EXTRACTA MATHEMATICAE Vol. 7, Núm. 2–3, 107 – 109 (1992)

On Neumann Operators [1]

TERESA BERMÚDEZ AND ANTONIO MARTINÓN

Dpto. de Análisis Matemático, Universidad de La Laguna, 38271 *La Laguna, Tenerife, Spain*

AMS *Subject Class.* (1991): 47B06 *Received February* 2, 1993

Let X be a complex Banach space and $L(X)$ the set of all (bounded linear) operators defined in X with values in X. For $T \in L(X)$ we consider the equation

$$x - Tx = y, \tag{1}$$

being $x, y \in X$. The Fredholm integral equation and the Volterra integral equation are examples of (1). We choose $x_0 := y$ and define $x_{n+1} := y + Tx_n$. If the sequence (x_n) is convergent, or equivalently if the Neumann series of T in y

$$\textstyle\sum_{n=0}^{\infty} T^n y \tag{2}$$

converges, then $x := \lim_{n\to\infty} x_n = \Sigma_{n=0}^{\infty} T^n y$ is a solution of (1). Obviously, if the Neumann series (2) is convergent then $\lim_{n\to\infty} T^n y = 0$. Several authors have studied the converse implication:

$$\lim_{n\to\infty} T^n y = 0 \quad \Rightarrow \quad \textstyle\sum_{n=0}^{\infty} T^n y \text{ converges.} \tag{3}$$

The first result about this is due to N. Suzuki [6], who proved that if T is a compact operator then the implication (3) is true. For some classes of operators other authors have given sufficient or equivalent conditions to the implication (3).

In this paper we give a sufficient condition for that T verifies the implication (3) such that the results of [6], [5], [2], [1] and [4] are particular cases of our sufficient condition.

NOTATIONS. In the following X will be a Banach space; $L(X)$ the class of all operators from X into X; I the identity operator of X; $\|T\|$ the norm, $N(T)$ the kernel and $R(T)$ the range of $T \in L(X)$. $T \in L(X)$ is called injection if T is injective and has closed range.

For any $T \in L(X)$, we consider the following subsets of X:

[1] Supported, in part, by the DGICYT Grant PB91–0307

$$C_T := \{ x \in X : \lim_{n\to\infty} T^n x = 0 \} \text{ and } S_T := \{ x \in X : \textstyle\sum_{n=0}^{\infty} T^n x \text{ converges} \}.$$

It is easy to prove that both subsets are linear subspaces of X. Obviously $S_T \subset C_T$. Moreover

$$x \in C_T \text{ and } (I-T)x = z \iff z \in S_T \text{ and } \textstyle\sum_{n=0}^{\infty} T^n z = x ,$$

hence $S_T = (I-T)C_T$.

DEFINITION 1. $T \in L(X)$ is called a Neumann operator on X if it verifies the implication (3), that is, if $C_T \subset S_T$. Denote by $N(X)$ the class of Neumann operators on X.

The class $N(X)$ has not good algebraic properties: it is not stable under sums, scalar multiples or products.

PROPOSITION 2. *Let* $T \in L(X)$. *The following assertions hold for* $k = 2,3,\ldots$

1) $C_{T^k} = C_T$
2) $S_{T^k} \subset S_T$
3) $T^k \in N(X) \Rightarrow T \in N(X)$.

In general the reciprocal of (3) is not true.

In the following theorem, we give a sufficient condition for $T \in N(X)$, from which we derive the results previously obtained by other authors.

THEOREM 3. *Let* $T \in L(X)$. *If there exists a non negative integer* k *such that* $R[(I-T)^k]$ *is closed and the restriction of* $I-T$ *to* $R[(I-T)^k]$ *is an injection, then* T *is a Neumann operator on* X.

Recall that $A \in L(X)$ is a chain–finite operator if there exists a non negative integer k such that $N(A^k) = N(A^{k+1})$ and $R(A^k) = R(A^{k+1})$.

COROLLARY 4. ([1]) *Let* $T \in L(X)$.

1) $I-T$ *is a chain–finite operator* $\Rightarrow T \in N(X)$.
2) $I-T$ *is an injection* $\Rightarrow T \in N(X)$.

Then Theorem 3 can be applied to operators T such that $I-T$ is not a chain–finite operator and $I-T$ is not an injection. For example, consider the operator $A \in L(c_0 \oplus c_0)$ defined by $A((x_n),(y_n)) := ((0,x_1,x_2,\ldots),(0))$, being $(0) = (0,0,\ldots)$. It is clear that A is neither injective nor a chain–finite operator, however $I-A$ verifies the Theorem 3, hence $I-A$ is a Neumann operator.

We denote by $Co(X)$ the class of all compact operators on X. The essential norm of $T \in L(X)$ is defined by $\|T\|_e := \inf\{\|T-K\| : K \in Co(X)\}$ and the radius of the essential spectrum of T by $r_e(T) := \lim \|T\|_e^{1/n}$. Recall that $T \in L(X)$ is a Riesz operator if $r_e(T)=0$. An operator $T \in L(X)$ is called quasi–Riesz if $r_e(T)<1$.

COROLLARY 5. *Let* $T \in L(X)$.

1) T *is a quasi–Riesz operator* $\Rightarrow$ $T \in N(X)$.

2) ([6]) $Co(X) \subset N(X)$.

3) ([4]) $\|T\|_e < 1 \Rightarrow T \in N(X)$.

4) ([5]) *If for some* $n=1,2,\ldots$ T^n *is a strictly singular operator or a strictly cosingular operator, then* $T \in N(X)$.

Finally our aim is to show that the result of Istratescu [2] is a particular case of Theorem 3. Recall that the Kuratowski measure of noncompacteness $\alpha(D)$ of a bounded subset $D \subset X$ is the infimum of the $\epsilon > 0$ such that D admits a finite cover by sets of diameter less than ϵ. Istratescu [2], [3] introduced the following concept: $T \in L(X)$ is called a locally power a–contraction, denoted by $T \in LPC_\alpha(X)$, if there is $\epsilon < 1$ such that for each bounded subset $D \subset X$, there exists $p=1,2,\ldots$ such that $\alpha(T^pD) \leqslant \epsilon\alpha(D)$. In [2, Theorem 3] is proved that $LPC_\alpha(X) \subset N(X)$. We give an alternative proof by showing that $T \in LPC_\alpha(X)$ if and only if T is a quasi–Riesz operator.

ACKNOWLEDGEMENT

The authors express thanks to Manuel Gonzáles for valuable discussion on some results of this paper.

REFERENCES

1. GONZÁLEZ, M. AND ONIEVA, V.M., On the convergence of Neumann series in Banach spaces, in "Actas XII Jornadas Luso–Espanholas de Matemáticas", Universidad de Minho, Braga, 1987, Vol. II, 335–338.

2. ISTRATESCU, V.I., On the convergence of Neumann series in Banach spaces, *Atti Acc. Lincei Rend.*, Ser. VIII **76** (1984), 14–18.

3. ISTRATESCU, V.I., "Introduction to Linear Operator Theory", Dekker, New York, 1981

4. MEDKOVA, D., On the convergence of Neumann series for noncompact operators, *Czechoslovak Mathematical Journal* **41**(116) (1991), 312–316.

5. SCHULZ, J., Uber die konvergenz der Neumannschen Reihe un linearen topologische Raumen, *Beitrage Anal.* **12** (1978), 177–183.

6. SUZUKI, N., On the convergence of Neumann series in Banach spaces, *Math. Ann.* **220** (1976), 143–146.

7. ZABREIKO, P.P., The domain of convergence of the method of succesive approximations for linear equations, (in Russian) *Dokl. Akad. Nauk. BSSR*, **29** (1985), 201–204.