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The Bidual of the Space of Polynomials on a Banach Space

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Questions concerning extensions of polynomials or analytic functions from a Banach space E to its bidual E'', and others about reflexivity and related properties on spaces of homogeneous polynomials $P^k(E)$, have recently spurred interest regarding the structure of the bidual of a space of polynomials [4], [11], [18]. In this paper we study the relationship between the bidual of $P^k(E)$ and the space of polynomials over E''. Just as in the one-dual case the Borel transform [14] provides a map relating $P^k(E)'$ with $P^k(E')$, we define a map β through which elements of the bidual of $P^k(E)$ may be viewed as polynomials over E'', and study this map to obtain information about $P^k(E)''$. Our definition and our subsequent study of β require a presentation of the space of polynomials over a Banach space as the dual of a space spanned by certain evaluation mappings. It is well-known that, for a Banach space E, the space $P^k(E)$ of all continuous k-homogeneous polynomials on E is a dual space (see [17], [16]). Here we develop a short way of seeing this, which will be useful in the sequel. Recall that if E and F are complex Banach spaces, a map P from E into F is said to be a continuous k-homogeneous polynomial if there exists a continuous k-linear map $A : E \times \cdots \times E \to F$ such that $P(x) = A(x, ..., x)$.

For a complex Banach space E, fix $k \in N$ and consider the map

$$\delta : E \longrightarrow P^k(E)'$$

defined by $\delta(x) = e_x$, where $e_x : P^k(E) \to \mathbb{C}$ is the evaluation at x given by $e_x(P) = P(x)$. Note that $\|e_x\| = \|x\|^k$. It is easy to check that δ is a continuous k-homogeneous polynomial, whose associated k-linear map is $\Delta(x_1, ..., x_k)(P) = A(x_1, ..., x_k)$, where A is the (unique) symmetric k-linear map corresponding to P.

Let S_k be the (not necessarily closed) linear subspace spanned by $\{e_x : x \in E\}$ in $P^k(E)'$. Each $s \in S_k$ admits a (non-unique) representation as $s = \sum_{j=1}^n e_{x_j}$, with $x_1, ..., x_n$ in E. We define the transpose of δ,

$$b : S_k' \longrightarrow P^k(E)$$

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by $b(T) = T \circ \delta$. We obtain that b is an isometric isomorphism from S_k' onto $P^k(E)$. Thus, we have the following

THEOREM 1. *With the preceding notations, the dual of S_k is isometrically isomorphic to the space $P^k(E)$.*

Also we can introduce the norm on S_k, given by $\|s\|_\pi = \inf\{\sum_{j=1}^n \|x_j\|^k\}$, where the infimum is taken over all the representations $s = \sum_{j=1}^n e_{x_j}$. It can be checked that $\|s\|_\pi = \|s\|$ for every $s \in S_k$. This shows that S_k can be identified with $\otimes_{k,s,\pi} E$, the k-fold symmetric tensor product of E, endowed with the projective norm (see [17], [16]), via the correspondence $e_x \longleftrightarrow x \otimes \cdots \otimes x$.

In [3], Aron and Berner found a way of extending any $\mathbb{C}$-valued k-homogeneous polynomial P defined on a Banach space E to a polynomial $\overline{P}$ on the bidual E'' (see also [19]). This provides a linear extension map

$$(AB) : P^k(E) \longrightarrow P^k(E'')$$

between the spaces of k-homogeneous polynomials on E and E'', which is given by $(AB)(P) = \overline{P}$. This map is continuous, and in fact it has been proved by Davie and Gamelin [8] that $\|\overline{P}\| = \|P\|$, with the usual norm on spaces of polynomials.

It should be mentioned here that, given any $u \in \mathrm{End}(E''|E)$ (endomorphims of E'' leaving fixed all points of E), one may modify the construction of (AB) obtaining a different extension map. The Aron-Berner map (AB) corresponds to $u =$*identity*.

As usual, we denote by $P_f^k(E)$ the linear subspace of $P^k(E)$ spanned by $\{\phi^n : \phi \in E'\}$ and $P_c^k(E)$ its completion in $P^k(E)$. A polynomial P in $P_f^k(E)$ (respectively, in $P_c^k(E)$) will be called of finite type (respectively, compact type). Note that (AB) maps polynomials of finite (compact) type into polynomials of finite (compact) type.

In [6], Aron and Schottenloher proved that $P^{k-1}(E)$ is isomorphic to a complemented subspace of $P^k(E)$. This also gives that $P_c^{k-1}(E)$ is isomorphic to a complemented subspace of $P_c^k(E)$. Using the fact that the Aron-Berner map respects this decomposition, we obtain the following result:

PROPOSITION 1. (1) *$P^k(E)$ is reflexive if and only if (AB) is weakly compact.* (2) *$P_c^k(E)$ is reflexive if and only if the restriction of (AB) to $P_c^k(E)$ is weakly compact.*

Since $P^k(E)$ can be considered as a subspace of $P^k(E'')$ via the Aron-Berner map, for $z \in E''$ the evaluation maps $e_z \in P^k(E'')'$ are also elements of $P^k(E)'$. The map β may be defined by transposing the polynomial map $E'' \longrightarrow P^k(E)'$ taking z into e_z. That is,

DEFINITION. The map $\beta : P^k(E)'' \longrightarrow P^k(E'')$ is given by $\beta(A)(z) = A(e_z)$.

Note that restricting β to the canonical inclusion of $P^k(E)$ in $P^k(E)''$ we obtain the Aron-Berner map (AB). Thus β is an extension of (AB). Also, β is a continuous linear operator of norm one.

In order to discuss other presentations of β, we will need some notation. As before, let S_k be the (non-closed) linear space spanned by the evaluations $\{e_z : z \in E''\}$. We denote the norm on $P^k(E)'$ by $|||\ |||$, and that on $P^k(E'')'$ by $\|\ \|$. Thus for any $s \in S_k$, $|||s||| \leq \|s\|$. We then have

$$(S_k, \|\ \|) \longrightarrow (S_k, |||\ |||) \hookrightarrow P^k(E)'.$$

Since the space on the left is a predual of $P^k(E'')$, by transposing we have

$$\beta : P^k(E)'' \xrightarrow{id} (S_k, |||\ |||)' \longrightarrow P^k(E'').$$

This is the definition of β used by Aron and Dineen in [4], in the case where polynomials over E are weakly continuous on bounded sets. Another presentation of β is the following. Denote by π the transpose of the canonical injection of $(S_k, \|\ \|)$ into $(S_k, \|\ \|)''$. Then by composing the maps

$$P^k(E)'' \xrightarrow{(AB)''} P^k(E'')'' \xrightarrow{\pi} P^k(E'')$$

one obtains the same map as above, that is, $\beta = \pi \circ (AB)''$. Note that if E is reflexive, then $\beta : P^k(E)'' \longrightarrow P^k(E)$ is the canonical projection from a third dual space onto the first dual. For non-reflexive E, since $P^k(E)$ is (through (AB)) a proper subspace of $P^k(E'')$, a non-zero linear form over the second space may be null on the first. Thus some caution might be in order when using phrases like "$s = 0$", "$s_1, \dots, s_n$ are linearly independent", or "the linear space spanned by e_z, $z \in E''$". Actually, when speaking of elements of S_k, it will follow from Proposition 2 that there is no ambiguity in such phrases.

PROPOSITION 2. *Let $z_1, \dots, z_k \in E''$ be given. If $\sum_{j=1}^n \overline{P}(z_j) = 0$ for all $P \in P_f^k(E)$, then $\sum_{j=1}^n Q(z_j) = 0$ for all $Q \in P^k(E'')$.*

In the proof of proposition above, we use the following lemma, which we believe has interest in itself and give in slightly greater generality.

LEMMA. *For $i = 0, \dots, n$, let $(\gamma_i, \varphi_i, \dots, \psi_i)$ be k-tuples of continuous linear functionals on X. If $\gamma_0\varphi_0 \cdots \psi_0 = \sum_{i=1}^n \gamma_i\varphi_i \cdots \psi_i$ then for any $\phi \in L_s^k(X')$*

$$\phi(\gamma_0, \varphi_0, \dots, \psi_0) = \sum_{i=1}^n \phi(\gamma_i, \varphi_i, \dots, \psi_i).$$

Now, we consider the initial topology on $P^k(E'')$ induced by the evaluations $\{e_z : z \in E''\}$ and call this the S_k-topology. Thus, a net (Q_i) of elements of $P^k(E'')$ is S_k-convergent to Q if for each $s = \sum_{j=1}^n e_{z_j} \in S_k$,

$$s(Q_i) = \sum_{j=1}^n Q_i(z_j) \quad \text{converges to} \quad \sum_{j=1}^n Q(z_j) = s(Q) \quad \text{with } i.$$

Remark 1. (1) By mimicry of the classical proof of the fact that the dual of (X', w^*) is X one obtains that any S_k-continuous linear functional on $P^k(E'')$ can be identified with an element of S_k, in other words, we have $(P^k(E''), S_k)' = S_k$.

(2) The S_k-topology is weaker than the w^*-topology $\sigma(P^k(E''), \overline{S_k})$, where $\overline{S_k}$ denotes the closure of S_k in $P^k(E'')'$. Nevertheless, it is easy to see that the S_k-topology coincides with the w^*-topology on bounded subsets of $P^k(E'')$.

The following theorem characterises the image of β in terms of the S_k-topology. Here B_P denotes the closed unit ball of $P^k(E)$ and B_Q will denote the closed unit ball of $P^k(E'')$.

THEOREM 2. *Let $Q \in P^k(E'')$. Then the following are equivalent:*

i) *$Q \in Im\beta$*

ii) *Q is the w^*-limit of a bounded net $(P_i) \subset P^k(E)$.*

iii) *Q is the S_k-limit of a bounded net $(P_i) \subset P^k(E)$.*

iv) *For some $c > 0$ and all $z_1, \dots, z_n \in E''$*

$$\left|\sum_{j=1}^{n} Q(z_j)\right| \leq c \sup_{P \in B_P} \left|\sum_{j=1}^{n} \overline{P}(z_j)\right|.$$

The constant c in the theorem depends on Q. In fact, $Q = \beta(A)$ for some A with $\|A\| \leq c$ if and only if Q is S_k-adherent to cB_P, if and only if c satisfies the inequality in iv). However, if β is surjective the same constant is good for all $Q \in B_Q$. Thus, the surjectivity of β may be expressed in any of the following equivalent ways.

COROLLARY 1. *The following are equivalent:*

i) *β is surjective.*

ii) *There is a constant C such that CB_P is w^*-dense in B_Q.*

iii) *There is a constant C such that for all $Q \in B_Q$ and $z_1, \dots, z_k \in E''$,*

$$\left|\sum_{j=1}^{n} Q(z_j)\right| \leq C \sup_{P \in B_P} \left|\sum_{j=1}^{n} \overline{P}(z_j)\right|.$$

iv) *$\overline{S_k} \cap Ker(AB)' = 0$. Here $(AB)'$ denotes the transpose of the Aron-Berner map.*

Note that condition ii) of the corollary is a Goldstine-type theorem for the inclusion $(AB) : P^k(E) \hookrightarrow P^k(E'')$. Condition iii) says that the $\|| \ \||$-norm and the $\| \ \|$-norm are equivalent over S_k. We prove below that a similar but weaker condition holds for any Banach space: $P^k(E)$ is S_k-dense in $P^k(E'')$.

THEOREM 3. *The following conditions hold for any Banach space E.*

(1) $P_f^k(E)$ *is* S_k*-dense in* $P^k(E'')$.

(2) $S_k \cap Ker(AB)' = 0$

We know of no case where the map β fails to be surjective. Next, using some ideas of [7], we give a criterion for the surjectivity of β. Recall that a Banach space X has the λ-approximation property if there is a net (T_i) of finite-rank operators on X such that $\|T_i\| \leq \lambda$ and $(T_i(x))$ is convergent to x for all $x \in X$.

COROLLARY 2. *Suppose that E'' has the λ-approximation property. Then $\lambda^k B_P$ is S_k-dense in B_Q and, therefore, β is surjective.*

Another positive result on the image of β is the following. Recall that a k-homogeneous polynomial P on a Banach space X is said to be nuclear if there exist a sequence $(\lambda_i) \in \ell^1$ and a bounded sequence $(\gamma_i) \subset X'$ such that $P(x) = \sum_{i\geq 1} \lambda_i \gamma_i^k(x)$ for all $x \in X$.

PROPOSITION 3. $P_N^k(E'') \subset \operatorname{Im}\beta$.

Recall that $\beta(A) = 0$ if and only if $A(e_z) = 0$ for all $z \in E''$. Thus $\operatorname{Ker}\beta = S_k^\perp$, and the injectivity of β is equivalent to the density of S_k in $P^k(E)'$. This condition is related to reflexivity of the space $P^k(E)$, as we see in the next proposition.

PROPOSITION 4. $P^k(E)$ *is reflexive if and only if E is reflexive and β is injective.*

The nuclear norm of $P \in P_f^k(E')$ is defined by $\|P\|_N = \inf\{\sum_{j=1}^\infty \|z_j\|^k\}$, where the infimum is taken over all possible representations $P = \sum_{j=1}^\infty z_j^k$, with $z_j \in E''$ and $\sum_{j=1}^\infty \|z_j\|^k < \infty$ (see [14], [9]). Since $P_N^k(E')$ is the completion of $P_f^k(E')$ under the norm $\|\cdot\|_N$, there exists a quotient map $H : \overline{S_k} \longrightarrow P_N^k(E')$ such that $H(e_z) = z^k$, for all $z \in E''$.

Following Dineen [9], we will say that $P \in P^k(E')$ is an integral polynomial if there exists a regular countably additive Borel measure of bounded variation μ on the compact set $(B_{E''}, w^*)$ such that

$$P(\gamma) = \int_{B_{E''}} z(\gamma)^k d\mu(z) \quad \text{for all } \gamma \in E'.$$

In this case, the integral norm of P is defined by $\|P\|_I = \inf\{\|\mu\|\}$, where the infimum is taken over all measures μ satisfying the definition. The space of integral polynomials is denoted by $P_I^k(E')$ and it follows from [9] that there is an isometric isomorphism $D : P_I^k(E') \longrightarrow P_c^k(E)'$, such that $D(P_z) = e_z$ for all $z \in E''$, where $P_z = z^k$ is the integral polynomial associated to the Dirac measure of z. We have that $P_N^k(E') \subset P_I^k(E')$ and $\|P\|_I \leq \|P\|_N$ for all $P \in P_N^k(E')$.

Thus, we obtain the map

$$\overline{S_k} \xrightarrow{H} P_N^k(E') \hookrightarrow P_I^k(E') \xrightarrow{D} P_c^k(E)'.$$

It is clear that $D \circ H$ is a bijection if and only if $P_N^k(E') = P_I^k(E')$ (e.g., when E'' has the Radon-Nikodým property [1]). In this case, $D \circ H$ is an isomorphism. For spaces E where every polynomial is weakly continuous on bounded sets, the map $D \circ H$ coincides with the map J_k defined in [4]. Consider the transpose of $D \circ H$, which we denote β_c, and the inclusion $i : P_c^k(E) \hookrightarrow P^k(E)$, with bitranspose i''. Then, we obtain that $\beta_c = \beta \circ i''$ and, using this, we arrive at the following:

THEOREM 4. *Suppose that E'' has the Radon-Nikodým property and the approximation property. Then*

(1) $\beta_c : P_c^k(E)'' \to P^k(E'')$ *is an isomorphism.*

(2) $\beta : P^k(E)'' \to P^k(E'')$ *is surjective.*

(3) $\beta : P^k(E)'' \to P^k(E'')$ *is injective if and only if $P_c^k(E) = P^k(E)$ (or, equivalently, every $P \in P^k(E)$ is weakly sequentially continuous). In this case, β is an isomorphism.*

Taking into account that reflexive Banach spaces verify the Radon-Nikodým property, some consequences of the preceding theorem can be formulated for this class of spaces.

COROLLARY 3. *For a Banach space E with the approximation property, the following conditions are equivalent, and imply that β is an isomorphism.*

i) $P^k(E)$ *is reflexive*

ii) $P_c^k(E)$ *is reflexive*

iii) E *is reflexive and* $P^k(E) = P_c^k(E)$

iv) E *is reflexive and every $P \in P^k(E)$ is weakly sequentially continuous.*

In what follows, we denote $\beta^k : P^k(E)'' \longrightarrow P^k(E'')$ the map β corresponding to the spaces of k-homogeneous polynomials.

Corollary 3 can be applied, for instance, to $E = \ell^p$, $1 < p < \infty$; here, we obtain that β^k is an isomorphism (in fact, the identity) if $k < p$, while β^k is not injective if $k \geq p$. This situation is typical, as we see in the proposition below.

PROPOSITION 5. *For a Banach space E, if $\beta^k : P^k(E)'' \longrightarrow P^k(E'')$ is an isomorphism, then for every $j < k$ the map $\beta^j : P^j(E)'' \longrightarrow P^j(E'')$ is an isomorphism.*

Note that if E has a quotient isomorphic to ℓ^p, and E'' has the Radon-Nikodým property and the approximation property it can be seen as in [15] that, for $k \geq p$, there exists on E a k-homogeneous polynomial that is not weakly sequentially continuous and, therefore, β^k is not injective for $k \geq p$. This can be applied to $E = L^p[0,1]$: since $L^p[0,1]$ $(1 < p < \infty)$ contains a complemented copy of ℓ^2, we obtain that β^k is not injective for $k \geq 2$.

Examples of Banach spaces satisfying the conditions in Corollary 3 for all $k \in \mathbb{N}$ are the original Tsirelson space, T^*, and any quotient of T^* having the approximation property (see [2], [5]). An example of non-reflexive Banach space for which $\beta^k : P^k(E) \longrightarrow P^k(E'')$ is an isomorphism for every k, is the Tsirelson*-James space $E = T_J^*$, constructed by Aron-Dineen in [4].

There is a close connection between weak sequential continuity of polynomials and the existence of upper and lower ℓ^p-estimates of sequences, as can be seen in [10], [4], [12] and [13]. This gives criteria for the map β^k to be an isomorphism. We have, for instance, the following.

Remark 2. Suppose that E'' has the Radon-Nikodým property and the approximation property. Then
(1) If no weakly null normalised sequence in E admits a lower ℓ^p-estimate, then β^k is an isomorphism for all $k < p$.
(2) If every weakly null sequence in E' has a subsequence with an upper $\ell^{p'}$-estimate and $\frac{1}{p'} + \frac{1}{p} = 1$, then β^k is not injective for $k \geq p$.
(3) If E' has type p' and $\frac{1}{p'} + \frac{1}{p} = 1$, then β^k is not injective for $k > p$.

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