EXTRACTA MATHEMATICAE Vol. 8, Núm. 2-3, 102 – 105 (1993)

Two Mappings Associated with Jensen's Inequality

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AMS *Subject Class.* (1991): 26D15 *Received February* 5, 1993

In Theory of Inequalities, the famous Jensen's discrete inequality

$$f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right] \leqslant \frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i) \tag{1}$$

valid for every convex function $f: C \subset X \longrightarrow \mathbb{R}$ (C is a convex subset of a linear space X) and for every $x_i \in C$ and $p_i \geqslant 0$ $(i = 1, \ldots, n)$ with $P_n := \Sigma_{i=1}^{n} p_i > 0$, plays such an important role that many mathematicians have tried not only to establish (1) in a variety of ways, but also to find different extensions, refinements and counterparts; see [1 – 10] where further references are given.

For a given convex function $f: C \subset X \longrightarrow \mathbb{R}$ and for $x_i \in C$, $p_i \geqslant 0$ $(i = 1, \ldots, n)$ with $P_n > 0$ we consider the mappings:

$$H: [0,1] \longrightarrow \mathbb{R}\,,\ \ H(t) := \frac{1}{P_n}\Sigma_{i=1}^{n} p_i f\left[t x_i + (1-t)\frac{1}{P_n}\Sigma_{j=1}^{n} p_j x_j\right]$$

and

$$F: [0,1] \longrightarrow \mathbb{R}\,,\ \ F(t) := (1/P_n^2)\ \Sigma_{i,j=1}^{n} p_i p_j f\big(t x_i + (1-t) x_j\big)\,.$$

The following theorem contains the main properties of these mappings.

THEOREM. *If $f: C \subset X \longrightarrow \mathbb{R}$ and x_i, p_i $(i = 1, \ldots, n)$ are as above, then*

(i) *H and F are convex on $[0,1]$.*

(ii) *We have the bounds:*

$$\inf_{t \in [0,1]} H(t) = H(0) = f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right] \tag{2}$$

$$\inf_{t \in [0,1]} F(t) = F(1/2) = (1/P_n^2)\ \Sigma_{i,j=1}^{n} p_i p_j f\big((x_i + x_j)/2\big) \tag{3}$$

$$\sup_{t \in [0,1]} H(t) = H(1) = \frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i) \tag{4}$$

$$\sup_{t \in [0,1]} F(t) = F(0) = F(1) = \frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i)\,. \tag{5}$$

(iii) *One has*

$$F(s+1/2)=F(1/2-s) \quad \textit{for all } s\in[0,1/2]\,.$$

(iv) *H is nondecreasing on* $[0,1]$, *F is nonincreasing on* $[0,1/2]$ *and nondecreasing on* $[1/2,1]$.

(v) *One has the inequalities*

$$f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right] \leqslant F(1/2) \tag{6}$$

and

$$\max\{H(t),H(1-t)\} \leqslant F(t) \quad \textit{for all } s\in[0,1]\,. \tag{7}$$

Proof. (i) Is obvious by the convexity of f.

(ii) Firstly, we shall prove the following inequalities:

$$\begin{aligned} f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right] &\leqslant H(t) \leqslant t\frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i) + (1-t)\, f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right] \leqslant \\ &\leqslant \frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i) \quad \text{for all } t\in[0,1]\,. \end{aligned} \tag{8}$$

By Jensen's inequality, we have

$$H(t) \geqslant f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right].$$

Using the convexity of f, we have

$$H(t) \leqslant t\frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i) + (1-t)\, f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right].$$

The last inequality in (8) is obvious observing that the mapping

$$g(t) := t\frac{1}{P_n}\Sigma_{i=1}^{n} p_i f(x_i) + (1-t)\, f\left[\frac{1}{P_n}\Sigma_{i=1}^{n} p_i x_i\right]$$

is nondecreasing in $[0,1]$.

Now, the bounds (2) and (4) are obvious.

Since f is convex on C, hence:

$$f\big(tx_i+(1-t)x_j\big) \leqslant tf(x_i)+(1-t)f(x_j)$$

for all $i,j\in\{1,\dots,n\}$ and $t\in[0,1]$. By multiplying with $p_ip_j \geqslant 0$ and summing over i and j to 1 at n, we get

$$\Sigma_{i,j=1}^{n} f\big(tx_i+(1-t)x_j\big)p_ip_j \leqslant P_n\,\Sigma_{i=1}^{n} p_i f(x_i)$$

which shows that $F(t) \leqslant F(0)=F(1)$ for all $t\in[0,1]$.

On the other hand, by the convexity of f, we also have

$$\frac{1}{2}\left[f(tx_i + (1-t)x_j) + f(tx_j + (1-t)x_i)\right] \geqslant f\big((x_i + x_j)/2\big)$$

for all $i,j \in \{1,\dots,n\}$ and $t \in [0,1]$. By multiplying this inequality with $p_i p_j \geqslant 0$ and summing over i and j to 1 at n, we derive

$$F(1/2) \leqslant F(t) \quad \text{for all} \quad t \in [0,1],$$

which proves the bounds (3) and (5).

(iii) Is obvious.

(iv) By the convexity of H and by (2), we have

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} \geqslant \frac{H(t_1) - H(0)}{t_1} \geqslant 0 \qquad \text{for } 0 < t_1 < t_2 < 1\,,$$

i.e., H is nondecreasing on $(0,1)$ and by (ii) also in $[0,1]$.

By the convexity of F and by (3), we have

$$\frac{F(t_2) - F(t_1)}{t_2 - t_1} \geqslant \frac{2\big(F(t_1) - F(1/2)\big)}{2t_1 - 1} \geqslant 0 \qquad \text{for } 1/2 < t_1 < t_2 < 1\,,$$

i.e., F is nondecreasing on $(1/2,1)$ and by (ii) also in $[1/2,1]$.

The fact that F is nonincreasing on $[0,1/2]$ follows by (iii).

(v) The inequality (6) follows by Jensen's inequality for double sums.

Now, let observe that

$$H(t) := \frac{1}{P_n}\Sigma_{i=1}^{n} p_i f\left[\frac{1}{P_n}\Sigma_{j=1}^{n}\big(tx_i + (1-t)x_j\big)p_j\right] \leqslant$$

$$\leqslant (1/P_n^2)\ \Sigma_{i,j=1}^{n} p_i p_j f\big(tx_i + (1-t)x_j\big) = F(t)$$

for all $t \in [0,1]$.

Since $H(1-t) \leqslant F(1-t) = F(t)$ for all $t \in [0,1]$, the statement (7) is proved. ∎

Remark. If we choose in the above inequalities: $f(x) = -\ln x$, $x > 0$; $f(x) = \|x\|^p$, $p \geqslant 1$, $x \in X$ ($(X, \|\ \|)$ is a normed space) or $f(x) = -\ln[x/(1-x)]$, $x \in (0,1/2]$ we can obtain some interesting results connected with arithmetic mean – geometric mean inequality, with generalized triangle inequality [2] and with Ky Fan's inequality [5], respectively.

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