EXTRACTA MATHEMATICAE Vol. 6, Núms. 2 y 3, 115 – 118 (1991)

Existence and Uniqueness for the Predator–Prey Model with Diffusion in the Scalar Case

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AMS *Subject Class.* (1980): 35G30, 35G20 *Received June* 24, 1991

We solve the problem of the existence and uniqueness of coexistence states for the classical predator–prey model of Lotka–Volterra with diffusion in the scalar case.

$$\mathscr{P}_b \qquad \begin{aligned} -u'' &= \lambda u - u^2 - buv \\ -v'' &= \mu v + cuv - v^2 \\ u(0) = u(1) &= v(0) = v(1) = 0\ , \end{aligned} \tag{1.1}$$

where b, c, λ, μ are real numbers with $b>0$ and $c>0$. Problem (1.1) models the behaviour of two interacting species on (0,1). From a biological point of view the real parameters describe, if positive, the net birth rates of the species and, if negative, the net death rates. The functions u and v represents the population density of both species. We are assuming logistic growth and that v preys on u.

In references [1]–[3], [5], [6] and [8]–[15] are obtained some existence and uniqueness results for the model (1.1) in a more general bounded domain $\Omega\subset\mathbb{R}^n$, with smooth enough boundary. The characterization of the values (λ,μ) for which (1.1) defined in $\Omega\subset\mathbb{R}^n$ has a coexistence state is well known (see [1], [2], [5]–[7] and [15]). The technical tool is fixed point index. There are partial answers to the uniqueness in [5], [8]–[10] and [13]. The result in [5] is of local nature, valid for the scalar model whenever one of the diffusion coefficients is small enough. We point out that the size of the diffusion coefficients for which uniqueness holds could depend on the values of λ, μ, b, c. For large domains is available the uniqueness result in [13]. In [9] was shown that the techniques introduced in [4] to obtain uniqueness work as well when dealing with predator–prey interactions. The uniqueness region built in [9] is a proper subset of the set of values of (λ,μ) for which there are coexistence states of (1.1). Monotone schemes techniques were used in [7] and [9] providing very restrictive results because for instance do not allow to μ be negative, being this one of the most interesting cases to consider from a biological point of view.

We conjecture that our restriction on the spatial dimension n is of technical nature. In fact, we only need that restriction to prove Lemma 1.2 (that Lemma proves that any coexistence state is not degenerate).

2. MAIN RESULT

For a sufficiently regular function $q:(0,1)\longrightarrow\mathbb{R}$, we shall denote by $\lambda_1(q)$ the first eigenvalue of $-(\cdot)''+q\cdot$ subject to homogeneus Dirichlet boundary conditions, by variational properties of eigenvalues

$$\lambda_1(q)=\inf_{H_0^1(0,1)}\left\{\int_0^1|u'|^2+\int_0^1 qu^2:\int_0^1 u^2=1\right\}$$

Consequently, $\lambda_1(q)$ is increasing and continuous. By the sake of briefness, we shall write λ_1 to mean $\lambda_1(0)$. Given $\gamma>\lambda_1$, we shall denote by θ_γ the unique positive solution of

$$-\theta_\gamma''=\gamma\theta_\gamma-\theta_\gamma^2 \quad \text{in } (0,1),\ \theta_\gamma(0)=\theta_\gamma(1)=0.$$

If $\gamma\leqslant\lambda_1$ then we consider $\theta_\gamma=0$.

THEOREM 1.1. *The problem* (1.1) *has exactly a coexistence state, i.e. a solution couple* (u,v) *componentwise positive in* $(0,1)$, *if and only if,* λ *and* μ *satisfy* $\lambda>\lambda_1(b\theta_\mu)$, $\mu>\lambda_1(-c\theta_\lambda)$.

The proof will be decompose in several steps, specifically continuation in the parameter b and nondegeneration of the coexistence states.

LEMMA 1.2. *Let* $(\bar{u}_0,\bar{v}_0)$ *be an arbitrary coexistence state of* $\mathcal{P}_b$. *Then the variational equation around* $(\bar{u}_0,\bar{v}_0)$, *given by*

$$-u''=\lambda u-2\bar{u}_0u-b\bar{v}_0u-b\bar{u}_0v,\quad -v''=\mu v+c\bar{u}_0v+c\bar{v}_0u-2\bar{v}_0v \quad in\ (0,1),$$
$$u(0)=u(1)=v(0)=v(1)=0,$$

has only the solution $(u,v)=(0,0)$.

LEMMA 1.3. *Assume that* $\lambda>\lambda_1(b\theta_\mu)$, $\mu>\lambda_1(-c\theta_\lambda)$ *and that* $\mathcal{P}_b$ *has an unique coexistence state, say* $(\bar{u}_0,\bar{v}_0)$. *Then there exists* $\epsilon_0=\epsilon(b,c,\lambda,\mu)>0$ *such that for every* $\epsilon<\epsilon_0$, *model* $\mathcal{P}_{b+\epsilon}$ *has an unique coexistence state, say* $(\bar{u}(\epsilon),\bar{v}(\epsilon))$. *Moreover* $(\bar{u}(0),\bar{v}(0))=(\bar{u}_0,\bar{v}_0)$ *and the mapping* $\epsilon\longrightarrow(\bar{u}(\epsilon),\bar{v}(\epsilon))$, *is* $\mathcal{C}^1$.

Sketch of the proof of theorem 1.1. The necessary part is known (see [1], [2],

[5]–[7] and [15]). We include a short proof of it. Assume that $b>0$, $c>0$ and $\mathscr{P}_b$ has a coexistence state, say (u,v). Then by applying well known results for the logistic equation it follows that $\lambda>\lambda_1(bv)$ and $\mu>\lambda_1(-cu)$. On the other hand, by standar comparison results we get $u\leqslant\theta_\lambda$ and $v\geqslant\theta_\mu$. Thus $\lambda>\lambda_1(bv)\geqslant\lambda_1(b\theta_\mu)$ and $\mu>\lambda_1(-cu)\geqslant\lambda_1(-c\theta_\lambda)$, because $-c<0$, and so the inequality concerning with the eigenvalues λ and μ is satisfied.

We emphasize the suficiency part.

Assume that $\lambda>\lambda_1(b\theta_\mu)$, $\mu>\lambda_1(-c\theta_\lambda)$. To show that $\mathscr{P}_b$ has an unique coexistence state we shall apply a continuation argument by using b as a parameter. For $b=0$ the system get uncoupled and so has an unique coexistence state. Now lemma 1.3 shows that if b is small enough then $\mathscr{P}_b$ has an unique coexistence state. Let b_s be the supreme of such a $b>0$, $b_s>0$. We claim that b_s is the unique value of b for which $\lambda>\lambda_1(b_s\theta_\mu)$. By an standar continuation argument we can continue the branch $(\bar{u}(b),\bar{v}(b))$ of coexistence states up to $b=b_s$. It is clear that $(\bar{u}(b_s),\bar{v}(b_s))$ is a coexistence state because otherwise $\lambda=\lambda_1(b_s\theta_\mu)$ or $\mu=\lambda_1(-c\theta_\lambda)$ should be satisfied. So $\mathscr{P}_b$ has at least a coexistence state. Therefore from the definition of b_s it follows that $\mathscr{P}_{b_s}$ must have at least two coexistence states. Now lemma 1.2 and the implicit function theorem, show that there are values $b<b_s$ for which $\mathscr{P}_b$ has at least two coexistence states, which contradicts the definition of b_s. ∎

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