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Atomic Mappings and Extremal Continua [1]

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The notion of atomic mappings was introduced by R.D. Anderson in [1] to describe special decompositions of continua. Soon, atomic mappings turned out to be important tools in continuum theory. In particular, it can be seen in [2] and [5] that these mappings are very helpful to construct some special, singular continua. Thus, the mappings have proved to be interesting by themselves, and several their properties have been discovered, e. g. in [6], [7] and [9]. The reader is referred to Table II of [9], p. 28, where relations between atomic and other classes of mappings are exhibited.

The paper is devoted to further study of various properties of atomic mappings. As the first main result of the paper it is shown that if a mapping between metric continua is atomic, then the inverse image of an irreducible subcontinuum of the range is an irreducible subcontinuum of the domain (Theorem 2). This result is then applied to investigate behaviour of extremal continua under atomic mappings. The concept of an extremal continuum has been introduced by M.A. Owens in [10] as a special kind of a terminal continuum. Mapping invariance of these continua was studied in the author's paper [3] (see also [4] for a related topic). The second main result of the paper supplies that study. Namely, it is shown that if a surjective mapping between continua is atomic, then the image of an extremal subcontinuum in the domain is an extremal subcontinuum in the range (Corollary 5).

Throughout the paper all spaces are metric and all mappings are continuous. A *continuum* means a compact connected space. If points a and b of a continuum X are such that no proper subcontinuum of X contains both a and b, then X is

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said to be *irreducible between a and b*. A continuum is said to be *irreducible* if it is irreducible between some pair of its points.

A proper subcontinuum S of a continuum X is called a *terminal continuum* in X provided whenever P and Q are proper subcontinua of X having union equal to X, such that both P and Q meet S, it follows that either $X = P \cup S$ or $X = Q \cup S$. If S is terminal in $I \cup S$, for every irreducible subcontinuum I in X such that $I \cap S \neq \emptyset \neq I \setminus S$, then S is called an *extremal continuum* in X (see [10], p. 264).

A mapping $f: X \longrightarrow Y$ from a continuum X onto a continuum Y is said to be:
– *atomic*, provided that, for each subcontinuum K of X, either $f(K)$ is degenerate, or $f^{-1}(f(K)) = K$;
– *monotone*, provided that the inverse image of each subcontinuum of Y is connected;
– *hereditarily monotone*, provided that, for each subcontinuum K of X, the partial mapping $f|_K : K \longrightarrow f(K)$ is monotone.

It is known (see [9], (4.14), p. 17) that

(1) *each atomic mapping of a continuum is hereditarily monotone.*

We intend to show the following result.

THEOREM 2. *If a surjective mapping $f: X \longrightarrow Y$ between continua X and Y is atomic, then*

(3) *for each irreducible subcontinuum J of Y the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Proof. Let J be a subcontinuum of Y irreducible between points p and q. By (1) the inverse image $f^{-1}(J)$ is a subcontinuum of X. Take points $a \in f^{-1}(p)$ and $b \in f^{-1}(q)$, and note that the continuum $f^{-1}(J)$ contains a continuum I that is irreducible between a and b (see [8], §48, I, Theorem 1, p. 192). Thus $f(I)$ is a subcontinuum of J containing both p and q, whence, by irreducibility of J, we have $f(I) = J$. Now, since f is atomic and $f(I)$ is nondegenerate, it follows that $I = f^{-1}(f(I)) = f^{-1}(J)$. The proof is complete. ∎

The next proposition was shown as Corollary 12 of [3], p. 279.

PROPOSITION 4. *If a surjective mapping $f: X \longrightarrow Y$ of a continuum X is hereditarily monotone and satisfies condition*

(3) *for each irreducible subcontinuum J of Y the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X,*

then for each extremal continuum S in X its image $f(S)$ is an extremal continuum in Y.

Thus, as a consequence of (1) and Theorem 2, we get a corollary.

COROLLARY 5. *If a surjective mapping between continua is atomic, then for each extremal continuum in the domain, its image is an extremal continuum in the range.*

Neither Theorem 2 nor Corollary 5 can be extended from atomic to hereditarily monotone mappings of continua, even if the domain is a simple triod, i.e., the union of three arcs emanating from a point which is the only common point of any two and all three of them. This can be seen by the following example.

Let a, b, c be complex numbers which are cubic roots of 1, and consider the simple triod

$$T = \{rz : z \in \{a,b,c\} \text{ and } r \in [0,1]\}.$$

Define $f: T \longrightarrow Y = \{rz : z \in \{b,c\} \text{ and } r \in [0,1]\}$ as a retraction which shrinks the arc $\{ra : r \in [0,1]\}$ to the origin. Then f is hereditarily monotone and it does not satisfy condition (3). Further, the singleton $\{a\}$ is an extremal continuum in T, while its image $f(\{a\}) = 0$ is not an extremal continuum in Y.

Finally, consider the well-known irreducible V–Λ–continuum X (named also an accordionlike continuum, see [8], §48, I, Example 5 and Fig. 3, p. 191, for a detailed definition), and let $f: X \longrightarrow [0,1]$ be the canonical (i.e. minimal) monotone mapping of X onto $[0,1]$. Then f is hereditarily monotone and satisfies condition (3), while is not atomic. Therefore, the converse implications to those of Theorem 2 and Corollary 5 are not true.

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