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A Real Nullstellensatz and Positivstellensatz for the Semipolynomials over an Ordered Field

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Let K be an ordered field and R its real closure. A semipolynomial will be defined as a function from R^n to R obtained by composition of polynomial functions and the absolute value. Every semipolynomial can be defined as a straight–line program containing only instructions with the following type: "polynomial", "absolute value", "sup" and "inf" and such a program will be called a semipolynomial expression. It will be proved, using the ordinary Real Positivstellensatz, a general Real Positivstellensatz concerning the semipolynomial expressions.

Firstly we recall the definitions of strong incompatibility and the general form for the Real Nullstellensatz in the polynomial case (see [4 or 5]). We consider an ordered field K, and X denotes a list of variables $X_1, X_2,..., X_n$. We then denote by $K[X]$ the ring $K[X_1, X_2,..., X_n]$. If F is a finite subset of $K[X]$, we let F^{*2} be the set of squares of elements in F, $\mathcal{M}(F)$ be the multiplicative monoid generated by $F \cup \{1\}$. $\mathcal{C}_p(F)$ is the additive monoid generated by elements of type pPQ^2 where p is positive in K, P is in $\mathcal{M}(F)$, Q is in $K[X]$ (positive cone generated by F). Finally, let $I(F)$ be the ideal generated by F.

DEFINITION 1. Consider 4 finite subsets of $K[X]$: $F_>$, $F_\geqslant$, $F_=$, $F_{\neq}$, containing polynomials for which we want respectively the sign conditions > 0, $\geqslant 0$, $= 0$, $\neq 0$. We say that $F = [F_>; F_\geqslant; F_=; F_{\neq}]$ is strongly incompatible in K if we have in $K[X]$ an equality of the following type:

$$S + P + Z = 0 \quad \text{with} \quad S \in \mathcal{M}(F_> \cup F_{\neq}^{*2}),\ P \in \mathcal{C}_p(F_\geqslant \cup F_>),\ Z \in I(F_=).$$

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The strong incompatibility implies that it is impossible to give the indicated signs to the polynomials in any ordered extension of K. If one considers the real closure R of K, the previous impossibility is testable by Hörmander's algorithm, for example. We use the following notation for a strong incompatibility:

$$\downarrow[S_1 > 0,\ldots, S_i > 0,\ P_1 \geqslant 0,\ldots, P_j \geqslant 0,\ Z_1 = 0,\ldots, Z_k = 0,\ N_1 \neq 0,\ldots, N_h \neq 0]\downarrow.$$

The different variants of the Nullstellensatz in the real case are consequences of the following general theorem (first proved in [6]) whose effective version can be found in [4]:

THEOREM 2. *Let K be an ordered field and R a real closed extension of K. For a fixed generalized sign conditions system on polynomials of $K[X]$ the strong incompatibility in K, the impossibility in R and the impossibility in all the ordered extensions of K are equivalent.*

A semipolynomial function with coefficients in K (a K-semipolynomial) from R^n to R is a function obtained by a finite repetition of composition of polynomials with coefficients in K and the function absolute value. A well-known proposition assures that the set of the K-semipolynomials agrees with the minimal sup-inf stable set of functions containing polynomials with coefficients in K (see for example [1]).

In order to obtain an explicit Positivstellensatz for the semipolynomials, we shall need firstly a notion for the algebraic identities concerning semipolynomials. As the semipolynomials do not have canonical representation, this question is a bit tricky but solved considering a new notion, the K-semipolynomial expression (shortly a K-spe). A K-spe $F(X_1, X_2,\ldots, X_n)$ is a straight-line program with the following structure:

– each instruction is an assignement $z_i \longleftarrow \ldots$ with the indexes i ordered in an increasing way (the last z_i is F),

– the instructions can have only the four following types

* $z_i \longleftarrow P(X_1, X_2,\ldots, X_n, z_{i_1},\ldots, z_{i_k})$ where $P \in K[X_1, X_2,\ldots, X_n, z_{i_1},\ldots, z_{i_k}]$ and every i_h is smaller than j,

* $z_j \longleftarrow |z_i|$ with $i < j$, $z_j \longleftarrow \sup\{z_{i_1},\ldots, z_{i_k}\}$ with every i_h smaller than j and $z_j \longleftarrow \inf\{z_{i1},\ldots, z_{i_k}\}$ with every i_h smaller than j.

It is clear that every K-semipolynomial can be obtained from a K-spe (we only need to replace every X_i by x_i and to execute the program). Moreover every

K–spe can be defined using only one of the three functions, absolute value, sup or inf.

A polynomial lying under a K–spe is, by definition, a polynomial in $K[X_1, X_2, \dots, X_n]$ obtained when the straight–line program given by the K–spe considered is executed in the following way:
–every instruction $z_j \longleftarrow |z_i|$ is replaced by $z_j \longleftarrow z_i$ or $z_j \longleftarrow -z_i$,
–every instruction $z_j \longleftarrow \sup\{z_{i_1},\dots,z_{i_k}\}$ is replaced by $z_j \longleftarrow z_{i_h}$ with $1 \leqslant h \leqslant k$,
–every instruction $z_j \longleftarrow \inf\{z_{i_1},\dots,z_{i_k}\}$ is replaced by $z_j \longleftarrow z_{i_h}$ with $1 \leqslant h \leqslant k$.

A K–semipolynomial expression F will be said formally null when all the polynomials lying under F are null. A K–spe G will be said interior to another K–spe F if (modulo a renumbering of the variables z_i in G) the straight–line program for G can be obtained from the one for F by deleting some instructions and if the straight–line program for G ends with an instruction absolue value, inf or sup. A polynomial lying under a K–spe G interior to F is said and interior polynomial lying under F.

A K–spe H is said a polynomial inside the context of the K–spe F if H is a polynomial in the variables X_i and in the K–spe interior to F. More precisely H must be written as the straight–line program associated to F plus instructions of polynomial type (indeed only one of such instructions would be sufficient). Remark that it is not forbidden to introduce new variables, i.e. not appearing in the F's context.

In a fixed context, we have the stronger notion of K–spe identical, as such K–spe defined by the same polynomials in the variables and in the K–spe interior to the context. In particular it is clear that the notion of K–spe identically null is stronger that the one of K–spe formally null.

All what follows will be applied on K–spe which are polynomials inside the context of a K–spe F fixed (we shall say, inside a fixed context). Let $\mathbb{H}$ be a system of generalized sign conditions on the K–spe F_i with $1 \leqslant i \leqslant t$. Next, we define in a recursive way which are the K–spe "evidently $= 0$, $\geqslant 0$ or > 0 under $\mathbb{H}$".

The K–spe evidently null under the hypothesis $\mathbb{H}$ are the K–spe equal to 0 in $\mathbb{H}$, the K–spe coming from polynomial instructions of the following type

$$z_j \longleftarrow P_1(X_1, X_2, \dots, X_n,\, z_{i_1},\dots,z_{i_k}) + P_k(X_1, X_2, \dots, X_n,\, z_{i_1},\dots,z_{i_k})$$

where the z_{i_h} are yet known as evidently null under $\mathbb{H}$ and the K–spe identical to another K-spe yet known as evidently null under $\mathbb{H}$.

The K-spe evidently positive or null under the hypothesis $\mathbb{H}$ are the K-spe >0 or $\geqslant 0$ in $\mathbb{H}$, every K-spe z_j obtained in the context by an absolute value instruction $z_j \longleftarrow |z_i|$, every K-spe of type $z_j - z_i$ where z_j is obtained in the context by a sup instruction $z_j \longleftarrow \sup\{z_{i_1},\dots,z_{i_k}\}$, every K-spe of type $z_j - z_i$ where z_j is obtained in the context by a inf instruction $z_j \longleftarrow \inf\{z_{i_1},\dots,z_{i_k}\}$, the square K-spe, i.e. the K-spe coming from an instruction $z_j \longleftarrow z_i^2$, the polynomials with positive coefficients in K in some K-spe $z_{j_1},\dots,z_{j_k}$ yet known as evidently $\geqslant 0$ under $\mathbb{H}$ and the K-spe identical to another K-spe yet known as evidently $\geqslant 0$ under $\mathbb{H}$.

A system of generalized sign conditions $\mathbb{H}$ is said strongly incompatible (in K and with the context fixed) if there exists a K-spe formally null, obtained as the sum of a K-spe evidently >0, a K-spe evidently $\geqslant 0$ and a K-spe evidently $=0$ (under $\mathbb{H}$).

Next, the K-semipolynomial expressions considered will be polynomials inside a fixed context and they will called K-spe. Also the strong incompatibilities will have their coefficients in K and in the fixed context (which implies that the functions absolute value, sup and inf can appear only as in the context).

THEOREM 3. *Let K be an ordered field and R a real closed field containing K. Let $\mathbb{H}$ be a system of generalized sign conditions defined on a finite family of K-spe in the variables $X_1, X_2,\dots, X_n$ (these K-spe are polynomials inside a fixed context). Then the system $\mathbb{H}$ is incompatible in R if and only if the system $\mathbb{H}$ is strongly incompatible in K (for the fixed context). More precisely*

If $\downarrow\mathbb{H}(X_1, X_2,\dots, X_n)\downarrow$ (in K) then the system $\mathbb{H}$ is incompatible in any ordered extension of K.

If for every $\underline{x} \in R^n$ the system $\mathbb{H}(\underline{x})$ is incompatible then $\downarrow\mathbb{H}(X_1, X_2,\dots, X_n)\downarrow$ (in K).

Proof. The first part in the statement of the theorem is trivial, it is enough to apply the definition of strong incompatibility introduced in the previous section.

To prove the second part, we shall reduce our problem to the ordinary Real Positivstellensatz. Firstly we introduce a formal variable z_j for every variable z_j in the context. So our system $\mathbb{H}$ can be rewritten as a system $\mathbb{H}'$ containing only polynomials in the variables X_i and z_j.

Now we define a polynomial system of generalized sign conditions $\mathbb{H}_c$ associated to the context in the following way:
– for every polynomial instruction $z_j \longleftarrow P_1(X_1, X_2, \ldots, X_n, z_{i_1}, \ldots, z_{i_k})$ we introduce in $\mathbb{H}_c$ the sign condition

$$z_j - P_1(X_1, X_2, \ldots, X_n, z_{i_1}, \ldots, z_{i_k}) = 0.$$

– for every absolute value instruction $z_j \longleftarrow |z_i|$ we introduce in $\mathbb{H}_c$ the sign conditions

$$z_j^2 - z_i^2 = 0 \qquad z_j \geqslant 0.$$

– for every sup instruction $z_j \longleftarrow \sup\{z_{i_1}, \ldots, z_{i_k}\}$ we introduce in $\mathbb{H}_c$ the sign conditions

$$(z_j - z_{i_1})(z_j - z_{i_2})\ldots(z_j - z_{i_k}) = 0, \quad z_j - z_{i_1} \geqslant 0, \quad z_j - z_{i_2} \geqslant 0, \ldots, z_j - z_{i_k} \geqslant 0.$$

– for every inf instruction $z_j \longleftarrow \inf\{z_{i_1}, \ldots, z_{i_k}\}$ we introduce in $\mathbb{H}_c$ the sign conditions

$$(z_j - z_{i_1})(z_j - z_{i_2})\ldots(z_j - z_{i_k}) = 0, \quad z_j - z_{i_1} \leqslant 0, \quad z_j - z_{i_2} \leqslant 0, \ldots, z_j - z_{i_k} \leqslant 0.$$

The system $[\mathbb{H}', \mathbb{H}_c]$ is incompatible in R because first, the system $\mathbb{H}$ is incompatible in R and second, every solution of the system $[\mathbb{H}', \mathbb{H}_c]$ provides a solution for $\mathbb{H}$. As all the elements involved in the system $[\mathbb{H}', \mathbb{H}_c]$ are polynomials, applying the ordinary Real Positivstellensatz (see theorem 2), we obtain a strong incompatibility

$$\downarrow [\mathbb{H}', \mathbb{H}_c] \downarrow \tag{1}$$

Now if we replace, in the algebraic identity obtained, every variable z_j by the corresponding K–spe then:
– the "strictly positive" part in (1) does not contain any generalized sign condition from $\mathbb{H}_c$ and provides a K–spe "evidently strictly positive" under $\mathbb{H}$,
– the "positive or null" part in (1) provides a K–spe "evidently positive or null" under $\mathbb{H}$,
– the "null" part in (1) can be separated in two pieces:
  * the first one is null under $\mathbb{H}'$ and provides a K–spe evidently null under $\mathbb{H}$,
  * the second one is null under $\mathbb{H}_c$ and provides a K–spe formally null (in the fixed context), which can be deleted.

So, deleting the last piece in the "null" part we obtain a K–spe which is equal to a K-spe identically null minus a K-spe formally null and so formally null, as we wanted to show. ∎

In particular if K is a discrete ordered field then theorem 3 together with the results in [4] show that for an incompatible system in R it is posible to construct in an effective way the corresponding strong incompatibility in K. The theorem 3 will provide an easier and more complete rational and continuous solution for the 17^{th} Hilbert's problem than the one appearing in [2] (see [3]).

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