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The Support of the Associated Measure to the Cowen's Tridiagonal Matrix

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1. Introduction

In this paper, we consider a class of three-term recurrence relations, whose associated tridiagonal matrices are subnormal operators. In these cases, there exist measures associated to the polynomials given by such relations. We study the supports of these measures.

Let $M_\lambda = (c_{ij}^\lambda)_{i,j=0}^\infty$ be a positive defined hermitian infinite matrix, generated by $c_{ij}^\lambda =< D_\lambda^j e_0, D_\lambda^i e_0 >$, where $e_0^t = (1,0,0,\dots)$ and $D_\lambda = T + \lambda T^*$, being

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{1+s} & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{1+s+s^2} & 0 & 0 & \dots \\ 0 & 0 & 0 & \sqrt{1+s+s^2+s^3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with $\lambda \in \mathbb{C}$ and $0 < s < 1$.

Cowen proved in [2] that D_λ is subnormal operator if and only if $\lambda = 0$ or $|\lambda| = s^{k/2}$, with $k \in \{0,1,2,\dots\}$. Therefore, in these cases it is possible to guarantee that M_λ is the matrix of the moments. In other words, there exists an only measure μ_λ with support Ω_λ such that

$$c_{i,j}^\lambda = \int_{\Omega_\lambda} z^i \bar{z}^j \, d\mu_\lambda(z).$$

The aim of this paper is the determination of the different supports Ω_λ for $\lambda = s^{k/2}$, $k = 0,1,2,\dots$, $0 < s < 1$.

2. Numerical Range and Spectrum of D

Proposition 1. *D_λ it is a bounded operator on ℓ^2, with norm* $\dfrac{1+s^{k/2}}{\sqrt{1-s}}$.

THEOREM 1. *Let be the following $n \times n$ tridiagonal matrix*

$$H_n = \begin{pmatrix} 0 & \beta_1 & 0 & \dots & 0 & 0 \\ \gamma_1 & 0 & \beta_2 & \dots & 0 & 0 \\ 0 & \gamma_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \beta_{n-1} \\ 0 & 0 & 0 & \dots & \gamma_{n-1} & 0 \end{pmatrix},$$

with $\beta_i, \gamma_i \in \mathbb{C}$, $i \in \mathbb{N}$. Then the boundary of its numerical rarge is the enveloping of the family of ellipses

$$\frac{x^2}{\left(\sum_{i=1}^{n-1} |x_i||x_{i+1}|(\beta_i + \gamma_i)\right)^2} + \frac{y^2}{\left(\sum_{i=1}^{n-1} |x_i||x_{i+1}|(\beta_i - \gamma_i)\right)^2} = 1$$

obtained for each $(x_1, x_2, \dots, x_n) \in \mathbb{C}^n$ verifying $|x_1|^2 + |x_2|^2 + \dots + |x_n|^2 = 1$.

Proof. We have $(x_1, x_2, \dots, x_n)H_n(x_1, x_2, \dots, x_n)^* = \sum_{j=1}^{n-1}[\beta_j \overline{x}_j x_{j+1} + \gamma_j \overline{\overline{x}_j x_{j+1}}]$ $= \sum_{j=1}^{n-1}(u_j + w_j i) = u + wi$, with $u, w \in \mathbb{R}$, where $\overline{x}_j x_{j+1} = a_j + b_j i$, $u_j = (\beta_j + \gamma_j)a_j$, $w_j = (\beta_j - \gamma_j)b_j$. So we have, for each $(|x_1|, |x_2|, \dots, |x_n|)$,

$$\frac{u_j^2}{(\beta_j + \gamma_j)^2} + \frac{w_j^2}{(\beta_j - \gamma_j)^2} = a_j^2 + b_j^2 = |x_j|^2 |x_{j+1}|^2.$$

∎

COROLLARY 1. *The closure of the numerical range $W(D_\lambda)$ of D_λ is*

$$\overline{W(D_\lambda)} = \{z \in \mathbb{C} \; : \; |z - c| + |z + c| \leq 2a\},$$

where $c = \dfrac{2s^{k/4}}{\sqrt{1-s}}$ and $a = \dfrac{1 + s^{k/2}}{\sqrt{1-s}}$.

In general, the closure of the numerical range includes the spectrum. In our case, as consequence of the previous result and for beins D_λ an hyponormal operator we can conclude that both sets are the same.

3. THE SUPPORT OF THE MEASURE μ_λ

In [2] it is obtained the normal extension $N : \mathcal{H}_{k+1} \longrightarrow \mathcal{H}_{k+1}$ of the operator $D_\lambda : \ell^2 \longrightarrow \ell^2$ for each value of $k \in \{0, 1, 2, \dots\}$. Concretely,

$$N = \begin{pmatrix} A_0 & B_1 & 0 & \dots & 0 & 0 \\ 0 & A_1 & B_2 & \dots & 0 & 0 \\ 0 & 0 & A_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_{k-1} & B_k \\ 0 & 0 & 0 & \dots & 0 & A_k \end{pmatrix},$$

where:

$$
\begin{aligned}
A_i &= s^{i/2}T + s^{(k-i)/2}T^* \qquad i = 0, 1, \dots k, \\
B_{i+1} &= \sqrt{(1 - s^{k-i})(1 + s + s^2 + \cdots + s^i)}\,\sqrt{T^*T - TT^*}\,, \quad i = 0, 1, \dots, (k-1)
\end{aligned}
$$

and $\mathcal{H}_{k+1}$ denotes the Hilbert space which is the direct sum of $k+1$ copies of ℓ^2. The support of μ_λ is the spectrum of the previous operator, and their determination is possible through the study of the spectra of their diagonal.

(In the following $\sigma_{ess}(A)$ will denote the essential spectrum of the operator A).

THEOREM 2. *A_i, $i = 0, 1, \dots, [k/2]$, is hyponormal. A_i, $i = [k/2] + 1, \dots, k$, is cohyponormal ($[k/2]$ denotes the integer part of $k/2$). As consequence, $\sigma(A_i) = \overline{W(A_i)}$, $\forall i = 0, 1, \dots, k$ and $\sigma_{ess}(A_i) = \sigma_{ess}(A_{k-i}) = \mathcal{E}_i$, $i = 0, 1, \dots, [k/2]$, where $\mathcal{E}_i$ is the ellipse*

$$\{z \in \mathbb{C} \,:\, |z - c_i| + |z + c_i| = 2a_i\},$$

with $c_i = \dfrac{2s^{k/4}}{\sqrt{1-s}}$ *and* $a_i = \dfrac{s^{i/2} + s^{(k-i)/2}}{\sqrt{1-s}}$.

Proof. $\sigma(A_i) = \overline{W(A_i)}$ is consequence of the Putnam inequality for hyponormal operators [3] and the second statement is inmediate from corollary 1. ∎

LEMMA 1. *N doesn't have isolated eigenvalues.*

Because the spectrum of an normal operator are constituted only by isolated eigenvalues and essential spectrum, it is sufficient to determine the last one. Moreover N is a compact perturbation of the operator

$$
N' = \begin{pmatrix}
A_0 & 0 & 0 & \dots & 0 & 0 \\
0 & A_1 & 0 & \dots & 0 & 0 \\
0 & 0 & A_2 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & A_{k-1} & 0 \\
0 & 0 & 0 & \dots & 0 & A_k
\end{pmatrix},
$$

so $\sigma_{ess}(N) = \sigma_{ess}(N')$.

PROPOSITION 2.

$$\sigma_{ess}(N') = \sigma_{ess}(A_0) \cup \sigma_{ess}(A_1) \cup \cdots \sigma_{ess}(A_k)$$

From them previous result it is obtained finally:

COROLLARY 2.

$$\operatorname{supp}(\mu_\lambda) = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \cdots \mathcal{E}_{[k/2]}.$$

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