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# The Generalizations of Integral Analog of the Leibniz Rule on the G–Convolutions

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An integral analog of the Leibniz rule for the operators of fractional calculus was considered in paper [1]. These operators are known to belong to the class of convolution transforms [2]. It seems very natural to try to obtain some new integral analog of the Leibniz rule for other convolution operators. We have found a general method for constructing such integral analogs on the base of notion of G–convolution [4]. Several results obtained by this method are represented in this article.

We introduce the following integral operators, which are particular cases of G–transform [3]:

a) The operator with Tricomi function $\Psi(a,b,x)$ in the kernel

$$(x^{\alpha}\Psi_b^a x^{-\alpha} f)(x) = x^{\alpha}\int_0^{\infty} e^{-x/t}\Psi(a,b,x/t)\, t^{-\alpha-1} f(t)\, dt.$$

b) The operator with algebraic function in the kernel

$$(x^{\alpha} Al_{\nu} x^{-\alpha} f)(x) = x^{\alpha}\int_x^{\infty} \frac{1}{\sqrt{1-x/t}}\Big[(1+\sqrt{1-x/t})^{\nu} + (1-\sqrt{1-x/t})^{\nu}\Big]\, t^{-\alpha-1} f(t)\, dt.$$

c) The operator of modified Meyer transformation

$$(x^{\alpha} K_{\nu} x^{-\alpha} f)(x) = x^{\alpha}\int_0^{\infty} K_{\nu}(2\sqrt{x/t})\, t^{-\alpha-1} f(t)\, dt,$$

where $K_{\nu}(x)$ is the Macdonald function.

*Remark* 1. The definition of the modified operator of fractional calculus $x^a I_-^b x^{a-b}$, the modified operator of Laplace transform $x^a \Lambda_+ x^{-a}$ and its inverse $x^a \Lambda_+^{-1} x^{-a}$ see in [2].

DEFINITION 1. ([3]) Let $c,\gamma \in \mathbb{R}$, and $2\,\mathrm{sign}(c) + \mathrm{sign}(\gamma) \geqslant 0$. Denote by $\overset{-1}{\mathfrak{M}}_{c,\gamma}(L)$ the space of functions $f(x)$, $x > 0$, representable in the form

$$f(x)=\frac{1}{2\pi i}\int_{\sigma} f^{*}(s)\,x^{-s}\,ds,\ x>0,$$

where $f^{*}(s)\,|s|^{\gamma}\,e^{\pi c|\mathrm{Im}\,s|}\in L(\sigma),\ \sigma=\{s:\mathrm{Re}\,s=1/2\}$.

DEFINITION 2. ([4]) Let $f\in\mathfrak{M}^{-1}_{c_1,\gamma_1}(L)$, $g\in\mathfrak{M}^{-1}_{c_2,\gamma_2}(L)$. Then the G–convolution of these functions is called the next integral

$$(f\star g)(x)=\frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_t}\frac{\Phi_1(s)\,\Phi_2(t)}{\Phi_3(s+t)}f^{*}(s)\,g^{*}(t)\,x^{-s-t}\,ds\,dt,$$

where functions $\Phi_i(\tau)$ have the following form

$$\Phi_i(s)=\frac{\prod\limits_{j=1}^{m_i}\Gamma(b_{j_i}+s)\prod\limits_{j=1}^{n_i}\Gamma(1-a_{j_i}-s)}{\prod\limits_{j=n_i+1}^{p_i}\Gamma(a_{j_i}+s)\prod\limits_{j=m_i+1}^{q_i}\Gamma(1-b_{j_i}-s)}\ ,$$

where $a_{j_i}$, $b_{j_i}$ – complex parameters and $\sigma_s=\{s:\mathrm{Re}\,s=1/2\}$, $\sigma_t=\{t:\mathrm{Re}\,t=1/2\}$.

We will use G–convolutions with $\Phi_1(\tau)=\Phi_2(\tau)\equiv1$ only in the further considerations.

THEOREM. *Suppose that* $f\in\mathfrak{M}^{-1}_{c_1,\gamma_1}(L)$ *and* $g\in\mathfrak{M}^{-1}_{c_2,\gamma_2}(L)$ *and G–convolution has the kernel* $\Phi_3(\tau)=H(\tau)$. *Then the following integral analogs of Leibniz rule hold true:*

*1) If* $H(\tau)=1/\Gamma(a+b-1+\tau)$, *then*

$$(f\star g)(x)=(x^{a+b-1}\Lambda_{+}^{-1}x^{-a-b+1}fg)(x)=$$

$$=\int_{-\infty}^{\infty}2^{2-a-b}(x^{a+\tau}\Lambda_{+}^{-1}x^{-a-\tau}f)(2x)(x^{b-\tau}\Lambda_{+}^{-1}x^{-b+\tau}g)(2x)\,d\tau$$

*under conditions:*

$$\mathrm{Re}(a+b)>1,\ 2\,\mathrm{sign}(c_i-1)+\mathrm{sign}(\gamma_i)\geqslant0,\ i=1,2.$$

*2) If* $H(\tau)=\Gamma(a+b+c+d-1+\tau)$, *then*

$$(f\star g)(x)=(x^{a+b+c+d-1}\Lambda_{+}x^{-a-b-c-d+1}fg)(x)=$$

$$= \int_{-\infty}^{\infty} \frac{1}{\Gamma(c-\tau)\,\Gamma(d-\tau)} (x^{a+c}\, \Psi^{c-d+1}_{\tau-d+1}\, x^{-a-c} f)(x)\, (x^{b+d}\, \Psi^{d-c+1}_{\tau-c+1}\, x^{-b-d} g)(x)\, \mathrm{d}\tau$$

*under conditions:*

$$\mathrm{Re}(a+b+c+d) > 1/2;\ \min\{\mathrm{Re}(a+c), \mathrm{Re}(a+d), \mathrm{Re}(b+c), \mathrm{Re}(b+d)\} > -1/2;$$

$$2\,\mathrm{sign}(c_1) + \mathrm{sign}(\gamma_1 - \mathrm{Re}(2a+c+d)) \geqslant 0;\ 2\,\mathrm{sign}(c_2) + \mathrm{sign}(\gamma_2 - \mathrm{Re}(2b+c+d)) \geqslant 0.$$

*3) If* $H(\tau) = \Gamma(a+b+\tau)/\Gamma(a+b+1/2+\tau)$, *then*

$$(f \star g)(x) = (x^{a+b}\, I_-^{1/2}\, \Lambda_+^{-1}\, x^{-a-b-1/2} fg)(x) =$$

$$= \int_0^{\infty} \frac{1}{2\pi^{3/2}} (x^{a-i\tau}\, Al_{2i\tau}\, x^{-a+i\tau} f)(x)\, (x^{b-i\tau}\, Al_{2i\tau}\, x^{-b+i\tau} g)(x)\, \mathrm{d}\tau$$

*under conditions:*

$$\mathrm{Re}(a+b) > 1/2,\ \mathrm{Re}(a) > -1/2,\ \mathrm{Re}(b) > -1/2$$

$$2\,\mathrm{sign}(c_i - 1) + \mathrm{sign}(\gamma_i - 1) \geqslant 0,\ i = 1,2.$$

*Remark* 2. The known Leibniz type rule for the operators of the fractional calculus [1] may be obtained by using this method if $H(\tau) = \Gamma(a-\tau)/\Gamma(b-\tau)$.

*Remark* 3. One can represent the last Leibniz type rule in the form

$$(f \star g)(x) = (x^{a+b}\, I_-^{1/2}\ x^{-a-b-1/2} fg)(x) =$$

$$= \int_0^{\infty} \frac{2}{\pi^{3/2}} (x^a\, K_{2i\tau}\, x^{-a} (t^{2a}\, \Lambda_+^{-1}\, t^{-2a} f(t^2))(\sqrt{t}/2))(x) \cdot$$

$$\cdot (x^b\, K_{2i\tau}\, x^{-b} (t^{2b}\, \Lambda_+^{-1}\, t^{-2b} g(t^2))(\sqrt{t}/2))(x)\, \mathrm{d}\tau.$$

*Remark* 4. The modified Meyer transformation in the poit $x = 1$ is known to be the Kontorovich–Lebedev transformation with respect to index $\nu = 2i\tau$ [3]. Consequently, one can consider the right part of the last equality (and the right parts of others Leibniz type rules) as integrals with respect to parameters of special functions in the kernels of operators.

## REFERENCES


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