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The Approximation of Continuous Linear Functionals

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Let $(X,\|\cdot\|)$ be a real normed linear space and consider the following norm derivatives [4, p. 35]:

$$(x,y)_i = \lim_{t\to 0-} (\|y+tx\|^2 - \|y\|^2)/2t,$$
$$(x,y)_s = \lim_{t\to 0+} (\|y+tx\|^2 - \|y\|^2)/2t.$$

Note that these mappings are well-defined on $X\times X$ and the following properties are valid [4, p. 35]:

(i) $(x,y)_i = -(-x,y)_s$ if x,y are in X;

(ii) $(x,x)_p = \|x\|^2$ for all x in X;

(iii) $(\alpha x,\beta y)_p = \alpha\beta(x,y)_p$ for all x,y in X and $\alpha,\beta \geqslant 0$;

(iv) $(\alpha x+y,x)_p = \alpha\|x\|^2 + (y,x)_p$ for all x,y in X and $\alpha\in\mathbb{R}$;

(v) $(x+y,z)_p \leqslant \|x\|\,\|z\| + (y,z)_p$ for all x,y,z in X;

(vi) the element x in X is Birkhoff orthogonal over y in X, i.e., $\|x+ty\|\geqslant\|x\|$ for all t in $\mathbb{R}$ iff $(y,x)_i \leqslant 0 \leqslant (y,x)_s$;

(vii) the space $(X,\|\cdot\|)$ is smooth iff $(y,x)_i = (y,x)_s$ for all x,y in X or iff $(\ ,\ )_p$ is linear in the first variable;

where $p=s$ or $p=i$.

For other properties of $(\ ,\ )_p$ see the papers [2] and [3] where further references are given.

Recall the famous theorem of Bishop – Phelps [1, p. 3]:

THEOREM 1. *Let C be a closed bounded convex set in the Banach space X, then the collection of functionals that achieve their maximum on C is dense in X^*.*

By the use of this result we have:

THEOREM 2. *Let X be a real Banach space. Then for every continuous linear functional $f:X\longrightarrow\mathbb{R}$ and for any $\epsilon>0$ there exists an element $u_{f,\epsilon}$ in X so that the following estimation:*

$$-\epsilon\|x\| + (x,u_{f,\epsilon})_i \leqslant f(x) \leqslant (x,u_{f,\epsilon})_s + \epsilon\|x\| \tag{1}$$

holds for all $x\in X$.

Proof. By the use of Bishop – Phelps theorem for the closed bounded convex set

$C=\overline{B}(0,1)=\{x\in X/\,\|x\|\leqslant 1\}$ it follows that the collection of functionals that achieve their norm on closed unit ball is dense in X^*, i.e., for every $f\in X^*$ and $\epsilon>0$ there is a continuous linear functional f_ϵ on X which achieve their norm on $\overline{B}(0,1)$ and so that

$$|f(x)-f_\epsilon(x)|\leqslant\epsilon\|x\| \text{ for all } x \text{ in } X. \tag{2}$$

Suppose $f_\epsilon\neq 0$ and let $v_{f,\epsilon}\in\overline{B}(0,1)\setminus\{0\}$ so that $f_\epsilon(v_{f,\epsilon})=\|f_\epsilon\|$. Then:

$$\|v_{f,\epsilon}\|\leqslant 1=f_\epsilon(v_{f,\epsilon})/\|f_\epsilon\|=f_\epsilon(v_{f,\epsilon}+\lambda y)/\|f_\epsilon\|\leqslant\|v_{f,\epsilon}+\lambda y\|$$

for all $\lambda\in\mathbb{R}$ and $y\in\operatorname{Ker}(f_\epsilon)$, i.e., $v_{f,\epsilon}\perp\operatorname{Ker}(f_\epsilon)$ and from (vi) we get:

$$(y,v_{f,\epsilon})_i\leqslant 0\leqslant(y,v_{f,\epsilon})_s \text{ for all } y\in\operatorname{Ker}(f_\epsilon).$$

Let $x\in X$ and put $y:=f_\epsilon(x)v_{f,\epsilon}-f_\epsilon(v_{f,\epsilon})x$. Then $y\in\operatorname{Ker}(f_\epsilon)$ for all $x\in X$ and then we obtain:

$$(f_\epsilon(x)v_{f,\epsilon}-f_\epsilon(v_{f,\epsilon})x,v_{f,\epsilon})_i\leqslant 0\leqslant(f_\epsilon(x)v_{f,\epsilon}-f_\epsilon(v_{f,\epsilon})x,v_{f,\epsilon})_s$$

for all $x\in X$. Since:

$$(f_\epsilon(x)v_{f,\epsilon}-f_\epsilon(v_{f,\epsilon})x,v_{f,\epsilon})_p=f_\epsilon(x)\|v_{f,\epsilon}\|^2-(x,f_\epsilon(v_{f,\epsilon})v_{f,\epsilon})_q$$

for all $x\in X$, where $p\neq q$, $p,q\in\{i,s\}$, we conclude, by the above inequalities, that:

$$(x,f_\epsilon(v_{f,\epsilon})v_{f,\epsilon}/\|v_{f,\epsilon}\|^2)_i\leqslant f_\epsilon(x)\leqslant(x,f_\epsilon(v_{f,\epsilon})v_{f,\epsilon}/\|v_{f,\epsilon}\|^2)_s$$

for all $x\in X$, from where results:

$$(x,u_{f,\epsilon})_i\leqslant f_\epsilon(x)\leqslant(x,u_{f,\epsilon})_s \tag{3}$$

for all x in X, where $u_{f,\epsilon}:=f_\epsilon(v_{f,\epsilon})v_{f,\epsilon}/\|v_{f,\epsilon}\|^2$.

Using (2) and (3) we obtain the desired estimation embodied in (1).

If $f_\epsilon=0$ then (1) also holds with $u_{f,\epsilon}=0$. The proof is finished. ∎

COROLLARY. *Let X be a smooth Banach space over the real number field and put $[x,y]:=(x,y)_i=(x,y)_s$, $x,y\in X$. Then for every continuous linear functional $f:X\longrightarrow\mathbb{R}$ and for any $\epsilon>0$ there exists an element $u_{f,\epsilon}$ in X so that the following approximation*

$$|f(x)-[x,u_{f,\epsilon}]|\leqslant\epsilon\|x\| \text{ for all } x \text{ in } X,$$

holds.

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