved that if A is a semi-prime Banach algebra with unit element, then $\overline{\mathrm{Socl}(A)} \cap \mathrm{Rad}(A) = \{0\}$, where the principal argument is that "the socle is a two-sided ideal of algebraic elements". The same result is proved in [2, p. 162] for a semi-prime non-radical Banach algebra that is also an annihilator algebra.

In this note we prove first a more general result on the two-sided ideals of a semi-prime Banach algebra, namely:

If L and M are two-sided ideals of a semi-prime Banach algebra A, then $L \cap M = \{0\}$ if and only if $\overline{L} \cap \overline{M} = \{0\}$. (Theorem 3.1).

In particular, since $\mathrm{Rad}(A)$ *is a closed two-sided ideal of A and* $\mathrm{Socl}(A)$ *is a two-sided ideal, the conditions* $\mathrm{Socl}(A) \cap \mathrm{Rad}(A) = \{0\}$ *and* $\overline{\mathrm{Socl}(A)} \cap \mathrm{Rad}(A) = \{0\}$ *are equivalent.*

Also, we give a elementary proof of the fact that $\mathrm{Socl}(A) \cap \mathrm{Rad}(A) = \{0\}$. (Theorem 3.2). The proof is based only on the definitions and elementary properties of the radical and socle.

Then we shall exploit the relation between radical and socle of a semi-prime Banach algebra.

It is well-known that semi-simple algebras are semi-prime (see [2, Proposition 5, p. 155]), and some examples of semi-prime radical Banach algebras exist (cf. [4], and [2, p. 255]).

Converse, as applications of above theorems, we prove that some class of semi-prime Banach algebras are semi-simple algebras. (Theorem 4.1, Corollary 4.1, Theorem 4.2).

2. Preliminaries

In this section we review some algebraic preliminaries which will be needed below. For notions and others results we refer to textbooks [8],[9] and [2].

Let A be an arbitrary algebra. Given a subset E of A, the left annihilator of E is the set $l(E)$ defined by $l(E) = \{x \in A : xE = \{0\}\}$. The set $\{x \in A : Ex = \{0\}\}$ is called right annihilator of E and it is denoted by $r(E)$. The sets $l(E)$ and $r(E)$ are two-sided ideals of A.

A minimal idempotent of A is a non-zero element $e \in A$ such that $e^2 = e$ and eAe is a division algebra.

An algebra A is semi-prime if $\{0\}$ is the only two-sided ideal of A with square equal to zero. If A is a semi-prime algebra, then $l(A) = r(A) = \{0\}$.

Let A be a semi-prime algebra with minimal one-sided (left or right) ideals. A left ideal L is a minimal left ideal of A if and only if $L = Ae$, where e is a minimal idempotent in A.

A right ideal R is a minimal right ideal of A if and only if $R = eA$, where e is a minimal idempotent in A.

The smallest left (right) ideal which contains every left (right) minimal ideals is called the left (right) socle of A. The left and right socles are both generated by minimal idempotents and therefore coincide, defining the socle.

Lemma 2.1. *Let A be a semi-prime algebra and let J be a two-sided ideal in A. Then $l(J) = r(J)$.*

Proof. We have $(Jl(J))^2 = J(l(J)J)l(J) = \{0\}$. Since A is semi-prime, it follows that $Jl(J) = \{0\}$, that is $l(J) \subseteq r(J)$. Similary, it is easy to see that $r(J) \subseteq l(J)$. Therefore $l(J) = r(J)$, for every two-sided ideal J of A. ∎

Lemma 2.2. *If J is a two-sided ideal of a semi-prime algebra A, then $J \cap l(J) = \{0\}$.*

Proof. We have $(J \cap l(J))^2 = (J \cap l(J))(J \cap l(J)) \subseteq l(J)J = \{0\}$. Since A is semi-prime, it follows that $J \cap l(J) = \{0\}$. ∎

3. The Main Results

We first establish a general result on the two-sided ideals in any semi-prime Banach algebra.

THEOREM 3.1. *Let A be a semi-prime Banach algebra, and let L and M be two-sided ideals of A. Then $L \cap M = \{0\}$ if and only if $\overline{L} \cap \overline{M} = \{0\}$. In particular,* $\mathrm{Socl}(A) \cap \mathrm{Rad}(A) = \{0\}$ *if and only if* $\overline{\mathrm{Socl}(A)} \cap \mathrm{Rad}(A) = \{0\}$.

Proof. It is clear that $\overline{L} \cap \overline{M} = \{0\}$ implies $L \cap M = \{0\}$. Conversely, suppose that $L \cap M = \{0\}$. Since L and M are two-sided ideals of A, we have $LM \subseteq L \cap M$.

By continuity of the product in A, it follows that $\overline{L} \subseteq l(\overline{M})$. Since A is semi-prime algebra we have $\overline{M} \cap l(\overline{M}) = \{0\}$. Thus, by $\overline{L} \cap \overline{M} \subseteq l(\overline{M}) \cap \overline{M}$, it follows that $\overline{L} \cap \overline{M} = \{0\}$. Therefore the conditions $L \cap M = \{0\}$ and $\overline{L} \cap \overline{M} = \{0\}$ are equivalent. It is well known that the radical of a Banach algebra is a closed two-sided ideal [9, Theorem (2.3.5) (i)] and the socle is a two-sided ideal. Thus, by the above argument, it follows that $\mathrm{Rad}(A) \cap \mathrm{Socl}(A) = \{0\}$ is equivalent to $\mathrm{Rad}(A) \cap \overline{\mathrm{Socl}(A)} = \{0\}$. ∎

Using only basic facts concerning the socle and the radical of a semi-prime Banach algebra, we give an elementary proof of the next theorem.

THEOREM 3.2. *If A is a semi-prime Banach algebra with one-sided minimal ideals, then* $\mathrm{Socl}(A) \cap \mathrm{Rad}(A)$ *is the zero ideal.*

Proof. Suppose that $\mathrm{Socl}(A) \cap \mathrm{Rad}(A) \neq \{0\}$. Let $a \in \mathrm{Socl}(A) \cap \mathrm{Rad}(A)$, $a \neq 0$. Then $a \in \mathrm{Socl}(A)$ implies that $a\mathrm{Socl}(A) \neq \{0\}$. Thus there exists a minimal idempotent element $e \in A$ such that $ae \neq 0$. Hence, by [2, Lemma 7, p. 155] or [9, Lemma (2.1.8), p. 45], aeA is a right minimal ideal. From $a \in \mathrm{Rad}(A)$ it follows that $aeA \subseteq \mathrm{Rad}(A)$. But this implies that $\mathrm{Rad}(A)$ contains an (minimal) idempotent element, which is imposible. This contradiction shows that $\mathrm{Socl}(A) \cap \mathrm{Rad}(A) = \{0\}$. ∎

From this theorem one can obtains the following interesting result.

COROLLARY 3.1. *If A is a radical semi-prime Banach algebra, then its socle is the zero ideal.*

Proof. We have $\mathrm{Socl}(A) = A \cap \mathrm{Socl}(A) = \mathrm{Rad}(A) \cap \mathrm{Socl}(A) = \{0\}$. ∎

4. Applications

As important applications of above theorems, we obtain conditions under which a semi-prime Banach algebra is a semi-simple Banach algebra.

In [9, p. 73 and p. 262] Banach algebras A such that $l(\mathrm{Socl}(A)) = \{0\}$ have been considered.

We have the following theorem.

THEOREM 4.1. *If A is a semi-prime Banach algebra such that the annihilator of its socle is zero, then A is a semi-simple Banach algebra.*

Proof. We have $\mathrm{Rad}(A)\mathrm{Socl}(A) \subseteq \mathrm{Socl}(A) \cap \mathrm{Rad}(A)$. From Theorem 3.2 it follows that $\mathrm{Rad}(A) \subseteq l(\mathrm{Socl}(A))$. Therefore $\mathrm{Rad}(A) = \{0\}$. ∎

Recall that an algebra A is said to be a prime if $aAb = \{0\}$ implies $a = 0$ or $b = 0$.

We can prove the following result.

COROLLARY 4.1. *If A is a prime Banach algebra with one-sided minimal ideals, then A is a semi-simple Banach algebra.*

Proof. It is known that a prime algebra is a semi-prime algebra. Let e be a minimal idempotent element of A. Since $l(\mathrm{Socl}(A)) \subseteq l(Ae) = \{0\}$, we can apply the Theorem 4.1. ∎

The algebra of compact operators on a Hilbert space, and annihilator or dual algebras are algebras with a dense socle (see, e.g. [1]).

We prove the next theorem.

THEOREM 4.2. *If A is a semi-prime Banach algebra with a dense socle, then A is a semi-simple Banach algebra.*

Proof. We have $\mathrm{Rad}(A) = A \cap \mathrm{Rad}(A) = \overline{\mathrm{Socl}(A)} \cap \mathrm{Rad}(A)$. Applying Theorem 3.1 and Theorem 3.2 it follows that $\mathrm{Rad}(A) = \{0\}$. ∎

Notice here that if A is a Banach algebra such that $l(A) = r(A) = \{0\}$ (e.g. algebras with identity, algebras with approximate identities, semi-prime algebra, etc.), and if $A = \mathrm{Socl}(A)$, then A is a finite dimensional algebra (see [5]).

For others related results on semi-prime Banach algebras the reader is referred to [6].

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