

Some Counterexamples concerning Sufficiency and Invariance*

R.H. BERK, A.G. NOGALES AND J.A. OYOLA

Dept. of Statistics, Rutgers University, New Brunswick, NJ 08903
Dpto. de Matemáticas, Universidad de Extremadura, 06071-Badajoz, Spain

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Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a statistical experiment (i.e., \mathcal{P} is a family of probability measures on the measurable space (Ω, \mathcal{A})) and G a group of bijective and bimeasurable maps of (Ω, \mathcal{A}) onto itself leaving the family \mathcal{P} invariant, that is, $gP \in \mathcal{P}, \forall P \in \mathcal{P}, \forall g \in G$, where gP is the probability measure on \mathcal{A} defined by $gP(A) = P(g^{-1}A), A \in \mathcal{A}$. If $P \in \mathcal{P}$, two events $B, C \in \mathcal{A}$ are said to be P -equivalent (and we shall write $B \stackrel{P}{\sim} C$) if $P(B\Delta C) = 0$; these events are said to be equivalent (we write $B \sim C$) if they are P -equivalent for all $P \in \mathcal{P}$. The null sets are the events equivalent to \emptyset . Let $\mathcal{A}_I = \{A \in \mathcal{A} : gA = A, \forall g \in G\}$ be the σ -field of G -invariant sets and $\mathcal{A}_A = \{A \in \mathcal{A} : gA \sim A, \forall g \in G\}$ the σ -field of \mathcal{P} -almost- G -invariant sets.

For two sub- σ -fields \mathcal{B}, \mathcal{C} of \mathcal{A} we shall write $\mathcal{B} \subseteq \mathcal{C}$ if for every $B \in \mathcal{B}$ there exists $C \in \mathcal{C}$ such that $B \sim C$; \mathcal{B} and \mathcal{C} will be said to be equivalent or \mathcal{P} -equivalent (and we shall write $\mathcal{B} \sim \mathcal{C}$) if $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{B}$. The sub- σ -fields \mathcal{B} and \mathcal{C} are said to be independent if they are P -independent for every $P \in \mathcal{P}$. A privileged dominating probability for the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability measure Q on (Ω, \mathcal{A}) of the form $Q = \sum_{n=1}^{\infty} a_n P_n$ such that $P \ll Q$ for all $P \in \mathcal{P}$, $\{P_n : n \in \mathbb{N}\} \subset \mathcal{P}$, $\sum_n a_n = 1$ and $a_n \geq 0, \forall n$. It is well known that a privileged dominating probability exists when the experiment is dominated. \mathcal{A}_S will always be a sufficient sub- σ -field of \mathcal{A} . The σ -fields $\mathcal{A}_{SI} = \{A \in \mathcal{A}_I : \exists B \in \mathcal{A}_S, P(A\Delta B) = 0, \forall P \in \mathcal{P}\}$ and $\mathcal{A}_{SA} = \{A \in \mathcal{A}_A : \exists B \in \mathcal{A}_S, P(A\Delta B) = 0, \forall P \in \mathcal{P}\}$ are also considered in Berk (1972).

Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be three sub- σ -fields of \mathcal{A} ; for $P \in \mathcal{P}$, the σ -fields \mathcal{B} and \mathcal{C} are said to be P -conditionally independent given \mathcal{D} , and we shall write $\mathcal{B} \perp_P \mathcal{C} | \mathcal{D}$,

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if

$$E_P(I_{B \cap C} | \mathcal{D}) \stackrel{P}{\sim} E_P(I_B | \mathcal{D}) \cdot E_P(I_C | \mathcal{D})$$

for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$. It is well known that $\mathcal{B} \perp_P \mathcal{C} | \mathcal{D}$ if and only if

$$E_P(I_C | \mathcal{B} \vee \mathcal{D}) \stackrel{P}{\sim} E_P(I_C | \mathcal{D}), \forall C \in \mathcal{C},$$

where $\mathcal{B} \vee \mathcal{D}$ is the smallest σ -field containing \mathcal{B} and \mathcal{D} . The σ -fields \mathcal{B} and \mathcal{C} are said to be conditionally independent given \mathcal{D} , and we shall write $\mathcal{B} \perp \mathcal{C} | \mathcal{D}$, if $\mathcal{B} \perp_P \mathcal{C} | \mathcal{D}, \forall P \in \mathcal{P}$. Other known concepts not defined here may be found in Lehmann (1986), for example.

The classical paper Hall, Wijsman and Ghosh (1965) investigates under which conditions the σ -field $\mathcal{A}_S \cap \mathcal{A}_I$ is sufficient for \mathcal{A}_I : it is shown that this is the case if $g\mathcal{A}_S = \mathcal{A}_S, \forall g \in G$ and $\mathcal{A}_S \cap \mathcal{A}_I \sim \mathcal{A}_S \cap \mathcal{A}_A$. The interesting analogous problem for almost-invariance is considered in Berk (1972) where it is shown that \mathcal{A}_{SA} is sufficient for \mathcal{A}_A if $g\mathcal{A}_S \sim \mathcal{A}_S, \forall g \in G$. A synonymous condition is that \mathcal{A}_S is equivalent to the σ -field induced by an almost-equivariant statistic (see Lemma 2 of Berk (1972)), and is satisfied if \mathcal{A}_S is minimal sufficient.

In this paper some concepts and examples are given to clarify certain results of the papers cited above.

Let us introduce a weaker notion of equivalence between σ -fields as follows: given two sub- σ -fields \mathcal{B} and \mathcal{C} of \mathcal{A} we will say that \mathcal{B} and \mathcal{C} are weakly- \mathcal{P} -equivalent if they are P -equivalent for all $P \in \mathcal{P}$. A σ -field will be said weakly- \mathcal{P} -trivial if it is weakly- \mathcal{P} -equivalent to the trivial σ -field. Using this weaker notion of triviality, a correct version of proposition (i) of Theorem 4 of Berk (1972) is as follows: the σ -fields \mathcal{A}_S and \mathcal{A}_A are independent if and only if they are conditionally independent given \mathcal{A}_{SA} and \mathcal{A}_{SA} is weakly- \mathcal{P} -trivial. The following counterexample shows a non trivial group for which \mathcal{A}_{SI} is not \mathcal{P} -equivalent to $\{\emptyset, \Omega\}$.

EXAMPLE 1. Let $\Omega = \{1, 2, 3, 4\}$, \mathcal{A} be the σ -field of all subsets of Ω , and $\mathcal{P} = \{P, Q\}$, where P is the uniform distribution on $\{2, 3, 4\}$ and Q is the probability measure concentrated at the point 1. The smallest σ -field \mathcal{A}_S containing the events $\{1\}$ and $\{2\}$ is sufficient for the experiment $(\Omega, \mathcal{A}, \mathcal{P})$. Let $G = \{I, g_1, g_2\}$ where I is the identity map on Ω , g_1 is the permutation $(1, 3, 4, 2)$ and $g_2 = (1, 4, 2, 3)$. We have that $\mathcal{A}_A = \mathcal{A}_I$ is the smallest σ -field including $\{1\}$ and \mathcal{A}_A and \mathcal{A}_S are independent, but $\mathcal{A}_{SI} = \mathcal{A}_{SA} = \mathcal{A}_A$ is not \mathcal{P} -equivalent to $\{\emptyset, \Omega\}$.

REMARK 1: It is not difficult to show that, replacing the independence of \mathcal{A}_S and \mathcal{A}_A by the stronger condition of independence of \mathcal{A}_S and \mathcal{A}_A for a privileged dominating probability, $\mathcal{A}_{SA} \sim \{\emptyset, \Omega\}$, and hence $\mathcal{A}_{SI} \sim \{\emptyset, \Omega\}$. The proof that independence for a privileged dominating probability implies independence when one of the σ -fields involved is sufficient provides in passing a converse to the well-known theorem of Basu, namely, any statistic independent of a sufficient statistic for a privileged dominating probability is ancillary. Example 1 also shows that this proposition is not true if we only assume independence.

We are now concerned with the relationship between the independence of \mathcal{A}_S and \mathcal{A}_A and the equivalence of \mathcal{A}_{SA} and \mathcal{A}_{SI} . A correct version of an assertion of Berk (1972) state that the independence of \mathcal{A}_S and \mathcal{A}_A implies that \mathcal{A}_{SA} is weakly- \mathcal{P} -equivalent to \mathcal{A}_{SI} ; in fact, it implies the weak \mathcal{P} -triviality of \mathcal{A}_{SA} . The condition $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$ is fulfilled if \mathcal{A}_S and \mathcal{A}_A are independent for a privileged dominating probability. It should be noted that while $\mathcal{A}_A \sim \mathcal{A}_I$ implies that $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$, it does not imply the stronger condition that $\mathcal{A}_S \cap \mathcal{A}_A \sim \mathcal{A}_S \cap \mathcal{A}_I$ as is shown in Example 1 of Landers and Rogge (1973).

The following counterexample shows that the independence of \mathcal{A}_S and \mathcal{A}_A is not a sufficient condition to have $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$. For the choice of the group of transformations in the two examples below, we make use of an idea due to Berk (1970).

EXAMPLE 2. Let E_1 and E_2 be disjoint intervals of \mathbb{R} , $\Omega = E_1 \cup E_2$, and \mathcal{A} be the Borel σ -field of Ω . Let $\mathcal{P} = \{U_1, U_2\}$, where U_i is the uniform distribution on E_i , $i = 1, 2$. The smallest σ -field \mathcal{A}_S containing E_1 and E_2 is sufficient (and complete) for the experiment considered. Let G be the group of all bijective maps of Ω onto itself moving at most a finite subset of Ω . We have that $\mathcal{A}_I = \mathcal{A}_{SI} = \{\emptyset, \Omega\}$, $\mathcal{A}_A = \mathcal{A}$, and \mathcal{A}_{SA} is the smallest σ -field including \mathcal{A}_S and the null sets. Hence \mathcal{A}_{SI} is not equivalent to \mathcal{A}_{SA} . Nevertheless, \mathcal{A}_S and \mathcal{A}_A are independent.

A right restatement of part (ii) of the theorem in Berk (1972) is as follows: under the assumption of weak- \mathcal{P} -equivalence of $\mathcal{A}_S \vee \mathcal{A}_I$ and \mathcal{A} , the independence of \mathcal{A}_S and \mathcal{A}_A implies the weak \mathcal{P} -equivalence of \mathcal{A}_A and \mathcal{A}_I . The next counterexample shows that we need not have equivalence of \mathcal{A}_A and \mathcal{A}_I , even if $\mathcal{A}_S \vee \mathcal{A}_I \sim \mathcal{A}$.

EXAMPLE 3. Let $\Omega = [0, 4] \times [0, 4]$, \mathcal{N} be the set of null Borel sets on Ω with respect to the Lebesgue measure, $A_1 = [0, 1] \times [0, 1]$, $A_2 = [2, 3] \times [2, 3]$, and \mathcal{A}

be the smallest σ -field containing \mathcal{N} , $[0, 2] \times [0, 2]$, $[2, 4] \times [2, 4]$ and $[1, 3] \times [1, 3]$. We shall write $U_i, i = 1, 2$ for the restriction to \mathcal{A} of the uniform distribution on A_i and $\mathcal{P} = \{U_1, U_2\}$. Let G be the group of all transformations on Ω moving at most a finite subset of Ω and leaving the set $[1, 3] \times [1, 3]$ invariant. Hence \mathcal{A}_I is the smallest σ -field including $[1, 3] \times [1, 3]$, and $\mathcal{A}_A = \mathcal{A}$. The smallest σ -field \mathcal{A}_S containing $[0, 2] \times [0, 2]$ and $[2, 4] \times [2, 4]$ is sufficient for the experiment $(\Omega, \mathcal{A}, \mathcal{P})$, is independent of \mathcal{A}_A , and satisfies $\mathcal{A} \sim \mathcal{A}_S \vee \mathcal{A}_I$. But $\mathcal{A}_A \not\sim \mathcal{A}_I$, since the event $[2, 3] \times [2, 3]$ is not equivalent to any event of \mathcal{A}_I .

REMARK 2: It is also claimed in Berk (1972) that under the hypothesis of conditional independence of \mathcal{A}_S and \mathcal{A}_A given \mathcal{A}_{SA} and $\mathcal{A} \sim \mathcal{A}_S \vee \mathcal{A}_I$, the propositions $\mathcal{A}_A \sim \mathcal{A}_I$ and $\mathcal{A}_{SA} \sim \mathcal{A}_{SI}$ are equivalent. The proof given there require the not-easily-checked condition " \mathcal{A}_I is sufficient for \mathcal{A}_A "; this condition (and, hence, $\mathcal{A}_A \sim \mathcal{A}_I$) is clearly satisfied in the dominated case. Another condition guaranteeing that \mathcal{A}_I is sufficient for \mathcal{A}_A is that the group acts transitively on the family \mathcal{P} (this means that $\mathcal{P} = \{gP : g \in G\}$) as is shown in Lemma 2 of Berk and Bickel (1968). The condition $\mathcal{A} \sim \mathcal{A}_S \vee \mathcal{A}_I$ can be replaced by $\mathcal{A}_A \subsetneq \mathcal{A}_S \vee \mathcal{A}_I$.

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