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Stability of the Local Spectrum

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1. INTRODUCTION

Let X be a complex Banach space and and let T be a (bounded linear) operator defined on X. For every $x \in X$, the operator T has associated a local spectrum $\sigma(x, T)$ which is an useful tool in the study of the structure of the spectrum and the invariant subspaces of T.

The problem we address ourselves is the detection of vectors y which have the same local spectrum than a fixed vector x, namely $\sigma(x, T) = \sigma(y, T)$. This problem has deserved the attention of several authors. In [2], Erdelyi and Lange prove that if T is an operator verifying the Single Valued Extension Property (hereafter referred to as SVEP) and $\widehat{x}_T$ is the local resolvent function of T in x, then $\sigma(\widehat{x}_T(\lambda), T) = \sigma(x, T)$ for all $\lambda \in \mathbb{C} \setminus \sigma(x, T)$. Moreover, if A is an operator which commutes with an operator T verifying the SVEP, then

$$\sigma(Ax, T) \subset \sigma(x, T), \tag{1}$$

for all $x \in X$. In particular, if A has an inverse, then the expresion (1) turns into an equality. It also follows, from the results derived by Bartle [1], that given $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\sigma((\lambda - T)^n x, T) \subset \sigma(x, T) \subset \sigma((\lambda - T)^n x, T) \cup \{\lambda\}. \tag{2}$$

Hence if $\lambda \notin \sigma(x, T)$, then $\sigma(x, T) = \sigma((\lambda - T)^n x, T)$. Finally McGuire [7] shows that if T is an operator in a complex separable Hilbert space H with an empty point spectrum, and f is an analytic function on an open set $\triangle(f)$ containing $\sigma(x, T)$, not identically zero on any component of $\triangle(f)$, then $\sigma(f[T]x, T) = \sigma(x, T)$, where $f[T]x$ is defined by using the "Cauchy formula" with the local resolvent of T in x (see below).

In this paper we give conditions implying the equality $\sigma(x,T) = \sigma(Ax,T)$ for certain operators A obtained from T by using the meromorphic functional calculus or the local functional calculus. Our results include that of [1], [2] and [7].

2. Preliminaries

Let X be a complex Banach space. We denote by $L(X)$ the class of all (bounded linear) operators on X, and by $C(X)$ the class of all closed operators with *domain* $D(T)$ and *range* $R(T)$ in X.

Given $T \in C(X)$, we have that λ belongs to $\rho(T)$, the *resolvent set* of T, if there exists $(\lambda - T)^{-1} \in L(X)$ such that $(\lambda - T)^{-1}(X) = D(T)$ and for every $x \in X$ we have $(\lambda - T)(\lambda - T)^{-1}x = x$. We denote by $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the *spectrum set* of T. Note that the set $\rho(T)$ is open and the *resolvent function* $\lambda \longrightarrow (\lambda - T)^{-1}$ is analytic in $\rho(T)$.

Likewise, for every $x \in X$ the local spectral theory is defined as follows. We say that $\lambda \in \rho(x,T)$, the *local resolvent set* of T in x, if there exists an analytic function $w : U \longrightarrow X$ defined on a neigbourhood U of λ, which satisfies the equation $(\mu - T)w(\mu) = x$, for every $\mu \in U$. We denote by $\sigma(x,T) := \mathbb{C} \setminus \rho(x,T)$ the *local spectrum* of T in x. Since w is not necessarily unique, a property is introduced to avoid this problem.

A closed linear operator $T : D(T) \subset X \longrightarrow X$ satisfies the SVEP if for every analytic function $h : \triangle(h) \longrightarrow X$ defined on an open set $\triangle(h) \subset \mathbb{C}$, the condition $(\lambda - T)h(\lambda) \equiv 0$ implies $h \equiv 0$. If T satisfies the SVEP, then for every $x \in X$ there exists a unique maximal analytic function $\widehat{x}_T : \rho(x,T) \longrightarrow X$, such that $(\lambda I - T)\widehat{x}_T(\lambda) = x$, for every $\lambda \in \rho(x,T)$. The function $\widehat{x}_T$ is called the *local resolvent*. See [2], [3] and [6] for further details.

For $T \in L(X)$, the *holomorphic functional calculus* is defined as follows [9]. Let f be an analytic function defined on an open set $\triangle(f)$ containing $\sigma(T)$. The operator $f(T) \in L(X)$ is defined by the "Cauchy formula"

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, T) d\lambda,$$

where Γ is the boundary of a Cauchy domain D such that $\sigma(T) \subset D \subset \triangle(f)$.

This definition may be extended to meromorphic functions. Let f be a meromorphic function in an open set $\triangle(f)$ containing $\sigma(T)$, such that the poles of f are not in the point spectrum $\sigma_p(T)$, and let $\alpha_1, \dots, \alpha_k$ be the poles of f in $\sigma(T)$, with multiplicity $n_1, \dots, n_k$, respectively. We consider the polynomial p given by $p(\lambda) = \prod_{i=1}^{k}(\alpha_i - \lambda)^{n_i}$. Note that $g(\lambda) := f(\lambda)p(\lambda)$ is an analytic function. In [4], Gindler defines a *meromorphic functional calculus* by $f\{T\} := g(T)p(T)^{-1} \in C(X)$. Clearly, the meromorphic calculus is an extension of the holomorphic calculus.

3. The local functional calculus

Let f be an analytic function defined on an open set $\triangle(f)$. For H a Hilbert space and $T \in L(H)$ an operator with empty point spectrum, McGuire [7] introduces a *local functional calculus* in which he defines $f[T]x$, for $x \in H$ with $\sigma(x,T) \subset \triangle(f)$, by

$$f[T]x = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\widehat{x}_T(\lambda)d\lambda, \tag{3}$$

where Γ is the boundary of a Cauchy domain D such that $\sigma(T) \subset D \subset \triangle(f)$.

Using this idea, for any $T \in L(X)$ we define an operator $f[T] : D(f[T]) \subset X \longrightarrow X$ with domain $D(f[T]) := \{x \in X \ : \ \sigma(x,T) \subset \triangle(f)\}$ and $f[T]x$ given by (3) for $x \in D(f[T])$. It is clear that $D(f[T])$ is a linear subspace of X and $f[T]$ is a linear operator.

Next, we give some results concerning the local functional calculus.

Proposition 1. *Let $T \in L(X)$ satisfy the SVEP and let f be an analytic function in $\triangle(f)$. Then the following assertions hold:*

(i) *If $S \in L(X)$ commutes with T, then S commutes with $f[T]$; i.e., $SD(f[T]) \subset D(f[T])$ and $Sf[T]x = f[T]Sx$ for all $x \in D(f[T])$.*

(ii) *If $x \in D(f[T])$ and $y := f[T]x$, then $f[T]\widehat{x}_T = \widehat{y}_T$ in $\rho(x,T)$, hence $\sigma(f[T]x,T) \subset \sigma(x,T)$.*

Assume $T \in L(X)$ satisfy the SVEP. Let $x \in X$, and let f, g analytic functions such that $x \in D(f[T]) \cap D(g[T])$. Clearly we have $(\alpha f + \beta g)[T]x = \alpha f[T]x + \beta g[T]x$, for all $\alpha, \beta \in \mathbb{C}$ and

$$(fg)[T]x = f[T]g[T]x = g[T]f[T]x. \tag{4}$$

Remark 2. Sometimes the results of evaluating $f[T]g[T]x$ and $(fg)[T]x$ are different, as it is showed by the following example: Let T be the operator in the Hilbert space $\ell_2(\mathbb{N})$ defined by $T(x_n) = (\frac{1}{n}x_n)$. Taking $x := (1,1,0,\dots)$, $f(\lambda) := \frac{1}{1-\lambda}$ and $g(\lambda) := 1-\lambda$, we obtain $f[T](I-T)x = (0,1,0,\dots)$ and $(fg)[T]x = x = (1,1,0,\dots)$.

Note that $x \notin D(f[T])$. So we cannot define $g[T]f[T]x$.

McGuire proved in [7] the equality (4) in the case X is a complex separable Hilbert and T has empty point spectrum.

4. Stability under the action of polynomials

Our first result is a characterization for the equality $\sigma(p(T)x, T) = \sigma(x, T)$, when p is a polynomial.

THEOREM 3. *Assume $T \in L(X)$ satisfy the SVEP. Let $x \in X$ and let $p(\lambda) = (\alpha_1 - \lambda)^{n_1} \dots (\alpha_p - \lambda)^{n_p}$ be a polynomial with $\alpha_i \neq \alpha_j$ for $i \neq j$. We have $\sigma(p(T)x, T) = \sigma(x, T)$ if and only if there is no $i \in \{1, \dots, p\}$ so that α_i is a pole of $\widehat{x}_T$ of order $\leq n_i$.*

Consequently, $\sigma(p(T)x, T) = \sigma(x, T)$ if no α_i is an isolated point of $\sigma(x, T)$.

COROLLARY 4. *Assume $T \in L(X)$ satisfy the SVEP. Let $p(\lambda)$ be a polynomial having no zeroes in $\sigma_p(T)$. Then $\sigma(p(T)x, T) = \sigma(x, T)$, for all $x \in X$. Consequently, if $y \in D(p(T)^{-1}) = R(P(T))$, then $\sigma(p(T)^{-1}y, T) = \sigma(y, T)$.*

COROLLARY 5. *Let $T \in L(X)$ satisfy the SVEP and let $x \in X$. If $p(\lambda)$ is a polynomial having no zeroes in $\sigma_p(T) \cap \sigma(x, T)$ then $\sigma(p(T)x, T) = \sigma(x, T)$.*

In general, the converse of the above corollary is not true, as shows the following example.

EXAMPLE 6. Let $B([0, 1])$ denote the Banach space of all bounded functions from [0,1] into $\mathbb{C}$, with the supremum norm. For $u \in B([0, 1])$ we define $(Tu)(s) = su(s)$ for all $s \in [0, 1]$. If $x(t)$ is given by

$$x(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} \\ 1, & \frac{1}{2} < t \leq 1, \end{cases}$$

then $\sigma(x, T) = [\frac{1}{2}, 1]$ and $1 \in \sigma(x, T) \cap \sigma_p(T)$. However for $p(\lambda) := 1 - \lambda$, we have

$$(I - T)x(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} \\ (1 - t), & \frac{1}{2} < t \leq 1, \end{cases}$$

hence $\sigma((I - T)x, T) = [\frac{1}{2}, 1] = \sigma(x, T)$. ∎

5. Stability under the action of analytic and meromorphic functions

The following Proposition gives a sufficient condition for the equality $\sigma(f[T]x,T)=\sigma(x,T)$, where f is a function of the local functional calculus.

Proposition 7. *Assume $T\in L(X)$ satisfy the SVEP. Let $x\in X$ and let f be an analytic function in a neighbourhood of $\sigma(x,T)$. If f has no zeroes in $\sigma(x,T)$, then*

$$\sigma(f[T]x,T)=\sigma(x,T).$$

Theorem 8. *(Stability of the local spectrum). Assume $T\in L(X)$ satisfy the SVEP. Let $x\in X$ and let f be a function analytic in a neighbourhood of $\sigma(x,T)$. Let $\alpha_1,\dots\alpha_p$ be the zeroes of f in $\sigma(x,T)$ with multiplicities $n_1,\dots,n_p$, respectively. Then it have $\sigma(f[T]x,T)=\sigma(x,T)$ if and only if there is no $i\in\{1,\dots,p\}$ so that α_i is a pole of $\widehat{x}_T$ of order $\le n_i$.*

The result [7, Theorem 1.5] of McGuire may be readily derived from the following Corollary.

Corollary 9. *Assume $T\in L(X)$ satisfy the SVEP. Let $x\in X$, and let f be an analytic function in $\sigma(x,T)$. If f has no zeroes in $\sigma_p(T)\cap\sigma(x,T)$, then $\sigma(x,T)=\sigma(f[T]x,T)$.*

The following Corollary gives caracterizations of when an analytic function f satisfies the equality $\sigma(f[T]x,T)=\sigma(x,T)$, for all $x\in D(f[T])$.

Corollary 10. *Assume $T\in L(X)$ satisfy the SVEP. If f is an analytic function which is not identically zero on any component of $\triangle(f)$ intersecting $\sigma(T)$, then the following assertions are equivalent:*

(i) *f has no zeroes in $\sigma_p(T)\cap\sigma(x,T)$, for all $x\in D(f[T])$.*

(ii) *$\sigma(f[T]x,T)=\sigma(x,T)$, for all $x\in D(f[T])$.*

(iii) *$f[T]$ is injective.*

In the following Corollary we give a necessary and sufficient condition for the stability of the local spectrum by the meromorphic calculus.

Notice that the result holds for all $x\in D(f\{T\})$, which in general includes properly $D(f[T])$.

COROLLARY 11. *Assume $T \in L(X)$ verifies the SVEP. Let f be a meromorphic function in an open set containing $\sigma(T)$, such that the poles of f are outside the point spectrum of T and f is identically zero in no component of $\triangle(f)$.*

Then $\sigma(f\{T\}x, T) = \sigma(x, T)$ for all $x \in D(f\{T\})$ if and only if f has no zeroes in $\sigma_p(T)$.

Finally we give a property similar to (2) for the operator $f[T]$.

PROPOSITION 12. *Assume $T \in L(X)$ satisfy the SVEP, and let f be an analytic function in $\sigma(x, T)$. Then $\sigma(x, T) \subset \sigma(f[T]x, T) \cup \{Z_x(f, T)\}$, where $Z_x(f, T)$ denotes the set of all zeros of f in $\sigma(x, T)$.*

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