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Shape Preserving Rational Cubic Interpolation

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0. INTRODUCTION

A rational cubic function, to produce a spline with shape control, was described and anlysed in [17]. It provides a C^2 computationally simpler alternative to the exponential spline–under–tension [1, 10, 13, 18]. Regarding shape characteristics, it has two shape control parameters associated with each interval which can be used to change the shape of the curve, both locally and globally, in a variety of ways. This rational spline scheme and other similar schemes like GC^2 Nielson's scheme [4], in general, do not preserve the shape of the given data. To achieve the shape preservation characteristics, the user needed to play with the shape parameters on distinct areas of the curves which, of course, is not a convenient job. In the last decade the problem of shape preserving interpolation has been considered by a number of authors. For example, monotonicity and/or convexity preserving scalar and parametric curves have been discussed in [2, 3, 5, 8, 9, 11, 12]. Their authors use rational quadratic/cubic/ rational cubic functions in the theory. For brevity, the reader is being refered to [6, 7, 14, 15, 16] for some other shape preserving schemes.

Being an important problem of the Scientists, it is useful to use the C^1 interpolation in [17] and develop a scheme so that the shape of the data is preserved. This paper is meant to give the parametric description of the shape preserving curves and begins with some preliminaries about the rational cubic interpolant. The shape parameters in the description of the interpolant are utilized and bounds are constructed on them, to produce pleasing graphical results, in Section 2. These bounds are dependent on the given data. The description of the tangent vectors, which are consistent and dependent on the given data, is also made in Section 2.

1. C^1 Piecewise Rational Cubic Hermite Interpolant

A piecewise rational cubic Hermite parametric function $P \in C^1[t_0, t_n]$, with parameters v_i, w_i, $i = 0,\dots,n-1$, is defined for $t \in [t_i, t_{i+1}]$, $i = 0,\dots,n-1$, by

$$P(t) = P_i(t; v_i, w_i) =$$

$$= \frac{(1-\theta)^3 F_{i+1} + \theta(1-\theta)^2(v_i F_i + h_i D_i) + \theta^2(1-\theta)(w_i F_{i+1} - h_i D_{i+1}) + \theta^3 F_i}{(1-\theta)^3 + v_i \theta(1-\theta)^2 + w_i \theta^2(1-\theta) + \theta^3} \quad (1)$$

where the notations F_i and $D_i \in \mathbb{R}^N$ are, respectively, the data values and the first derivative values at the knots t_i, $i = 0,\dots,n$ with $t_0 < t_1 < ,\dots, < t_n$, $h_i = t_i - t_{i-1}$, $\theta = (t - t_{i-1})/h_i$ and $v_i, w_i \geqslant 0$.

The function $P(t)$ has the Hermite interpolation properties that

$$P(t_i) = F_i \quad \text{and} \quad P^{(1)}(t_i) = D_i, \quad i = 0,\dots,n. \quad (2)$$

The v_i and w_i, $i = 0,\dots,n-1$, will be used so that the shape of the data is preserved. The case $v_i = w_i = 3$, $i = 0,\dots,n-1$, is that of cubic Hermite interpolation and the restriction $v_i, w_i \geqslant 0$ ensures a positive denominator in (1).

For $v_i, w_i \neq 0$, (1) can be written in the form

$$P_i(t; v_i, w_i) =$$

$$= R_0(\theta; v_i, w_i) F_{i-1} + R_1(\theta; v_i, w_i) V_i + R_2(\theta; v_i, w_i) W_i + R_3(\theta; v_i, w_i) F_i, \quad (3)$$

where

$$V_i = F_{i-1} + h_i D_{i-1}/v_i, \quad W_i = F_i - h_i D_i / w_i, \quad (4)$$

and $R_j(\theta; v_i, w_i)$, $j = 0, 1, 2, 3$, are appropriately defined rational functions with

$$\textstyle\sum_{j=0}^{3} R_j(\theta; v_i, w_i) = 1. \quad (5)$$

Moreover, these functions are rational Bernstein–Bézier weight functions which are non–negative for $v_i, w_i > 0$. Thus in $\mathbb{R}^N$, $N > 1$ and for $v_i, w_i > 0$, we have:

PROPOSITION 1. (*Convex hull property*) *The curve segment P_i lies in the convex hull of the control points $\{F_{i-1}, V_i, W_i, F_i\}$.*

It is well known that rational Bézier curves enjoy the variation diminishing property:

PROPOSITION 2. (*Variation diminishing property*) *The curve segment P_i crosses any (hyper) plane of dimension $N-1$ no more times than it crosses the* *control polygon joining* F_{i-1}, V_i, W_i, F_i.

A proof of Proposition 2 can be found in [17].

2. DERIVATIVE APPROXIMATIONS AND THE SHAPE CONSTRAINTS

In order to construct C^1 shape preserving rational cubic interpolation curve $P(t)$ dfined by (3) the first thing is to determine the tangent vectors D_i, $i = 0,\ldots,n$, at interpolating points, then constraints on the shape parameters will be imposed appropriately. Let

$$\left.\begin{aligned} F_i &= (x_i\ , y_i\ ) \\ D_i &= (D_i^x\ , D_i^y\ ) \\ \Delta_i &= (\Delta_i^x\ , \Delta_i^y\ ) \end{aligned}\right\} , \tag{6}$$

where

$$\Delta_i{}^x = (x_i - x_{i-1})/h_i\ ,\ \ \Delta_i{}^y = (y_i - y_{i-1})/h_i\ . \tag{7}$$

Let

$$\left.\begin{aligned} \beta_{1,i} &= \Delta_i^x\,\Delta_{i+1}^y - \Delta_i^y\,\Delta_{i+1}^x \\ \beta_{2,i} &= D_i^x\,\Delta_i^y - D_i^y\,\Delta_i^x \\ \beta_{3,i} &= \Delta_i^x\,D_{i+1}^y - \Delta_i^y\,D_{i+1}^x \\ \beta_{4,i} &= D_i^x\,D_{i+1}^y - D_i^y\,D_{i+1}^x \end{aligned}\right\} . \tag{8}$$

Let $\Delta_i = (F_i - F_{i-1})/h_i$, $i = 1,\ldots,n$, then the tangent vectors D_i*'s* at t_i*'s* will be defined as:

$$D_i = \alpha_i \Delta_i + \beta_i \Delta_{i+1}\ ,\ \ \alpha_i, \beta_i > 0\ ,\ \ i = 1,\ldots,n-1\ , \tag{9}$$

where $\alpha_i = 0$ if and only if F_i, F_{i+1}, F_{i+2} are collinear and $\beta_i = 0$ if and only if F_{i-2}, F_{i-1}, F_i are collinear. The α_i*'s* and β_i*'s* are defined as follows:

For an open curve define

$$\alpha_i = |\beta_{1,i+2}| \big/ (|\beta_{1,i}| + |\beta_{1,i+2}|) \ \text{ and } \ \beta_i = 1 - \alpha_i\ ,\ \ i = 2,\ldots,n-2\ . \tag{10}$$

The tangent vectors D_0, D_1, D_{n-1} and D_n can by determined by the formulae given in [6].

For a closed curve, F_i*'s* are considered cyclic, i.e., $F_{i+n} = F_i$, for all i. Thus formulae (10) are extended for $i = 0,\ldots,n$.

Now we come to the constraints on the shape parameters. It was shown in [17] that these shape parameters control the curve in different ways provided the derivatives are bounded. Our problem is now to seek for appropriate minimal bounds on v_i*'s* and w_i*'s* so that the resultant curve preserves the shape of the data.

If the control polygon joining $F_{i-2}, F_{i-1}, F_i, F_{i+1}$ is convex, then so is the curve segment (1). In case the control polygon joining $F_{i-2}, F_{i-1}, F_i, F_{i+1}$ is not convex, then the curve segment (1) will also be not convex and it will have only one inflection point but not any singular point or any extra inflection point. If F_{i-2}, F_{i-1} and F_i (or F_{i-1}, F_i and F_{i+1}) are collinear, then the curve segment (1) degenerates into a linear segment.

Consider the equation

$$\Delta_i = \hat{\alpha}_i D_{i-1} + \hat{\beta}_i D_i . \tag{11}$$

It can be solved for $\hat{\alpha}_i$'s and $\hat{\beta}_i$'s, if we are given curve segment (1). Obviously there are following four possible cases:

CASE 1. If $\hat{\alpha}_i \hat{\beta}_i$ is positive, then their solution determines the contraints on shape parameters as follows:

$$v_i \geqslant \beta_{4,i} / \beta_{2,i} \ , \ \ w_i \geqslant \beta_{4,i} / \beta_{3,i} \, . \tag{12}$$

CASE 2. If $\hat{\alpha}_i \hat{\beta}_i$ is negative, then solution will be determined by replacing D_{i-1} by $\hat{D}_{i-1}$, where $\hat{D}_{i-1}$ is the reflection in the line containing F_{i-1} and F_i. The constraints on shape parameters are then obtained as follows:

$$v_i = \hat{\beta}_{4,i} / \beta_{2,i} \ , \ \ w_i = \hat{\beta}_{4,i} / \beta_{3,i} \, . \tag{13}$$

CASE 3. If $\hat{\alpha}_i \hat{\beta}_i$ vanishes, then the constraints are $v_i = 3 = w_i$.

CASE 4. Fourth possiblity is, obviously, that there is no solution for $\hat{\alpha}_i$ and $\hat{\beta}_i$. This happens when the tangent vectors D_{i-1} and D_i are collinear. The curve segment (1) will have only one inflection provided these derivatives are in the same direction and hence the user can utilize the values in (13). In case of opposite direction but same absolute value, the values $v_i = 1 = w_i$ will produce a conic segment.

3. SOME REMARKS

This section is devoted for some very important remarks about the theory developed in Section 2. These remarks are as follows:

Remark 1. In the formulae (12) and (13), the shape parameters v_i, w_i have been determined by looking at the components $\beta_{1,i}$, $\beta_{2,i}$, $\beta_{3,i}$, $\beta_{4,i}$, of the vector products $\Delta_i \times \Delta_{i+1}$, $D_i \times \Delta_i$, $\Delta_i \times D_{i+1}$, $D_i \times D_{i+1}$ respectively. It should be noted that the signs of these vector quantities are significant for the shape construction of the picture which has been explained above.

Remark 2. The restriction $v_i = w_i = r_i$ (say), in Case 1 above, recovers the convexity preserving method of Sarfraz in [14] as the constraints

$$r_i \geqslant \max\{\beta_{4,i}/\beta_{2,i}\ ,\ \beta_{4,i}/\beta_{3,i}\}\ , \tag{14}$$

which are similar to those in [14], can be obtaind from (12).

Remark 3. The scheme can also be considered for the data when it arises from a function, i.e., for the scalar case. The restriciton $v_i = w_i = r_i$ (say) is also required in this case and it can be considered as an application of interpolation scheme $(t, p(t))$ in $\mathbb{R}^2$ to the values $(t_i, F_i) \in \mathbb{R}^2$ and derivatives $(1, D_i) \in \mathbb{R}^2$, $i = 0, \ldots, n$. It can also be noted that $\Delta_i = (1, \Delta_i)$. Therefore the convexity constraints (12), in this case, are respectively

$$r_i \geqslant \max\left\{\frac{D_{i+1} - D_i}{\Delta_i - D_i}, \frac{D_{i+1} - D_i}{D_{i+1} - \Delta_i}\right\}. \tag{15}$$

These constraints are same as in [3]. Hence the convexity preseving method of Delbourgo and Gregory [3], for scalar curves can be obtained as a very special case. Similarly, monotonicity of [3] can also be achieved as the special case of this scheme.

4. Conclusions and Suggestions

C^1 rational cubic Hermite interpolant with two shape parameters has been utilized to obtain a C^1 convexity and/or monotonicity preserving plane curve method. Data dependent shape constraints are derived on the shape parameters to assure the shape preservation of the data. Choice of the tangent vectors, which are consistent and dependent on the data, has also been made. The shape preserving scheme, in this paper, not only recovers the curve methods in [3, 14], but also gives alternative to the methods in [5, 6, 7, 11].

The shape preserving scheme has also been looked at the aspect of producing conic sections. The constraints for parabolic, hyperbolic and elliptic arcs can be discovered as a subsequent work, which is under consideration of the author. Moreover, circular arcs for this rational interpolation scheme can also be investigated.

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