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A Characterization of Reflexivity in the Terms of the Existence of Generalized Centers

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If X is a Banach space, A is a bounded set in X, $x \in X$, we put $f_\infty(x) = \sup_{a\in A}\|x-a\|$. The *Chebyshev radius* of A is the number $r_\infty(A) = \inf f_\infty(X)$. The point x is a *Chebyshev center* of A if $f_\infty(x) = r_\infty(A)$.

It is not difficult to prove (using a weak–compactness argument) that if X is reflexive then each bounded set $A \subset X$ admits at least one Chebyshev center. S. Konyagin [3] proved that if X is nonreflexive then it can be equivalently renormed so that a three–point set $A \subset X$ have no Chebyshev center.

Analogously, we can define $f_1(x) = (1/n)\Sigma_{i=1}^n\|x-a_i\|$ and $r_1(A) = \inf f_1(X)$ if $A = \{a_1,\dots,a_n\} \subset X$ is a finite set (the normalizing factor $1/n$ is not essential for our purposes). Following [1], we call a *median* of A each point $x \in X$ such that $f_1(x) = r_1(A)$.

Again it is easy to prove that in reflexive spaces medians of finite sets always exist. In [1], Section 7, the authors ask whether the Konyagin's result holds also for medians. The aim of the present paper is to give an affirmative answer, even for a general class of centers defined by monotone symmetric norms on $\mathbb{R}^3$. The presented proof is a generalization of Konyagin's idea in [3]. Let us remark that our Lemma 1 and Lemma 2 about norms on $\mathbb{R}^n$ may be of independent interest.

As usual, if $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ we put $|x| = (|x_1|,\dots,|x_n|)$ and $\|x\|_\infty = \max\{|x_1|,\dots,|x_n|\}$. We consider the coordinate–wise ordering on $\mathbb{R}^n$: $(x_1,\dots,x_n) \leqslant (y_1,\dots,y_n)$ iff $x_i \leqslant y_i$ holds for all $i = 1,\dots,n$.

A norm π on $\mathbb{R}^n$ is *symmetric* if $\pi(x_1,\dots,x_n) = \pi(x_{p(1)},\dots,x_{p(n)})$ whenever p is a permutation of $\{1,\dots,n\}$ and $(x_1,\dots,x_n) \in \mathbb{R}^n$; and π is *monotone* if $\pi(x) \leqslant \pi(y)$ whenever $|x| \leqslant |y|$, $x,y \in \mathbb{R}^n$.

Before defining generalized centers let us state two lemmas on symmetric, resp. symmetric and monotone norms on $\mathbb{R}^n$. The proof of the first one was suggested by E. Casini.

LEMMA 1. *Let π be a symmetric norm on $\mathbb{R}^n$, and $(x_1,\dots,x_n)\in\mathbb{R}^n$ be such that $x_1 \geqslant x_2$. Then*

$$\pi(x_1,x_2,x_3,\dots,x_n) \leqslant \pi(x_1+\epsilon, x_2-\epsilon, x_3,\dots,x_n)$$

for any $\epsilon > 0$. Moreover, if π is strictly convex then the inequality becomes strict.

Proof. Denote

$$x = (x_1, x_2, x_3,\dots,x_n)$$
$$y = (x_1+\epsilon, x_2-\epsilon, x_3,\dots,x_n)$$
$$z = (x_1-\epsilon, x_2+\epsilon, x_3,\dots,x_n)$$
$$v = (1,-1,0,\dots,0).$$

Then $y = x+\epsilon v$ and $z = x-(\epsilon+x_1-x_2)v$. Consequently, x is a nontrivial convex combination of the points y, z. The assertion of Lemma 1 follows from the convexity (resp. strict convexity) of π and from $\pi(y)=\pi(z)>0$. ∎

LEMMA 2. *Let π be a symmetric monotone norm on $\mathbb{R}^n$, $x,y\in\mathbb{R}^n$, $|x|\leqslant|y|$ and $\|x\|_\infty < \|y\|_\infty$. Then $\pi(x)<\pi(y)$.*

Proof. Suppose $x\neq 0$ (for $x=0$ the assertion is trivial).

(a) At first, let us prove Lemma 2 for

$$x = (x_1, x_2, x_3,\dots,x_n) \geqslant 0$$
$$y = (y_1, x_2, x_3,\dots,x_n)$$
$$y_1 > x_1 = \|x\|_\infty.$$

It is possible to assume that $x_i > 0$ for each $i=1,\dots,n$ (for otherwise we can forget about the coordinates with $x_i = 0$ and solve the problem in a space of smaller dimension). Choose $\epsilon > 0$ and $\delta > 0$ so small that

$$(1+\delta)(x_1+(n-1)\epsilon) \leqslant y_1 \quad \text{and}$$
$$0 \leqslant (1+\delta)(x_i-\epsilon) \leqslant x_i \quad \text{for } i=2,\dots,n.$$

Using $n-1$ times Lemma 1 (for the first and i-th coordinates, $i=2,\dots,n$) and the monotonicity of π we get

$$\begin{aligned}\pi(x_1, x_2, \dots, x_n) &\leqslant \pi(x_1 + (n-1)\epsilon, x_2 - \epsilon, \dots, x_n - \epsilon) \\ &< (1+\delta)\pi(x_1 + (n-1)\epsilon, x_2 - \epsilon, \dots, x_n - \epsilon) \\ &\leqslant \pi(y_1, x_2, \dots, x_n).\end{aligned}$$

(b) We are ready to prove the general case. It is possible to assume that $\|y\|_\infty = |y_1|$. The assertion follows easily from (a):

$$\begin{aligned}\pi(x) &\leqslant \pi(\|x\|_\infty, |x_2|, \dots, |x_n|) < \pi(\|y\|_\infty, |x_2|, \dots, |x_n|) \\ &\leqslant \pi(|y_1|, |y_2|, \dots, |y_n|) = \pi(y). \quad \blacksquare\end{aligned}$$

Let us define generalized radius and center of finite sets. For the first time this definition appears in [2] by R. Durier. It is easy to see that if π is the ℓ_∞– or ℓ_1–norm on $\mathbb{R}^n$ then π–centers are simply the Chebyshev centers or medians, respectively.

DEFINITION. Let π be a norm on $\mathbb{R}^n$. For a subset $A = \{a_1, \dots, a_n\}$ of a normed linear space X, the number

$$r_\pi(A) := \inf_{x \in X} \pi(\|x - a_1\|, \dots, \|x - a_n\|)$$

is called the *π–radius* of A. Any point $x_0 \in X$ with $\pi(\|x_0 - a_1\|, \dots, \|x_0 - a_n\|) = r_\pi(A)$ is called a *π–center* of A.

We shall consider norms π on $\mathbb{R}^3$ and three–point sets A only. The following simple lemma about π–centers of a concrete three–point set in the plane (equipped with the max–norm) describes a situation which we are going to "imitate" in the proof of our main result.

LEMMA 3. *Let π be a symmetric monotone norm on $\mathbb{R}^3$. If $(x,0)$ is a π–center of the set*

$$A_0 = \left\{(0,1), (0,-1), (\tfrac{3}{2},0)\right\}$$

in $(\mathbb{R}^2, \|.\|_\infty)$ then $\frac{1}{2} \leqslant x \leqslant 1$.

Proof. Denote

$$f(x) := \pi(\|(x,0) - (0,1)\|_\infty, \|(x,0) - (0,-1)\|_\infty, \|(x,0) - (\tfrac{3}{2},0)\|_\infty)$$

$$= \pi(\max\{|x|,1\},\max\{|x|,1\}, |x-\tfrac{3}{2}|).$$

If $x < \frac{1}{2}$ then $|x-\frac{3}{2}| > 1$. Lemma 2 implies

$$f(\tfrac{1}{2}) = \pi(1,1,1) < \pi(1,1,|x-\tfrac{3}{2}|) \leqslant f(x).$$

So $(x,0)$ cannot be a π–center of A_0.

If $x > 1$ we can write $x = 1+\epsilon$ where $\epsilon > 0$. By Lemma 2 and Lemma 1 we obtain

$$f(1) = \pi(1,1,\tfrac{1}{2}) < \pi(1+\epsilon,1,\tfrac{1}{2}) \leqslant \pi(1+\epsilon,1+\epsilon,\tfrac{1}{2}-\epsilon) = f(x)\ .$$

Neither in this case $(x,0)$ is a π– center of A_0. ∎

It is easy to prove, using a weak–compactness argument, that for any monotone norm π the π–centers always exist in reflexive Banach spaces. The main result of this paper concerns the opposite implication: given a symmetric monotone norm π, the existence of π–centers of all three–point sets for every equivalent renorming implies reflexivity. The main idea of the proof comes from [3] where a particular case (Chebyshev centers) was proved.

THEOREM. *Let X be a nonreflexive Banach space. There exists a three–point set $A \subset X$ with the following property: for each symmetric monotone norm π on $\mathbb{R}^3$ there exists an equivalent norm $|||.|||$ on X such that A has no π–center in $(X,|||.|||)$.*

Proof. Choose a closed subspace Y of codimension one in X, and a point $x_1 \in X \setminus Y$. Since Y is nonreflexive, by the James theorem there exists a norm–one functional $y^* \in Y^*$ that does not attain its maximum value on the unit ball of Y. Choose $y_1 \in Y$ with $y^*(y_1) = 1$ and define

$$A = \left\{x_1, -x_1, \tfrac{3}{2}y_1\right\}.$$

Let π be a symmetric monotone norm on $\mathbb{R}^3$. We shall define an equivalent norm $|||.|||$ on X with which no π–center of A exists.

Let $A_0 \subset (\mathbb{R}^2, \|.\|_\infty)$ be as in Lemma 3. Denote

$$\lambda = \min\{x \in \mathbb{R} \mid (x,0) \text{ is a } \pi\text{–center of } A_0\}.$$

(Of course A_0 must have a π–center of the form $(x,0)$ by symmetry: the set of its π–centers is nonempty, convex and symmetric with respect to the x–axis.)

Lemma 3 implies $1/2 \leqslant \lambda \leqslant 1$. Let $\alpha > 0$ be so small that

$$\alpha\left[\lambda + \tfrac{3}{2}\|y_1\|\right] \leqslant \tfrac{3}{2} - \lambda$$

where $\|.\|$ is the original norm on X. Consequently the number

$$\beta = 1 - \alpha\lambda$$

is positive since $\beta \geqslant (3/2) - \lambda - \alpha\lambda \geqslant \alpha(3/2)\|y_1\|$.

Let us define an equivalent norm on X by the formula

$$|||y + tx_1||| = \max\{\alpha\|y\| + \beta|t|, |y^*(y)|, |t|\}.$$

Put

$$F(x) = \pi(|||x - x_1|||, |||x + x_1|||, |||x - \tfrac{3}{2}y_1|||).$$

For $y \in Y$, we have

$$F(y) = \pi(\max\{\alpha\|y\| + \beta, |y^*(y)|, 1\}, \max\{\alpha\|y\| + \beta, |y^*(y)|, 1\},$$
$$\max\{\alpha\|y - \tfrac{3}{2}y_1\|, |y^*(y) - \tfrac{3}{2}|\}).$$

Let f be as in Lemma 3, i.e. $f(\xi) = \pi(\max\{|\xi|, 1\}, \max\{|\xi|, 1\}, |\xi - 3/2|)$, and $(\xi, 0)$ is a π-center of A_0 iff f attains its minimum at ξ. For simplicity denote

$$R = r_\pi(A) \quad \text{and} \quad r = r_\pi(A_0).$$

For any $0 < \epsilon < \lambda$ choose $y_\epsilon \in Y$ such that $\|y_\epsilon\| = \lambda$ and $y^*(y_\epsilon) = \lambda - \epsilon$. Considering that $\alpha\lambda + \beta = 1$ and $\alpha\|y_\epsilon - 3/2y_1\| \leqslant \alpha(\lambda + 3/2\|y_1\|) < 3/2 - \lambda < 3/2 - \lambda + \epsilon$, we can compute

$$\begin{aligned} F(y_\epsilon) &= \pi(\max\{\lambda - \epsilon, 1\}, \max\{\lambda - \epsilon, 1\}, \max\{\alpha\|y_\epsilon - 3/2y_1\|, 3/2 - \lambda + \epsilon\}) \\ &\leqslant \pi(\max\{\lambda, 1\}, \max\{\lambda, 1\}, 3/2 - \lambda + \epsilon) \\ &\leqslant \pi(\max\{\lambda, 1\}, \max\{\lambda, 1\}, 3/2 - \lambda) + \epsilon\pi(0,0,1) \\ &= f(\lambda) + \epsilon\pi(0,0,1) = r + \epsilon\pi(0,0,1). \end{aligned}$$

Then $R \leqslant r$ since ϵ can be taken arbitrary small.

The monotonicity and symmetry of π easily imply that the function F is convex and satisfies $F(y + tx_1) = F(y - tx_1)$ for all $y \in Y$, $t \in \mathbb{R}$. Consequently it suffices to compute the π-radius of A as the greatest lower bound of $F(Y)$, and to prove that A has no π-center in Y. Suppose on the contrary that a point $y \in Y$ is a π-center of A. Then

$$r \geqslant R = F(y) \geqslant \pi(\max\{|y^*(y)|,1\}, \max\{|y^*(y)|,1\}, |y^*(y) - \tfrac{3}{2}|) = f(y^*(y)) \geqslant r.$$

This implies that $R = r$ and $(y^*(y),0)$ is a π-center of A_0. By Lemma 3

$$\tfrac{1}{2} \leqslant \lambda \leqslant y^*(y) \leqslant 1.$$

Hence

$$r \geqslant R = F(y) = \pi(\max\{\alpha\|y\| + \beta,1\}, \max\{\alpha\|y\| + \beta,1\}, \max\{\alpha\|y - \tfrac{3}{2}y_1\|, \tfrac{3}{2} - y^*(y)\})$$

$$\geqslant \pi(1,1,\tfrac{3}{2} - y^*(y)) = f(y^*(y)) = r.$$

Therefore the last inequality is in fact equality. Denoting $\xi = \max\{\alpha\|y\| + \beta, 1\}$, $\eta = \max\{\alpha\|y - 3/2y_1\|, 3/2 - y^*(y)\}$ and $\vartheta = 3/2 - y^*(y)$, we have the following situation:

$$\pi(\xi,\xi,\eta) = \pi(1,1,\vartheta) \quad \text{and} \quad (\xi,\xi,\eta) \geqslant (1,1,\vartheta).$$

Lemma 2 and the inequality $\vartheta \leqslant 3/2 - 1/2 = 1$ imply

$$\alpha\|y\| + \beta \leqslant \xi \leqslant \|(\xi,\xi,\eta)\|_\infty = \|(1,1,\vartheta)\|_\infty = 1.$$

But this implies $\|y\| \leqslant (1-\beta)/\alpha = \lambda$, a contradiction with the fact that $y^*(y) \geqslant \lambda$, $\|y^*\| = 1$ and y^* does not attain its norm. ∎

Our Theorem and the remark after Lemma 3 give the following characterizations of reflexivity. (The equivalence of (i), (ii) and (v) for $\pi = \|\cdot\|_\infty$ follows also from [3].)

COROLLARY. *Let $\mathscr{P}$ be the set of all symmetric monotone norms on $\mathbb{R}^3$. Let a $\pi_0 \in \mathscr{P}$ be given. Let X be a Banach space and $\mathscr{N}$ be the set of all equivalent norms on X. Then the following assertions are equivalent.*

(i) X is reflexive.

(ii) For each $\nu \in \mathscr{N}$ every bounded $B \subset X$ has a Chebyshev center in (X,ν).

(iii) For each $\nu \in \mathscr{N}$ every finite set $F \subset X$ has a median in (X,ν).

(iv) For each $\nu \in \mathscr{N}$ and each $\pi \in \mathscr{P}$ every three-point set $A \subset X$ has a π-center in (X,ν).

(v) For each $\nu \in \mathscr{N}$ every three-point set $A \subset X$ has a π_0-center in (X,ν).

(vi) For each three-point set $A \subset X$ there exists $\pi \in \mathscr{P}$ such that A has a π-center in (X,ν) whenever $\nu \in \mathscr{N}$.

Remark. Note that the proof of our Theorem gives a possibility to construct concrete examples of three–point sets with no π–center. For example, if π is the ℓ_1–norm on $\mathbb{R}^3$ and $X = \ell_1$, put $Y = \{(\eta_1, \eta_2, \dots) \in \ell_1 \mid \eta_1 = 0\}$, $y^* = (0, 3/4, 4/5, \dots, (n+1)/(n+2), \dots) \in \ell_\infty$, $y_1 = (0, 4/3, 0, 0, \dots)$. It is easy to see that $\lambda = 1$ in this case, so the inequality $\alpha(\lambda + 3/2\|y_1\|) \leqslant 3/2 - \lambda$ becomes $\alpha(1+2) \leqslant 1/2$; thus we can take $\alpha = 1/6$. Then $\beta = 5/6$. By the proof of Theorem, the set

$$A = \{(1,0,0,\dots), (-1,0,0,\dots), (0,2,0,0,\dots)\}$$

has no median in $(\ell_1, |||\cdot|||)$ where

$$|||(\eta_1, \eta_2, \dots)||| = \max\left\{\frac{5}{6}|\eta_1| + \frac{1}{6}\sum_{n=2}^{\infty}|\eta_n|,\ \left|\sum_{n=2}^{\infty}\frac{n+1}{n+2}\eta_n\right|,\ |\eta_1|\right\}.$$

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