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Some Results on Weighted Fréchet and LB–Spaces of Moscatelli Type [1]

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The structure of the weighted Fréchet and LB–spaces of Moscatelli type appears when one combines both the structure of the Köthe sequence spaces [3] and the structure of Fréchet and LB–spaces of Moscatelli type, introduced by Moscatelli in 1980 [12] and developed by Bonet and Dierolf in [4,5]. The theory of this new structure includes both theories.

The main motivation for our research on these spaces are the questions which remain open in the theory of LB–spaces. The most important one is the question posed by Grothendieck [8] asking whether every regular LB–space is complete. This question is answered positively in our present frame here.

This paper is a summary of some results appearing in [11] which were divided into three sections. First we introduced the weighted LB–spaces of Moscatelli type and studied strictness, regularity and bounded retractivity. We also proved that these inductive limits are regular if and only if they are complete (under mild additional assumptions). Then we defined the weighted Fréchet spaces of Moscatelli type and investigated when they are Montel, Schwartz and when they satisfy property (Ω_φ) or property (DN_φ) of Vogt. We finished by establishing a certain duality between the weighted Fréchet and LB–spaces of Moscatelli type.

1. Weighted LB-spaces of Moscatelli Type

1.1. Definitions and Preliminaires. In what follows, $(L, \| \ \|)$ will denote a normal Banach sequence space, i.e. a Banach sequence space which satisfies:

(α) $\varphi \subset L \subset \omega$ algebraically and the inclusion $(L, \|.\|) \longrightarrow \omega$ is continuous,

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where $\omega = \Pi_{k\in\mathbb{N}} \mathbb{K}$ and $\varphi = \oplus_{k\in\mathbb{N}} \mathbb{K}$.

(β) $\forall a = (a_k)_{k\in\mathbb{N}} \in L$, $\forall b = (b_k)_{k\in\mathbb{N}} \in \omega$ such that $|b_k| \leqslant |a_k|$, $\forall k \in \mathbb{N}$, we have $b \in L$ and $\|b\| \leqslant \|a\|$.

Clearly every projection onto the first n coordinates $p_n : \omega \longrightarrow \omega$, $(a_k)_{k\in\mathbb{N}} \longrightarrow ((a_k)_{k<n}, (0)_{k>n})$ induces a norm–decreasing endomorphism on L.

We shall also consider on $(L, \| \|)$ the following properties:

(γ) $\|a\| = \lim_n \|p_n(a)\| \ \ \forall a \in L$.

(δ) If $a \in \omega$, $\sup_n \|p_n(a)\| < \infty$, then $a \in L$ and $\|a\| = \lim_n \|p_n(a)\|$.

(ϵ) $\lim_n \|a - p_n(a)\| = 0 \ \ \forall a \in L$.

Unexplained notation as in [9, 13].

Following the classical notations (see [3]), given a strictly positive Köthe matrix $A = (a_n)_{n\in\mathbb{N}}$ on $\mathbb{N}$, that is $0 < a_n(k) \leqslant a_{n+1}(k) \quad (n, k \in \mathbb{N})$, we shall denote by $V = (v_n)_{n\in\mathbb{N}}$ the associated decreasing sequence of strictly positive weights with $v_n = (1/a_n)$ $(n \in \mathbb{N})$ and by $\overline{V}$ be the family $\overline{V} = \{\overline{v} = (\overline{v}(k)) \in \omega : \sup_{k\in\mathbb{N}} (\overline{v}(k)/v_n(k)) < \infty, \ \forall n \in \mathbb{N}\}$. We shall always assume without loss of generality that for every $\overline{v} \in \overline{V}$, we have $\overline{v}(k) > 0 \quad (k \in \mathbb{N})$.

In order to define the weighted LB–spaces of Moscatelli type, we would like to make the following three conventions for this first section:

– $(L, \|.\|)$ will denote a normal Banach sequence space with property (γ).

– $V = (v_n)_{n\in\mathbb{N}}$ will stand for a decreasing sequence of strictly positive weights.

– $(X_k, r_k)_{k\in\mathbb{N}}$ and $(Y_k, s_k)_{k\in\mathbb{N}}$ will represent two sequences of Banach spaces such that for each $k \in \mathbb{N}$, Y_k is a subspace of X_k and $s_k \geqslant r_k | Y_k$ (in consequence, $B_k := \{y \in Y_k : s_k(y) \leqslant 1\} \subset \{x \in X_k : r_k(x) \leqslant 1\} =: A_k$).

For every $n \in \mathbb{N}$, the space

$$L(v_n, (X_k, r_k)_{k<n}, (Y_k, s_k)_{k>n}) =$$

$$= \{(x_k)_{k\in\mathbb{N}} \in \prod_{k<n} X_k \times \prod_{k>n} Y_k : (v_n(k) r_k(x_k))_{k<n}, v_n(k) s_k(x_k))_{k>n}) \in L\}$$

provided with the norm: ${}_n\|(x_k)_{k\in\mathbb{N}}\| = \|(v_n(k) r_k(x_k))_{k<n}, v_n(k) s_k(x_k))_{k>n})\|$

is a Banach space, the inclusion

$$L(v_n, (X_k)_{k<n}, (Y_k)_{k>n}) \longrightarrow L(v_{n+1}, (X_k)_{k<n+1}, (Y_k)_{k>n+1})$$

is continuous and the unit ball of the first space is contained in the unit ball of the second one. Now the inductive limit

$$k(V,L,(X_k),(Y_k)) := \mathrm{ind}_n L(v_n,(X_k)_{k<n},(Y_k)_{k\geq n})$$

is the LB–space of Moscatelli type with regards to $(L,\|\ \|)$, $V=(v_n)_{n\in\mathbb{N}}$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$.

Recall that if $X_k = Y_k = \mathbb{K}$ and $L = \ell_p$ we obtain the Köthe co–echelon spaces $k(V,\ell_p) = \mathrm{ind}_n \ell_p(v_n)$ [3] and for the case $v_n(k) = 1$ $(k,n \in \mathbb{N})$ we get the LB–spaces of Moscatelli type [4].

Now given $(\epsilon_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $\epsilon_k > 0$ $(k \in \mathbb{N})$ and $\bar{v} \in \overline{V}$, $p_{\epsilon_k,\bar{v}}$ will denote the Minkowski functional of $\epsilon_k A_k + (1/\bar{v}(k))B_k$. Then $p_{\epsilon_k,\bar{v}}$ is a norm on X_k which is equivalent to r_k. Therefore $(X_k, p_{\epsilon_k,\bar{v}})$ is a Banach space. Since $\sup_k(\bar{v}(k)/v_n(k)) < \infty$ $(n \in \mathbb{N})$, for arbitrary $(\epsilon_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $\epsilon_k > 0$ $(k \in \mathbb{N})$, $\bar{v} \in \overline{V}$ and $n \in \mathbb{N}$ the space $L(v_n,(X_k)_{k<n},(Y_k)_{k\geq n})$ is continuously injected in the Banach space

$$L((X_k,p_{\epsilon_k,\bar{v}})_{k\in\mathbb{N}}) = \{(x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} X_k : (p_{\epsilon_k,\bar{v}}(x_k))_{k\in\mathbb{N}} \in L\}$$

endowed with the corresponding norm:

$$(x_k)_{k\in\mathbb{N}} \in L((X_k,p_{\epsilon_k,\bar{v}})_{k\in\mathbb{N}}) \longrightarrow \|(p_{\epsilon_k,\bar{v}}(x_k))_{k\in\mathbb{N}}\|$$

Consequently, if we define

$$K(\overline{V},L,(X_k),(Y_k)) :=$$
$$\mathrm{proj}(L((X_k,(p_{\epsilon_k,\bar{v}})_{k\in\mathbb{N}}) \ : \ (\epsilon_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}},\ \epsilon_k > 0\ (k \in \mathbb{N}),\ \bar{v} \in \overline{V})$$

we have the continuous inclusion: $k(V,L,(X_k),(Y_k)) \subset K(\overline{V},L,(X_k),(Y_k))$.

For the case $X_k = Y_k = \mathbb{K}$ $(k \in \mathbb{N})$ and $L = \ell_p$, the space $K(\overline{V},L,(X_k),(Y_k))$ coincides with the well–known projective hull $K(\overline{V},\ell_p)$ of the Köthe co–echelon space $k(V,\ell_p)$ [3].

PROPOSITION 1.2. *$k(V,L,(X_k),(Y_k))$ is boundedly retractive if and only if the following two conditions hold:*

i) There exists $m \in \mathbb{N}$ such that Y_k is a topological subspace of X_k $(k \geqslant m)$.

ii) $\mathrm{ind}_n L(v_n)$ is boundedly retractive.

PROPOSITION 1.3. *The following statements are equivalent:*

i) *$k(V,L,(X_k),(Y_k))$ is strict.*

ii) *For every $k \in \mathbb{N}$, Y_k is a topological subspace of X_k and for every $n \in \mathbb{N}$, there is $M_n > 0$ with $(1/v_n(k)) \leqslant (M_n/v_1(k))$ $(k \in \mathbb{N})$.*

Regarding Grothendieck's question [8] whether regularity implies completeness for LB–spaces, we have:

PROPOSITION 1.4. *If* $(L,\| \|)$ *is a step in the sense of* [14], *that is* $(L,\| \|)$ *is perfect,* $(\ell_1,\| \|_1) \subset (L,\| \|) \subset (\ell_\infty,\| \|_\infty)$ *and* $(L,\beta(L,L^X))$ *is a Banach space, where* L^X *denotes the* α*-dual of* L *with property* (ϵ), *then* $k(V,L,(X_k),(Y_k))$ *is regular if and only if it is complete.*

In the cases $(L,\| \|)=(c_0,\| \|_\infty)$ or $(L,\| \|)=(\ell_\infty,\| \|_\infty)$ the equivalence between regularity and completeness holds too.

2. WEIGHTED FRÉCHET SPACES OF MOSCATELLI TYPE

2.1. DEFINITIONS AND PRELIMINAIRES. In this section 2, we would like to make the following conventions:

– $(L,\|\cdot\|)$ will be a normal Banach sequence space with property (γ).

– $A=(a_n)_{n\in\mathbb{N}}$ will stand for a strictly positive Köthe matrix.

– $(Y_k,s_k)_{k\in\mathbb{N}}$ and $(X_k,r_k)_{k\in\mathbb{N}}$ will represent two sequences of Banach spaces and $f_k:Y_k\longrightarrow X_k$ will be a continuous linear mapping such that $f_k(B_k)\subset A_k$ (resp., B_k) stands for the unit ball of X_k (resp., Y_k), $(k\in\mathbb{N})$.

Now, for every $n\in\mathbb{N}$, we define:

$$G_n=L(a_n,(Y_k)_{k<n},(X_k)_{k>n}):=$$

$$=\{(x_k)_{k\in\mathbb{N}}\in\Pi_{k<n}Y_k\times\Pi_{k>n}X_k:((a_n(k)s_k(x_k))_{k<n},(a_n(k)r_k(x_k))_{k>n})\in L\})$$

provided with the norm: $\|(x_k)_{k\in\mathbb{N}}\|_n=\|((a_n(k)s_k(x_k))_{k<n},(a_n(k)r_k(x_k))_{k>n})\|$. Clearly G_n is a Banach space $(n\in\mathbb{N})$. We put $g_n:G_{n+1}\longrightarrow G_n$, $(x_k)_{k\in\mathbb{N}}\longrightarrow ((x_k))_{k<n},f_n(x_n),(x_k))_{k>n})$ $(n\in\mathbb{N})$. Clearly g_n is a continuous linear mapping $(n\in\mathbb{N})$ and we define the weighted Fréchet space of Moscatelli type with regards to A, $(L,\|\cdot\|)$, $(Y_k,s_k)_{k\in\mathbb{N}}$ $(X_k,r_k)_{k\in\mathbb{N}}$ and f_k: $Y_k\longrightarrow X_k$ $(k\in\mathbb{N})$ by

$$G=\lambda(A,L,(Y_k),(X_k)):=\mathrm{proj}_{n\in\mathbb{N}}(G_n,g_n).$$

As in [5], it is easy to check that G coincides algebraically with

$$\{y=(y_k)_{k\in\mathbb{N}}\in\Pi_{k\in\mathbb{N}}Y_k:(f_k(y_k))_{k\in\mathbb{N}}\in\mathrm{proj}_n L(a_n,(X_k)_{k\in\mathbb{N}}))\}$$

and G has the initial topology with regards to the inclusion $j:G\longrightarrow\Pi_{k\in\mathbb{N}}Y_k$ and the linear mapping $\tilde{f}:G\longrightarrow\mathrm{proj}_n L(a_n,(Y_k)_{k\in\mathbb{N}}))$, $(x_k)_{k\in\mathbb{N}}\longmapsto(f_k(y_k))_{k\in\mathbb{N}}$. We can always assume without loss of generality that $f_k(Y_k)$ is dense in X_k $(k\in\mathbb{N})$.

If $X_k=Y_k=\mathbb{K}$ $(k\in\mathbb{N})$, we shall write $\lambda(A,L)$, following the classical notations.

Recall that if $X_k = Y_k = \mathbb{K}$ $(k \in \mathbb{N})$ and $L = \ell_p$ we obtain the Köthe echelon spaces [3] and in the case $a_n(k) = 1$ $(k,n \in \mathbb{N})$, we get the LB–spaces of Moscatelli type [4].

LEMMA 2.2. *a)* $\lambda(A,L,(Y_k),(X_k))$ *is a complemented subspace of*

$$\lambda(A,L,F) := \{(x^n)_{n\in\mathbb{N}} \in F^{\mathbb{N}} : (a_m(n) r(x^n))_{n\in\mathbb{N}} \in L, \forall r \in cs(F), \forall m \in \mathbb{N}\}$$

where F *is the Fréchet space of Moscatelli type with regards to* $(L,\|\cdot\|)$, $(Y_k,s_k)_{k\in\mathbb{N}}$, $(X_k,r_k)_{k\in\mathbb{N}}$ *and* $(f_k)_{k\in\mathbb{N}}$ *(see [4]).*

b) The sectional subspace $(\lambda(A,L))_J := \{(\alpha_k)_{k\in\mathbb{N}} \in \lambda(A,L) : \alpha_k = 0, \forall k \notin J\}$, *with* $J = \{k \in \mathbb{N} : f_k(Y_k) \not\equiv 0\}$ *of the Köthe echelon space is algebraically and topologically isomorphic to a complemented subspace of* $\lambda(A,L,(Y_k),(X_k))$.

PROPOSITION 2.3. *Let* $J = \{k \in \mathbb{N} : f_k(Y_k) \not\equiv 0\}$ *and consider the sectional subspace* $(\lambda(A,L))_J$. *Then,*

i) $\lambda(A,L,(Y_k),(X_k))$ *is Montel (resp., Schwartz) if and only if* $(\lambda(A,L))_J$ *is Montel (resp., Schwartz) and* Y_k *is finite dimensional for all* $k \in \mathbb{N}$.

ii) $\lambda(A,L,(Y_k),(X_k))$ *has property* (DN_φ) *(resp., property* (Ω_φ)*) if and only if* $(\lambda(A,L))_J$ *and* F *have property* (DN_φ) *(resp., property* (Ω_φ)*) where* F *is the Fréchet space of Moscatelli type with regards to* $(L,\|\cdot\|)$, $(Y_k,s_k)_{k\in\mathbb{N}}$, $(X_k,r_k)_{k\in\mathbb{N}}$ *and* $(f_k)_{k\in\mathbb{N}}$.

The properties (Ω_φ) and (DN_φ) were introduced by D. Vogt in [15] as follows:

For an increasing continuous function $\varphi : (0,\infty) \longrightarrow (0,\infty)$ we say that a Fréchet space with a basis of zero–neighbourhoods $\{U_n\}_{n\in\mathbb{N}}$ has property (Ω_φ) if

$$\forall p\ \exists q\ \forall k\ \exists C > 0\ \forall r > 0 : U_q \subset C\varphi(r) U_k + r^{-1} U_p$$

and a Fréchet space with a fundamental sequence of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ satisfies property (DN_φ) if:

$$\exists n_0\ \forall m\ \exists n \in \mathbb{N}, \exists C > 0 : \forall x \in F, \forall r > 0 : \|x\|_m \leqslant C\varphi(r)\|x\|_{n_0} + r^{-1}\|x\|_n.$$

These two conditions play an important role in [10,16]. By [10], a Fréchet space is quasinormable if and only if it has (Ω_φ) for some φ. By [17], a Fréchet space F has (Ω_φ) for $\varphi(k) = 1$ $(k \in \mathbb{N})$ if and only if F'' is a quojection and this is equivalent to the fact that F does not satisfy the condition (*) of Bellenot and Dubinsky (cf. [1]). Property (DN_φ) is related with some normability conditions (see [15]).

3. DUALITY

Let $G=\lambda(A,L,(Y_k),(X_k))$ be the weighted Fréchet space of Moscatelli type with regards to $(L,\|\cdot\|)$ with (ϵ), $A=(a_n)_{n\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(f_k)_{k\in\mathbb{N}}$, each f_k having dense range and $f_k(B_k)\subset A_k$ $(k\in\mathbb{N})$. Because of lemma 2.2 in [5] we may naturally identify (algebraically and topologically) the strong dual of $G_n:=L(a_n,(Y_k,s_k)_{k<n},(X_k,r_k)_{k>n})$ with $H_n:=L'(v_n,(Y_k',s_k')_{k<n},(X_k',r_k')_{k>n})$ and define the weighted LB space of Moscatelli type: $H:=\text{ind}_{n\in\mathbb{N}}H_n=k(V,L,(Y_k'),(X_k'))$ with regards to the duals.

PROPOSITION. *There is an identity map $I:H\longrightarrow G_\beta'$ which is continuous. Moreover H is the bornological space associated to G_β' and G_β' coincides topologically with $K(\overline{V},L',(Y_k'),(X_k'))$.*

COROLLARY 3.2. *G is distinguished if and only if the corresponding weighted LB−space with regards to the duals $k(V,L',(Y_k'),(X_k'))$ satisfies $k(V,L',(Y_k'),(X_k'))=K(\overline{V},L',(Y_k'),(X_k'))$ topologically.*

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