EXTRACTA MATHEMATICAE Vol. 7, Núm. 1, 5 – 8 (1992)

# On Herstein's Theorems Relating Modularity in A and $A^{(+)}$ [1]

J. A. ANQUELA [2]

*Dpto. de Matemáticas, Univ. Zaragoza,* 50009 *Zaragoza, Spain*

AMS *Subject Class.* (1980): 17C50

*Received March* 5, 1992

## 0. INTRODUCTION AND PRELIMINARIES

In this paper we will examine the relationship between modularity in the lattices of subalgebras of $A$ and $A^{(+)}$, for $A$ an associative algebra over an algebraically closed field. To this aim we wil construct an ideal which measures the modularity of an algebra (not necessarily associative) in §1, examine modular associative algebras in §2, and prove in §3 that the ideal constructed in §1 coincides for $A$ and $A^{(+)}$. We will also examine some properties of the ideal mentioned in §1 when the algebras involved are Jordan.

The usual terms in lattice theory can be found in [6].

Let $J$ be an algebra over a field. We will put $L(J)$ for its lattice of subalgebras, where $\leqslant, \wedge, \vee$ are naturally defined ($\leqslant$ is the inclusion relation). We define the length of $J$ (and put $\ell(J)$) as the length of $L(J)$. It is very easy to prove that if $J$ is a solvable algebra, then its length and dimension coincide. In general we obviously have that $\ell(J) \leqslant \dim_F J$, where $F$ is the ground field.

DEFINITION. An algebra $J$ is called modular (semimodular), if $L(J)$ is modular (semimodular).

The easiest examples of modular algebras are trivial algebras since the lattice of submodules of a module is a modular lattice.

## 1. THE MODULAR RADICAL

Let $J$ be an algebra. We define $M_1(J)$ as the sum of the ideals of $J$ which are modular algebras (we will say modular ideals). For any $\alpha$ ordinal we will

---

[1] This paper will appear in the *Journal of Algebra.*

[2] This paper has been written under the direction of Professor Santos González and it is a part of the author's Doctoral Thesis. The author has been partially supported by the Ministerio de Educación y Ciencia (F.P.I. Grant) and the Diputación General de Aragón.

construct $M_\alpha(J)$. Supposing we have already constructed the $M_\beta(J)$'s for all $\beta < \alpha$,

– if $\alpha$ is not a limit ordinal $M_\alpha(J)$ is such that $M_\alpha(J)/M_{\alpha-1}(J) = M_1(J/M_{\alpha-1}(J))$,

– if $\alpha$ is a limit ordinal we will put $M_\alpha(J) = \cup_{\beta<\alpha} M_\beta(J)$.

We now define $M(J)$ as the union (or the limit) of the $M_\alpha(J)$'s for all $\alpha$. We will say an algebra $J$ to be $M$–semisimple if $M(J)=0$ and $M$–radical if $M(J)=J$.

THEOREM. *Let $J$ be an algebra. The quotient $J/M(J)$ has no nonzero modular ideals. Moreover, $M(J)$ is the smallest ideal in $J$ with this property.*

PROPOSITION. *Let $J$ be a finite–dimensional Jordan algebra. Then $J/M(J)$ is semisimple (i.e.* $\mathrm{Jac}(J)=0$*).*

DEFINITION. Let $J$ be a Jordan algebra. A non–zero idempotent $e$ in $J$ will be called a modular idempotent when it is completely primitive (i.e. $J_1(e)$, the (1)–component in the Pierce decomposition of $J$ with respect to $e$, is a division algebra), $J_{1/2}(e)=0$ and $J_1(e)$ is modular.

THEOREM. *Let $J$ be a finite–dimensional Jordan algebra and $N$ be its nilpotent radical. Put*

$$J/N = (J/N)_1(e_1') \oplus \cdots \oplus (J/N)_1(e_r') \oplus S,$$

*with $e_1',\ldots,e_r'$ modular orthogonal idempotents in $J/N$, $S$ semisimple such that its directs summands which are division algebras are not modular. Then*

$$J/M(J) \cong S,$$

$$M(J) = \langle J_1(e_1),\ldots,J_1(e_r),N\rangle,$$

*where the $e_i$'s are orthogonal idempotents in $J$ such that $e_i + N = e_i'$ for all $i$.*

When the ground field $F$ is algebraically closed and the Jordan algebra $J$ is finite–dimensional an idempotent $e$ is modular if and only if it is completely primitive and $J_{1/2}(e) \doteq 0$. In these conditions $J/M(J)$ is the sum of the simple components of $J/N$ not isomorphic to $F$ and $M(J)$ is generated by $N$ and the idempotents which lift the idempotents of $J/N$ in the simple components of $J/N$ isomorphic to $F$.

## 2. MODULAR ASSOCIATIVE ALGEBRAS

PROPOSITION. *A modular associative algebra is algebraic. If it is nil then it is locally nilpotent.*

THEOREM. *Let $A$ be an associative nilalgebra. Then the following are equivalent:*

*i) $A$ is modular.*

*ii) $A$ is semimodular.*

*iii) Given a pair of subalgebras of $A$, $S$ and $T$, $S+T$ is a subalgebra of $A$.*

THEOREM. *Let $A$ be a semimodular associative algebra over an algebraically closed field. Then $A/\mathrm{Nil}(A)$ is isomorphic to $0$, $F$ or $F\oplus F$.*

COROLLARY. *Let $A$ be a semimodular finite-dimensional associative algebra over $F$, an algebraically closed field. Then $\ell(A)=\dim_F A$.*

THEOREM. *Let $A$ be an associative algebra over an algebraically closed field. Then the following are equivalent:*

*i) $A$ is modular.*

*ii) $A$ is semimodular.*

*iii) Given a pair of subalgebras of $A$, $S$ and $T$, $S+T$ is a subalgebra of $A$.*

## 3. MODULARITY IN $A$ AND $A^{(+)}$

Throughout this section all algebras will be considered over algebraically closed fields of characteristic not two.

THEOREM. *If $A$ is a modular associative algebra, then $A^{(+)}$ is a modular Jordan algebra.*

COROLLARY. *Let $J$ be a special Jordan algebra which is $M$-semisimple. Then, there exists an associative envelope for $J$, which is $M$-semisimple.*

THEOREM. *Let $A$ be an associative algebra. Then $M(A)=M(A^{(+)})$.*

## REFERENCES


1. AMAYO, R.K. AND SCHWARTZ, J., Modularity in Lie algebras, *Hiroshima Math. J.* **10** (1980), 311–322.
2. ANQUELA, J.A., Modular nilpotent Jordan algebras, to appear in the "Proceedings of the Jornadas sobre modelos algebraicos no asociativos y sus aplicaciones, Univ. Zaragoza, April 1989", Nova Science Publishers, Inc.

3. ANQUELA, J.A., On modular Jordan algebras, to appear in the "Proceedings of the Fifth International Conference on Hadronic Mechanics and Nonpotential Interactions, Univ. of Northern Iowa, August 1990", Nova Science Publishers, Inc.
4. ELDUQUE, A.C., On semimodular Mal'cev algebras, *Arch. Math.* **50** (1988), 328 – 336.
5. ELDUQUE, A.C., A note on modularity in Mal'cev algebras, *Arch. Math.* **50** (1988), 424 – 428.
6. GRÄTZER, G., "General Lattice Theory", Mathematische Reihe. Band 52, Birkhauser Verlag Basel und Stuttgart, 1978.
7. HERSTEIN, I., "Noncommutative Rings", The Carus Mathematical Monographs 15, The Mathematical Association of America Inc., 1968. Distributed by John Wiley and Sons Inc.
8. JACOBSON, N., "Structure and Representations of Jordan Algebras", Amer. Math. Soc. Colloquium Publications vol. XXXIX. A.M.S. Providence, Rhode Island, 1968.
9. KOLMAN, B., Semimodular Lie algebras, *J. Sci. Hiroshima Univ. Ser.* **29** (1965), 149 – 163.
10. MCCRIMMON, K., On Herstein's theorems relating Jordan and associative algebras, *J. Alg.* **13** (1969), 382 – 392.
11. ZELMANOV, E.I., Prime Jordan algebras II, *Sib. Math. J.* **24**(1) (1983), 89 – 104.