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Minimax Principle for Stochastic Differential Games

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The equations of motion for continuous stochastic differential games for two players are a set of stochastic differential equations:

$$\mathrm{d}x = f(x,t,u^1(x,t),u^2(x,t)) + \sigma(x,t,u^1(x,t),u^2(x,t))\,\mathrm{d}W \tag{1}$$

where $x \in \mathbb{R}^n$, W is an m–dimensional Wiener process; $u^i(x,t)$ the strategy of player i, $i=1,2$, is a r_i–dimensional vector, u^i may or not be subject to restrictions; f is a vector function and σ is a $n \times m$ matrix function. Having fixed an initial state $x(t_0) = x_0$ together with the strategies of players, one obtains, assuming that (1) has an unique solution, a stochastic trajectory for the state $x(t)$. The payoffs to each player depend on the strategies employed, and for player i is:

$$J^i(x_0,t_0,u^1,u^2) = \mathrm{E}_{x_0,t_0}\left[\int_{t_0}^{\tau} L^i(\xi,t,u^1,u^2)\,\mathrm{d}t + F^i(\xi(\tau),\tau)\right]$$

where $i=1,2$, ξ is the solution of (1) with the initial condition $\xi(t_0) = x_0$ and τ is the stopping time of the game, which may be fixed or may be determined by the time at which $(\xi(t),t)$ leaves a certain $(n+1)$–dimensional domain. Each player attempts to maximize his own payoffs by choosing his own strategy. For two–person zero–sum stochastic games the payoff to player 2 is the opposite of player 1. By calling the payoff to player 1 $J = J^1$, then player 1 attemtps to maximize J by choosing $u^1(x,t)$ and player 2 tries to minimize J by the choice of $u^2(x,t)$. The problem thus consists in finding strategies u^{1*}, u^{2*} that will form a saddle point for the functional J:

$$J(x_0,t_0,u^1,u^{2*}) \leqslant J(x_0,t_0,u^{1*},u^{2*}) \leqslant J(x_0,t_0,u^{1*},u^2)$$

u^i may vary in a certain class $\mathscr{U}^i$ of admissible strategies for player i, $i=1,2$. Having fixed strategies u^1, u^2 and state x at time s, let us set:

$$J(x,s,u^1,u^2) = \mathrm{E}_{x,s}\left[\int_s^\tau L(\xi,t,u^1,u^2)\,\mathrm{d}t + F(\xi(\tau),\tau)\right]$$

where ξ is the solution of (1) with the condition $\xi(s)=x$, and τ denotes the exit time of $(\xi(t),t)$, after moment s, from an open set $Q \subset \mathbb{R}^{n+1}$.

THEOREM. (Minimax Principle) *If one has a function $V(x,s)$ such that*

i) For each $u^1 \in \mathcal{U}^1$ and $u^2 \in \mathcal{U}^2$, the function V is in the domain of the weak infinitesimal operator of ξ, denoted by $\overline{A}_s^{u^1,u^2}$.

ii) For $x,s \in Q$:

$$-\frac{\partial V}{\partial s}(x,s) = \max_{u^1 \in \mathcal{U}^1} \inf_{u^2 \in \mathcal{U}^2}\left\{\overline{A}_s^{u^1,u^2}(V)(x,s) + L(x,s,u^1(x,s),u^2(x,s))\right\} =$$

$$= \min_{u^2 \in \mathcal{U}^2} \sup_{u^1 \in \mathcal{U}^1}\left\{\overline{A}_s^{u^1,u^2}(V)(x,s) + L(x,s,u^1(x,s),u^2(x,s))\right\} =$$

$$= \overline{A}_s^{u^{1*},u^{2*}}(V)(x,s) + L(x,s,u^{1*}(x,s),u^{2*}(x,s))$$

where $u^{1*} \in \mathcal{U}^1$, $u^{2*} \in \mathcal{U}^2$ is a saddle point.

iii) $V(x,s) = F(x,s)$ for $(x,s) \in Q^c$.

Then u^{1*},u^{2*} are the optimal strategies for players and

$$V(x,s) = J(x,s,u^{1*},u^{2*}).$$

Proof. It is based in the properties of weak infinitesimal operator for the stochastic differential process ξ and a Dynkin's formula (see [2, Vol.1] and also [3, theorem 5.5.2]).

It is known (see for example [1], [5], [4], etc.) that if $V(x,s)$ is a class $C^{2,1}(Q)$, then $\overline{A}_s^{u^1,u^2}$ is the partial differential operator:

$$\overline{A}_s^{u^1,u^2}(V)(x,s) = V_x(x,s)f(x,s,u^1(x,s),u^2(x,s)) + \tfrac{1}{2}\mathrm{Tr}\left[V_{xx}(x,s)\,a\right]$$

where

$$a = \sigma((x,s,u^1(x,s),u^2(x,s))\,\sigma((x,s,u^1(x,s),u^2(x,s))^{\mathrm{T}}.$$

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