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Weak Uniform Continuity and Weak Sequential Continuity of Continuous n–Linear Mappings Between Banach Spaces

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In this paper it is shown that the class $L^n_{WU}(E_1,E_2,\dots,E_n;F)$ of weakly uniformly continuous n–linear mappings from $E_1 \times E_2 \times \cdots \times E_n$ to F on bounded sets coincides with the class $L^n_{WSC}(E_1,E_2,\dots,E_n;F)$ of weakly sequentially continuous n–linear mappings if and only if for every Banach space F, each Banach space E_i for $i=1,2,\dots,n$ does not contain a copy of ℓ_1.

Here, the above mentioned classes are subspaces of the space $L^n(E_1,E_2,\dots,E_n;F)$ of all continuous n–linear maps on $E_1 \times E_2 \times \cdots \times E_n$ to F, endowed with the norm

$$\|A\| = \sup\{\|A(x_1,x_2,\dots,x_n)\| : x_i \in E_i,\ \|x_i\| \leqslant 1,\ 1 \leqslant i \leqslant n\}.$$

In fact,

$L^n_{WU}(E_1,E_2,\dots,E_n;F) = \{A \in L^n(E_1,E_2,\dots,E_n;F)$: for all balls $B(E_i) \subset E_i$, and for every $\epsilon > 0$, there exists $\delta_i > 0$ and finite subsets $\Phi_i \subset E_i^*$ such that $x_i, y_i \in B(E_i)$ with $|\phi(y_i - x_i)| < \delta_i$ $(\phi \in \Phi_i^*,\ i=1,2,..,n)$ then $\|A(y_1,y_2,\dots,y_n) - A(x_1,x_2,\dots,x_n)\| \leqslant \epsilon\}$.

$L^n_{WSC}(E_1,E_2,\dots,E_n;F) = \{A \in L^n(E_1,E_2,\dots,E_n;F)$: for all bounded sequences (x_i^m) in E_i for which $\phi(x_i^m - x_i) \longrightarrow 0$ for some $x_i \in E_i$ for which $(\phi \in E_i^*,\ i=1,2,..,n)$, $\|A(x_1^m,x_2^m,\dots,x_n^m) - A(x_1,x_2,\dots,x_n)\| \longrightarrow 0\}$.

Also we need to consider the subspace $L^n_{WC}(E_1,E_2,\dots,E_n;F)$ of weak Cauchy continuous n–linear maps given by

$L^n_{WC}(E_1,E_2,\dots,E_n;F) = \{A \in L^n(E_1,E_2,\dots,E_n;F)$: for all bounded sequences (x_i^m) in E_i for which $(\phi(x_i^m))$ is Cauchy $(\phi \in E_i^*, i=1,2,..,n)$, $(A(x_1^m,x_2^m,\dots,x_n^m))$ is Cauchy in $F\}$.

From the above definitions, for Banach spaces E,F, it is true that the class

$L_{WSC}(E,F)$ contains the class $L_{WU}(E,F)$. So, the key idea in the proof of the necessity is to show that for each fixed $i=1,2,..,n$, the class $L_{WSC}(E_i,F)$ is contained in the class $L_{WU}(E_i,F)$ and appeal to the result that the classes $L_{WU}(E,F)$ and $L_{WSC}(E,F)$ coincide if and only if E does not contain a copy of ℓ_1 due to Aron et al. [1].

Indeed, assume the condition that

$$L^n_{WU}(E_1,E_2,...,E_n;F)=L^n_{WSC}(E_1,E_2,...,E_n;F).$$

Let y_j be in $B_1(E_j)$ and y_j^* be in $B_1(E_j^*)$ such that $y_j^*(y_j)=1$ for all j different from a fixed i. As in [2], consider the map $p:L(E_i,F)\longrightarrow L^n(E_1,E_2,...,E_n;F)$ given by

$$p(A)(e_1,e_2,...,e_n)=y_1^*(e_1)\cdots y_{i-1}^*(e_{i-1})\cdot y_{i+1}^*(e_{i+1})\cdots y_n^*(e_n)\,A(e_i).$$

The map p is injective. Now, it is shown that if T is in $L_{WSC}(E_i,F)$ then T is in $L_{WU}(E_i,F)$. Observe pT is in $L^n_{WSC}(E_1,E_2,...,E_n;F)$. In fact, let e_k^j be weakly null in E_j for $j=1,2,...,n$. Then

$$p(T)(e_k^1,...,e_k^j,...,e_k^n)=y_1^*(e_k^1)\cdots y_{i-1}^*(e_k^{i-1})\cdot y_{i+1}^*(e_k^{i+1})\cdots y_n^*(e_k^n)\,Te_k^i.$$

Note e_k^j is weakly null in E_j implies $y_j^*(e_k^j)$ is null for all $j=1,2,...,i-1,i+1,...,n$. Also $T\in L_{WSC}(E_i,F)$ means Te_k^i is null sequence in F. Thus, $p(T)(e_k^1,...,e_k^n)$ is norm null in F. By given condition $pT\in L^n_{WU}(E_1,E_2,...,E_n;F)$. That is, for every $\epsilon>0$, there exist finite subsets $\Phi=(\Phi_1,\Phi_2,...,\Phi_n)\subset E_1^*\times E_2^*\times\cdots\times E_n^*$ and $\delta_i>0$ for all $i=1,2,..,n$ such that if x_i,x_i' in $B(E_i)$ with $|\phi(x_i'-x_i)|<\delta_i$ for all $\phi\in\Phi_i$ for all $i=1,2,...,n$, then

$$\|pT(x_1',...,x_n')-pT(x_1,...,x_n)\|<\epsilon,$$

that is

$$\|\,y_1^*(x_1')\cdots y_{i-1}^*(x_{i-1}')\cdot y_{i+1}^*(x_{i+1}')\cdots y_n^*(x_n')\,(Tx_i'-Tx_i)\,\|<\epsilon.$$

Now, for the choice $x_j'=x_j=y_j$ for all $j=1,2,...,i-1,i+1,...,n$, and the earlier choice of y_j, the above gives $\|Tx_i'-Tx_i\|<\epsilon$ which implies $T\in L_{WU}(E_i,F)$. Thus for each $i=1,2,...,n$, we have $L_{WSC}(E_i,F)\subset L_{WU}(E_i,F)$. Hence as pointed out earlier, from the result due to Aron et al. [1], it follows that E_i does not contain ℓ_1 for each $i=1,2,...,n$.

As for the sufficiency part, first we observe that $L^n_{WU}(E_1,E_2,...,E_n;F)\subset L^n_{WSC}(E_1,E_2,...,E_n;F)$ is always true from their definitions. So let $A\in L^n_{WSC}(E_1,E_2,...,E_n;F)$. Then we claim that the associated maps

$A_i \in L(E_i, L^{n-1}(E_1, E_2, \ldots, E_n; F))$ given by

$$A(e_i)(e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n) = A(e_1, e_2, \ldots, e_n)$$

are compact under the assumption that E_i does not contain ℓ_1 for each $i = 1, 2, \ldots, n$. Indeed, let i be any fixed number from 1 to n. Then

$$A \in L^n_{WSC}(E_1, E_2, \ldots, E_n; F) \Rightarrow A_i \in L_{WSC}(E_i, L^{n-1}(E_1, E_2, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n; F)).$$

Therefore $A_i \in L_{WC}(E_i, L^{n-1}(E_1, E_2, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n; F))$ [3]. Now E_i does not contain ℓ_1 implies that A_i is compact. Thus for eacch $i = 1, 2, \ldots, n$, A_i is compact. Then, it is an easy consequence that $A \in L^n_{WU}(E_1, E_2, \ldots, E_n; F)$ [3].

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