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Some Results on Convergence Linked to a Perturbed Boundary Optimal Control System

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0. Introduction and Position of the Problem

This is an extract of some results taken from recent works (cf. [1] and [2]), where we have considered the system: $(P_1)(u_\epsilon)$ and (P_2):

$$(P_1)(u_\epsilon) \qquad \begin{cases} -\Delta y_\epsilon = 0 & \text{on } \Omega \\ \dfrac{\partial}{\partial v} y_\epsilon + \epsilon y_\epsilon = u_\epsilon & \text{in } \Gamma = \partial\Omega \\ \displaystyle\int_\Gamma y_\epsilon \, d\Gamma = 0; \quad y_\epsilon \in H^1(\Omega) \end{cases}$$

$$(P_2) \qquad J_\epsilon(u_\epsilon) = \min\{J_\epsilon(v);\ v \in \mathcal{U}_{ad}\}$$

where $\frac{\partial}{\partial v} y_\epsilon$ is the normal derivative of y_ϵ, Ω is a regular and bounded open set in the Euclidean space $\mathbb{R}^n$ with $\Gamma = \partial\Omega$ its boundary assumed to be smooth. We denote u_ϵ the optimal control solution of the problem (P_2).

$$J\epsilon(v) := \int_\Gamma (y_\epsilon(v) - z_1)^2 \, d\Gamma + \int_\Gamma \left(\tfrac{\partial}{\partial v} y_\epsilon(v) - z_2\right)^2 d\Gamma, \tag{0.1}$$

$y_\epsilon(v)$ is a solution of $(P_1)(v)$, $v \in \mathcal{U}_{ad}$, $\mathcal{U}_{ad}$ is a closed linear subspace of $\mathcal{U}$ with finite dimension (cf. [1]) or infinite dimension (cf. [2]) where:

$$\mathcal{U} := \left\{ v \in L^2(\Gamma) : \int_\Gamma v \, d\Gamma = 0 \right\}, \tag{0.2}$$

z_1 and z_2 are fixed functions in the space $L^2(\Gamma)$ (decision functions).

We expose here the results obtained in [1] and [2] concerning the existence of the state $y_\epsilon(u_\epsilon)$ and control u_ϵ and study their convergence.

In the first section, we assume that $\mathcal{U}_{ad}$ is of finite dimension. We prove in that case that the state $y_\epsilon(u_\epsilon)$ converges in the Sobolev space $H_1(\Omega)$ and that the optimal control u_ϵ exists and converges in $L^2(\Gamma)$.

We end this extract with some concluding remarks.

1. THE FINITE DIMENSIONAL CASE

1.1. EXISTENCE OF THE PERTURBED STATE AND CONTROL FOR THE SYSTEM: $(P_1)(u_\epsilon)$ AND (P_2). The space of admissible controls $\mathcal{U}_{ad}$ will be a linear subspace of $\mathcal{U}$ with finite dimension $m \geq 1$. $H^1(\Omega)$ is the usual Sobolev space with its scalar product and associated norm. We look for solutions (i.e.: the states of) the system $(P_1)(u_\epsilon)$ in the space:

$$V := \left\{ y \in H^1(\Omega) : \int_\Gamma y \, d\Gamma = 0 \right\}. \tag{1.1}$$

For the existence of the state we have the following theorem:

THEOREM 1.1. *For all $v \in \mathcal{U}_{ad}$, there exists a unique solution of the problem $(P_1)(u_\epsilon)$ denoted by $y_\epsilon(v)$ in the space V.*

The proof of this theorem is classic and based on the variational formulation of the problem $P_1(v)$.

For the existence of the optimal control we have the following theorem:

THEOREM 1.2. *There exists a non vanishing subset X_ϵ of $\mathcal{U}_{ad}$, such that for all $u_\epsilon \in X_\epsilon$, we have:*

$$J_\epsilon(u_\epsilon) = \min\{J_\epsilon(v); v \in \mathcal{U}_{ad}\}. \tag{1.2}$$

Proof. To prove the existence of X_ϵ, it suffices to prove that the following conditions are satisfied (cf. [7]):

(i) The map $v \longrightarrow J_\epsilon(v)$ is strictly convex and l.s.c. (i.e. lower semi-continuous) on the space $\mathcal{U}_{ad}$.

(ii) For all sequence (v_n) of elements in $\mathcal{U}_{ad}$ such that $\|v_n\|_{L^2(\Gamma)} \rightarrow +\infty$ (when $n \rightarrow +\infty$).

The map is differentiable on $L^2(\Gamma)$ then it is continuous so the condition (i) is satisfied. The condition (ii) results from the next Lemma. ∎

LEMMA 1.1. *The map $\mathcal{B}_\epsilon : \mathcal{U} \longrightarrow L^2(\Gamma)$ which associates to each $v \in \mathcal{U}$ the element $\mathcal{B}_\epsilon(v) := y_\epsilon(v)_{|\Gamma}$ is a linear, bounded and injective map into $L^2(\Gamma)$.*

Remark 1.1. Since $\mathcal{U}_{ad}$ is a finite dimensional space, to decide the unicity of solution of (P_2), one can use the Hessian function associated to J_ϵ, after fixing (for example) an orthonormal basis of $\mathcal{U}_{ad}$.

1.2. STUDY OF THE CONVERGENCE OF THE STATE y_ϵ AND CONTROL u_ϵ. The main result of this section is the following theorem:

THEOREM 1.3. *We have the following statements:*

(i) *The control u_ϵ converges strongly on the space $L^2(\Gamma)$ to $u \in \mathcal{U}_{ad}$, satisfying: $J(u) = \min\{J(v); v \in \mathcal{U}_{ad}\}$; where $J(v) := \int_\Gamma (y(v) - z_1)^2\, d\Gamma + \int_\Gamma (v - z_2)^2\, d\Gamma$ and $y(v)$ is the solution of the problem:*

$$(P_2)(v) \qquad \begin{cases} -\Delta y(v) = 0 & \text{on } \Omega \\ \dfrac{\partial}{\partial v} y(v) = v & \text{in } \Gamma = \partial\Omega \\ \displaystyle\int_\Gamma y(v)\, d\Gamma = 0; \quad y(v) \in H^1(\Omega) & \end{cases}$$

(ii) *The state y_ϵ converges strongly in the space $H^1(\Omega)$ to the state $y(u)$ solution of the system $(P_4)(u)$.*

The proof will be given in more general case in the second section (see theorem 2.2).

2. THE INFINITE DIMENSIONAL CASE

2.1. EXISTENCE OF THE PERTURBED STATE AND CONTROL FOR THE SYSTEM: $(P_1)(u_\epsilon)$ AND (P_2). The space of admissible controls $\mathcal{U}_{ad}$ will be an arbitrary closed linear subspace of $\mathcal{U}$ with infinite dimension. Then, for all $v \in \mathcal{U}_{ad}$, there exists a unique solution of the problem $(P_1)(v)$ denoted by $y_\epsilon(v)$ in the space V.

To prove the existence of the optimal control (theorem 2.1) we need the following proposition.

PROPOSITION 2.1. *The map $\mathcal{B}_\epsilon : \mathcal{U}_{ad} \longrightarrow L^2(\Gamma)$ which associates to each $v \in \mathcal{U}_{ad}$ the element $\mathcal{B}_\epsilon(v) := y_\epsilon(v)_{|\Gamma}$ is linear, bounded and injective map into $L^2(\Gamma)$. If the space $\mathcal{B}_\epsilon(\mathcal{U}_{ad})$ is closed in $L^2(\Gamma)$, then there exists a constant $C_\epsilon > 0$ (in fact, $C_\epsilon = (\|\mathcal{B}_\epsilon^{-1}\|)^{-1}$ where $\mathcal{B}_\epsilon^{-1}$ is the operator inverse of $\mathcal{B}_\epsilon$ defined on the range $\mathcal{B}_\epsilon(\mathcal{U}_{ad})$ such that:*

$$(2.1) \qquad C_\epsilon \|v\|_{L^2(\Gamma)} \leq \|y_\epsilon(v)\|_{L^2(\Gamma)} \quad \text{for all } v \in \mathcal{U}_{ad}.$$

For the proof see [2].

THEOREM 2.1. *Let $\mathcal{U}_{ad}$ a closed linear subspace of infinite dimension in the space $\mathcal{U}$. If the map $\mathcal{B}_\epsilon : \mathcal{U}_{ad} \longrightarrow L^2(\Gamma)$ is of closed range, then there exists a non vanishing subset X_ϵ of $\mathcal{U}_{ad}$, such that for all $u_\epsilon \in X_\epsilon$, we have:*

$$J_\epsilon(u_\epsilon) = \min\{J_\epsilon(v);\ v \in \mathcal{U}_{ad}\}$$

Remark 2.1. We have made the assumption on $\mathcal{B}_\epsilon$ (defined on $\mathcal{U}_{ad}$ to be of closed range, this is not always the case. But if $\mathcal{U}_{ad}$ is of finite dimension (for example) this is true, in [2] we will give another proof not needing this assumption. (cf. [2]).

2.2. STUDY OF THE CONVERGENCE OF THE STATE y_ϵ AND CONTROL u_ϵ. The main result of this section is the following theorem.

THEOREM 2.2. *We suppose that the map $\mathcal{B}_\epsilon : \mathcal{U}_{ad} \longrightarrow L^2(\Gamma)$ is of closed range. Then we have the following statements:*

(i) *The control u_ϵ converges weakly in the space $L^2(\Gamma)$ to $u \in \mathcal{U}_{ad}$, satisfying:*

$$J(u) = \min\{J(v); v \in \mathcal{U}_{ad}\},$$

where $J(v) = \int_\Gamma (y(v) - z_1)^2 d\Gamma + \int_\Gamma (v - z_2)^2 d\Gamma$ and $y(v)$ is the solution of the problem:

$$(P_2)(v) \qquad \begin{cases} -\Delta y(v) = 0 & \text{on } \Omega \\ \dfrac{\partial}{\partial v} y(v) = v & \text{in } \Gamma = \partial\Omega \\ \displaystyle\int_\Gamma y(v)\, d\Gamma = 0; \quad y(v) \in H^1(\Omega) \end{cases}$$

(ii) *The state y_ϵ converges strongly in the space $H^1(\Omega)$ to the state $y(u)$, solution of the system $(P_4)(u)$.*

Proof. As the control 0 is in the space $\mathcal{U}_{ad}$, we have:

$$J_\epsilon(u_\epsilon) \le J_\epsilon(0) = \|z_1\|^2_{L^2(\Gamma)} + \|z_2\|^2_{L^2(\Gamma)};$$

then there exists two constants $C_1 > 0$ and $C_2 > 0$ such that:

$$(2.2) \qquad \|y_\epsilon\|_{L^2(\Gamma)} \le C_1 \quad \text{and} \quad \|u_\epsilon\|_{L^2(\Gamma)} \le C_2$$

(y_ϵ denotes the solution of the problem $(P_1)(u_\epsilon)$). Then we can say that (u_ϵ) converges weakly in $L^2(\Gamma)$ to an element $u \in \mathcal{U}_{ad}$. Using the variational formulation of the problem $(P_1)(u_\epsilon)$, we deduce that there exists a constant $C_3 > 0$ such that: $\|y_\epsilon\|_{H^1(\Omega)} \le C_3$ independently of ϵ. Consequently the state y_ϵ converges weakly in the space $H^1(\Omega)$ to an element $y(u)$ (denoted by y) which is solution of the problem:

$$(P_3)(u) \qquad \begin{cases} -\Delta y = 0 & \text{on } \Omega \\ \dfrac{\partial}{\partial v} y = u & \text{in } \Gamma = \partial\Omega \\ \displaystyle\int_\Gamma y\, d\Gamma = 0; \quad y \in H^1(\Omega) \end{cases}$$

In order to prove the strong convergence of the state y_ϵ to y in $H^1(\Omega)$, it suffices to prove that $\|\nabla y - \nabla y_\epsilon\|_{L^2(\Omega)}$ converges to 0, when $\epsilon \to 0$. We have:

$$(2.3) \qquad \|\nabla y - \nabla y_\epsilon\|^2_{L^2(\Omega)} = \int_\Omega |\nabla y_\epsilon|^2\, dx - 2\int_\Omega \nabla y \nabla y_\epsilon\, dx + \int_\Omega |\nabla y|^2\, dx$$

We remark that an application of Green formula to the problem $(P_1)(u_\epsilon)$ gives:

$$\int_\Omega |\nabla y_\epsilon|^2\, dx = -\epsilon \int_\Gamma y_\epsilon^2\, d\Gamma + \epsilon \int_\Gamma y_\epsilon u_\epsilon\, d\Gamma.$$

Since $\|y_\epsilon\|_{H^1(\Omega)} \leq C_3$ and (u_ϵ) converges weakly in $L^2(\Gamma)$ to u, then by the trace theorem (cf. [9]), we can assert that: $\int_\Omega |\nabla y_\epsilon|^2\, dx$ converges to $\int_\Gamma yu\, d\Gamma$ when $\epsilon \to 0$.

Consequently, $\|\nabla y - \nabla y_\epsilon\|^2_{L^2(\Omega)}$ converges (when $\epsilon \to 0$) to

$$\int_\Gamma yu\, d\Gamma - \int_\Omega |\nabla y|^2\, dx,$$

and this quantity vanishes because y is solution of $(P_4)(u)$.

Again by the trace theorem and the continuity of the norm in $L^2(\Gamma)$, we obtain that for all $v \in \mathcal{U}_{ad}$

$$J(u) = \lim_{\epsilon \to 0} J_\epsilon(u_\epsilon) \leq \lim_{\epsilon \to 0} J_\epsilon(v) = J(v). \tag{2.4}$$

This completes the proof. ∎

Remark 2.2. Let $\{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ be an orthonormal basis of the space $L^2(\Gamma)$. If the space $\mathcal{U}_{ad}$ is included in ℓ^1 (i.e. the space of $u = \sum u_i \phi_i$ with $\sum |u_i| < \infty$) then (cf. [5]) the control u_ϵ converges strongly to u in $\mathcal{U}_{ad}$. We left open the problem to prove the strong convergence of optimal control (u_ϵ) in general case: this is made in [2].

3. Conclusion

We have established the existence of the state and the control for the perturbed boundary optimal control system: $(P_1)(u_\epsilon)$ and (P_2) (under the assumption that $\mathcal{B}_\epsilon(\mathcal{U}_{ad})$ is closed) for a functional cost J_ϵ which is not strictly convex and defined on the boundary. We have considered $\mathcal{U}_{ad}$ the space of admissible controls as a linear subspace of $\mathcal{U}$. Then one can replace (in all former statements) $\mathcal{U}_{ad}$ by $\mathcal{W} + \mathcal{U}_{ad}$ where $\mathcal{W}$ is a closed and bounded convex set in $\mathcal{U}$. It is interesting to look for other convex sets of admissible controls for which our techniques work.

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