EXTRACTA MATHEMATICAE Vol. 9, Núm. 1, 28 – 31 (1994)

Compact Cosymplectic Manifolds of Positive Constant φ–Sectional Curvature [1]

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AMS *Subject Class.* (1991): 53C15, 53C55, 57R30 *Received March* 2, 1994

It is well known that the curvature properties of a compact orientable Riemannian manifold affect its topological structure ([7]). If the Riemannian manifold is endowed with an extra geometrical structure (Kähler or cosymplectic) we can define a special type of sectional curvature and derive new topological properties.

Let V be an almost Hermitian manifold with a metric h and almost complex structure J. Denote by $\mathfrak{X}(V)$ the Lie algebra of vector fields on V. The Kähler 2–form Ω is defined by $\Omega(X,Y)=h(X,JY)$ for $X,Y\in\mathfrak{X}(V)$. An almost Hermitian manifold (V,J,h) is said to be Kähler if $[J,J]=0$ and $d\Omega=0$.

The sectional curvature of the J–invariant planes is called holomorphic sectional curvature (see [9]). For any positive number k, the complex projective space $P_m(\mathbb{C}^{m+1})$ carries a complete Kähler metric of constant holomorphic sectional curvature k [9]. We denote by $P_m(\mathbb{C}^{m+1})(k)$ the Kähler manifold with this structure.

A map F between the almost Hermitian manifolds (V,J,h) and (V',J',h') is said to be a holomorphic isometry if F is an isometry which verifies $F_*\circ J = J'\circ F_*$.

For compact Kähler manifolds we have the two following results (see [4,5,8]):

THEOREM A. *A compact Kähler manifold with positive definite Ricci tensor is simply connected.*

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[1] Supported by the "Consejería de Educación del Gobierno de Canarias" and DGICYT–SPAIN, Proyecto PB91–0142.

THEOREM B. *A compact Kähler manifold with positive constant holomorphic sectional curvature is holomorphically isometric to a complex projective space of positive constant holomorphic sectional curvature.*

The odd-dimensional counterpart of Kähler manifolds are cosymplectic manifolds. Let (M,φ,ξ,η,g) be an almost contact metric manifold. Then, we have

(1) $$\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi)=1, \quad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all $X,Y\in\mathfrak{X}(M)$, I being the identity transformation. The fundamental 2-form Φ of M is defined by $\Phi(X,Y)=g(X,\varphi Y)$, for $X,Y\in\mathfrak{X}(M)$. An almost contact metric manifold (M,φ,ξ,η,g) is said to be cosymplectic if $[\varphi,\varphi]=0$ and $d\eta=0$, $d\Phi=0$ [1].

On an almost contact metric manifold (M,φ,ξ,η,g) we denote by $\mathfrak{F}$ the foliation defined by the vector field ξ and by $\mathfrak{F}^\perp$ the distribution determined by the normal bundle of $\mathfrak{F}$, i.e., $\mathfrak{F}^\perp$ is the distribution given by $\eta=0$.

A mapping F between the almost contact metric manifolds (M,φ,ξ,η,g) and $(M',\varphi',\xi',\eta',g')$ is said to be an almost contact isometry if F is an isometry which verifies $F_*\circ\varphi = \varphi'\circ F_*$ and $F^*\eta' = \eta$. The above conditions imply that $F_*\xi = \xi'$.

Let (M,φ,ξ,η,g) be an almost contact metric manifold and x a point of M. A plane section π in the tangent space to M at x, T_xM, is called a φ-section if there exists a unit vector u in T_xM orthogonal to ξ_x such that $\{u,\varphi_x u\}$ is an orthonormal basis of π. Then the sectional curvature $K_x u = R_x(u,\varphi_x u,u,\varphi_x u)$ is called a φ-sectional curvature.

Denote by $T\mathfrak{F}^\perp$ the vector subbundle of the tangent bundle of M which consists of the tangent vectors to the distribution $\mathfrak{F}^\perp$ and, by $T_x\mathfrak{F}^\perp$ the fiber of $T\mathfrak{F}^\perp$ over x, for a point x of M. Let S be the Ricci curvature tensor of M. Then S is said to be transversally positive definite if S_x is positive definite on the subspace $T_x\mathfrak{F}^\perp$ for all $x\in M$.

On a cosymplectic manifold (M,φ,ξ,η,g) the vector ξ is parallel [1]. Thus, if S is the Ricci tensor of M then $S(\xi,\xi)=0$, which implies that S cannot be positive definite. On the other hand, from the results of [6], we deduce that a cosymplectic manifold of positive constant φ-sectional curvature has transversally positive definite Ricci tensor.

The canonical example of simply connected cosymplectic manifold is given by the product of a simply connected Kähler manifold with $\mathbb{R}$. In fact, a complete simply connected cosymplectic manifold is almost contact isometric to the product of a complete simply connected Kähler manifold with $\mathbb{R}$ [2].

The natural example of compact cosymplectic manifold is given by the product of a compact Kähler manifold (V,J,h) with the circle S^1. The cosympletic structure (φ,ξ,η,g) on the product manifold $M=V\times S^1$ is defined by

$$\varphi = J\circ(\mathrm{pr}_1)_*, \quad \xi=\frac{E}{c}, \quad \eta=c(\mathrm{pr}_2)^*(\theta), \quad g=(\mathrm{pr}_1)^*(h)+c^2(\mathrm{pr}_2)^*(\theta\otimes\theta), \tag{2}$$

where $\mathrm{pr}_1: M \longrightarrow V$ and $\mathrm{pr}_2: M \longrightarrow S^1$ are the projections of $V\times S^1$ onto the first and second factor respectively, θ is the length element of S^1, E is its dual vector field and c is a real number, $c\neq 0$ [2]. If the Kähler manifold (V,J,h) is of constant holomorphic sectional curvature k then (M,φ,ξ,η,g) is a cosymplectic manifold of constant φ-sectional curvature k. Thus, for all positive real k, the manifold $P_m(\mathbb{C}^{m+1})(k)\times S^1$ is a compact cosymplectic manifold of constant φ-sectional curvature k.

We remark that if $b_1(M)$ is the first Betti number of a compact cosymplectic manifold M then, since $b_1(M)\geqslant 1$ ([2,3]), we have that the fundamental group of M is infinite. Therefore, we conclude that a compact simply connected manifold cannot admit a cosymplectic structure. Moreover, there exists compact cosymplectic manifolds which are not topologically a global product of a Kähler manifold with the circle S^1 [3].

We need a modification of the notion of almost contact isometry. Let $F: M \longrightarrow M'$ be a diffeomorphism between two cosymplectic manifolds (M,φ,ξ,η,g) and $(M',\varphi',\xi',\eta',g')$, and let $\mathfrak{F}^\perp$ (respectively, $(\mathfrak{F}')^\perp$) be the foliation on M (respectively, M') given by $\eta=0$ (respectively, $\eta'=0$). F is said to be transversally holomorphic isometric if $F^*\eta'=\eta$ and for all $x\in M$ the mapping $F_{|\mathfrak{F}_x^\perp}: \mathfrak{F}_x^\perp \longrightarrow (\mathfrak{F}')_{F(x)}^\perp$ is a holomorphic isometry between the Kähler manifolds $\mathfrak{F}_x^\perp$ and $(\mathfrak{F}')_{F(x)}^\perp$, being $\mathfrak{F}_x^\perp$ (respectively, $(\mathfrak{F}')_{F(x)}^\perp$) the leaf of the foliation $\mathfrak{F}^\perp$ (respectively, $(\mathfrak{F}')^\perp$) over x (respectively, $F(x)$). It is clear that an almost contact isometry is transversally holomorphic isometric.

The following two results are the cosymplectic version of Theorems A and B:

THEOREM A'. *The fundamental group of a compact cosymplectic manifold with transversally positive definite Ricci tensor is isomorphic to* $\mathbb{Z}$.

THEOREM B'. *Let* (M,φ,ξ,η,g) *be a* $(2m+1)$*-dimensional compact cosymplectic manifold with positive constant* φ*-sectional curvature* k*. Then there exists a diffeomorphism* $F : M \longrightarrow P_m(\mathbb{C}^{m+1})(k)\times S^1$ *of* M *onto the product manifold* $P_m(\mathbb{C}^{m+1})(k)\times S^1$ *which is transversally holomorphic isometric.*

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