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The Number of Semisimple Rings of Order at most x [1]

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It is a well–known fact that every finite Abelian group can be represented as a direct product of cyclic groups of prime power order, moreover, the representation is unique except for possible rearrangements of the factors. Let the arithmetical function $a(n)$ denote the number of nonisomorphic Abelian groups of order n. It is seen that $a(n)$ is equal to the number of ways in which n can be expressed by $n = n_1 n_2^2 n_3^3 \ldots$, where $n_1, n_2, \ldots$ are positive integers. This shows that $a(n)$ is connected with some divisor function and with the divisor problems.

The problem of estimating the average order of $a(n)$ was raised by E. Erdös and G. Szekeres in 1935. In 1981 G. Kolesnik proved the following estimate which is, at present, the best known one

$$\sum_{n<x} a(n) = c_1 x + c_2 x^{1/2} + c_3 x^{1/3} + O(x^{97/381} \log^{35} x) \tag{1}$$

$$c_j = \prod_{\substack{k=1 \\ k \neq j}}^{\infty} \zeta\left(\frac{k}{j}\right) \quad , \quad j = 1,2,3.$$

We will consider here the arithmetical function $S(n)$ which denotes the number of nonisomorphic finite semisimple rings with n elements. $S(n)$ is also a multiplicative function being $S(n) = 1$ when n is a square–free positive integer. The Dirichlet series associated with this function may be represented as product of the Riemann zeta function; thus we have for $\mathrm{Re}\, s > 1$

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{r=1}^{\infty} \prod_{m=1}^{\infty} \zeta(rm^2 s). \tag{2}$$

Obviously $S(n)$ can be expressed by means of $a(n)$ using the properties of the Dirichlet series

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(3) $$\sum_{n=1}^{\infty}\frac{S(n)}{n^s}=\prod_{r\geq 1}^{\infty}\zeta(rs)\prod_{m\geq 2}^{\infty}\zeta(rm^2s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}\sum_{m=1}^{\infty}\frac{S_4^*(m)}{m^s}$$

where the last series is convergent in $\operatorname{Re} s > 1/4$. Thus we have

$$S(n)=\sum_{d\delta=n}a(d)S_4^*(\delta)\,.$$

Therefore, the function $S(n)$ is connected with the function $a(n)$ and consequently with the divisor problems. The problem of estimating the error $-$term $R(x)$ in the asymptotic formula for $\Sigma_{n<x}S(n)$ has been considered by several authors. J. Knopfmacher in [2] and J. Duttlinger in [1] studied $S(n)$ and obtained some asymptotic formulae with an error$-$term of the form $R(x)=O(x^a\log^b x)$. Thus, in 1972 J. Knopfmacher proved that

(4) $$\sum_{n<x}S(n)=\alpha_1 x+\alpha_2 x^{1/2}+O(x^{1/3}\log^2 x)$$

where α_1,α_2 are constants. J. Duttlinger in 1974 obtained

(5) $$\sum_{n<x}S(n)=\alpha_1 x+\alpha_2 x^{1/2}+\alpha_3 x^{1/3}+O(x^{7/27}\log^2 x)$$

$$\alpha_j=\prod_{\substack{r=1\\ rm^2\neq j}}^{\infty}\prod_{m=1}^{\infty}\zeta\left(\frac{rm^2}{j}\right).$$

It seems natural to ask whether the bound given by G. Kolesnik for $\Sigma_{n<x}a(n)$ function is also satisfied for $\Sigma_{n<x}S(n)$. In this paper we will obtain an asymptotic formula for $\Sigma_{n<x}S(n)$ which improves the previously known one and its O$-$term has the same order that the given one by Kolesnik for $\Sigma_{n<x}a(n)$.

THEOREM 1. *For* $x\geq 2$

(6) $$\sum_{n<x}S(n)=c_1B_1x+c_2B_2x^{1/2}+c_3B_3x^{1/3}+O(x^{97/381}\log^{35}x)$$

where

$$c_j=\prod_{\substack{k=1\\ k\neq j}}^{\infty}\zeta\left(\frac{k}{j}\right)\quad,\quad B_j=\sum_{n=1}^{\infty}\frac{S_4^*(n)}{n^{1/j}}=\prod_{r\geq 1}\prod_{m\geq 2}\zeta\left(\frac{rm^2}{j}\right),\quad j=1,2,3.$$

Note that the exponent $a=97/381$ is close to $1/4$, so one may expect that

$$\sum_{n<x}S(n)=c_1B_1x+c_2B_2x^{1/2}+c_3B_3x^{1/3}+c_4B_4x^{1/4}+o(x^{1/4}).$$

Before to prove this theorem we will give the following result:

LEMMA. *Let $S_4^*(n)$ be the arithmetical function such that for* $\operatorname{Re} s > 1/4$

$$\sum_{n=1}^{\infty} \frac{S_4^*(n)}{n^s} = \prod_{r>1} \prod_{m>2} \zeta(rm^2 s).$$

Then, for $\operatorname{Re} s > 1/4$

$$\sum_{n<x} \frac{S_4^*(n)}{n^\sigma} = \sum_{n=1}^{\infty} \frac{S_4^*(n)}{n^\sigma} + O(x^{1/4-\sigma}).$$

Proof.

$$\sum_{n=1}^{\infty} \frac{S_4^*(n)}{n^s} = \prod_{r>1} \zeta(4rs) \prod_{m>3} \zeta(rm^2 s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{4s}} \sum_{m=1}^{\infty} \frac{S_9^{\oplus}(m)}{m^s}$$

where $a(n)$ is the number of nonisomorphic abelian groups with n elements and the second series is convergent for $\sigma > 1/9$.

Therefore,

$$\sum_{n<x} S_4^*(n) = \sum_{n^4 m<x} a(n) S_9^{\oplus}(m) = \sum_{m<x} S_9^{\oplus}(m) \sum_{n \leqslant (x/m)^{1/4}} a(n) =$$

$$O\left[x^{1/4} \sum_{m<x} \frac{S_9^{\oplus}(m)}{m^{1/4}}\right] = O(x^{1/4}).$$

Now, let $\sigma > 1/4$ then

$$\sum_{n>x} \frac{S_4^*(n)}{n^\sigma} = O\left[\int_x^{\infty} \frac{dt}{t^{\sigma+1-1/4}}\right] = O(x^{1/4-\sigma}).$$

So, Lemma is proved.

Proof of Theorem 1. From (3) we deduce that

$$\sum_{n<x} S(n) = \sum_{nm<x} a(n) S_4^*(m) = \sum_{n<x} S_4^*(n) \sum_{m<x/n} a(m).$$

By Lemma and the estimate (1)

$$\sum_{n<x} S(n) = c_1 \sum_{n=1}^{\infty} \frac{S_4^*(n)}{n} x + c_2 \sum_{n=1}^{\infty} \frac{S_4^*(n)}{n^{1/2}} x^{1/2} +$$

$$+ c_3 \sum_{n=1}^{\infty} \frac{S_4^*(n)}{n^{1/3}} x^{1/3} + O\left[x^{97/381} \log^{35} x \sum_{n=1}^{\infty} \frac{S_4^*(n)}{n^{97/381}}\right] =$$

$$c_1 B_1 x + c_2 B_2 x^{1/2} + c_3 B_3 x^{1/3} + O(x^{97/381} \log^{35} x).$$

Thus the theorem is proved. Analogously we can also prove the following result:

THEOREM 2. *Let $a(n)$ the number of nonisomorphic abelian groups with n elements. If*

$$\sum_{n<x} a(n) = c_1 x + c_2 x^{1/2} + c_3 x^{1/3} + O(x^a \log^b x) \tag{7}$$

where c_1, c_2, c_3 are constants and $1/3 > a > 1/4$, $b > 0$ positive constants, then we have

$$\sum_{n<x} S(n) = c_1 B_1 x + c_2 B_2 x^{1/2} + c_3 B_3 x^{1/3} + O(x^a \log^b x). \tag{8}$$

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