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Note on the Asymptotic Behaviour of the Interface of some Nonlinear Diffusion Problems

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This note presents some results concerning the behaviour of the interface of the following problem:

(1) $u_t = (u^m)_{xx} + (C/(x+a))(u^m)_x$ for $(x,t) \in S = (0,+\infty) \times (0,+\infty)$,

(2) $u(x,0) = u_0(x)$ for $x \in (0,+\infty)$,

(3) $u(0,t) = u_1(t)$ for $t \in (0,+\infty)$,

where $m > 1$, $C \geqslant 0$ and $a > 0$. We shall denote the above problem by $P(m,C;u_0,u_1)$. Troughout this paper we make the following assumptions:

(4) $u_0 \in L^\infty(0,\infty)$, $\operatorname{ess\,inf} u_0 \geqslant 0$, $u_0 \equiv 0$ a.e. on (α,∞) $(\alpha \geqslant 0)$,
$u_1 \in L^\infty(0,\infty)$, $\operatorname{ess\,inf} u_1 \geqslant \beta > 0$.

In the case $C=0$ equation (1) becomes the one–dimensional porous medium equation ([2],[3],[10]). If $C = N-1$ then (1) is the radial version of the N–dimensional porous medium equation $u_t = \Delta(u^m)$ transformed by introducing the translated spatial variable ([7]). Especially, the problem $P(2,1;0,1)$ describes the radially symmetrical infiltration into an unsaturated soil, when the level of water in a cylindrical reservoir is constant ([9]). The question of interest is the range of infiltrating water.

Under assumptions (4) the problem $P(m,C;u_0,u_1)$ has a unique weak solution $u = u(x,t)$ ([6],[7],[8]). The function u is nonnegative, bounded and continuous on S, and u satisfies an appropriate integral identity instead of (1). However u is the classical solution for those points $(x,t) \in S$ at which $u(x,t) > 0$.

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Moreover, if we define $\zeta(t)=\sup\{x\in(0,\infty): u(x,t)>0\}$ $(t>0)$, then $0<\zeta(t)<\infty$ for $t>0$ and ζ is a Lipschitz continuous nondecreasing function. The curve $x=\zeta(t)$ is called the interface or the free boundary of $P(m,C;u_0,u_1)$.

We know that in the case of $P(m,0;0,1)$ the interface has the form

(5) $$\zeta(t)=c_0(m)t^{\frac{1}{2}},$$

where the constant $c_0(m)>0$ depends on m ([1],[5],[11],[12],[16],[18],[19]).

Using some integral equations method ([4],[13],[14],[15]) we construct in [17] a so–called weak subsolution of $P(m,C;u_0,u_1)$. With the help of this subsolution we prove the following asymptotic result ([17]):

THEOREM. *Let $C\in[0,1]$. If ζ is the interface of the problem $P(m,C;u_0,u_1)$ then*

(6) $$\log\zeta(t)\sim\frac{1}{2}\log t \quad as\ t\longrightarrow\infty.$$

The behaviour of the interface in the case $C>1$ is still under considerations.

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