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On Cyclic Cubic Fields

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In 1960, Godwin [3] states a conjecture about units in totally real cubic fields. Given such a field K, if R denotes its ring of integers, he defines

$$S(\alpha) = \tfrac{1}{2}[(\alpha-\alpha')^2 + (\alpha-\alpha'')^2 + (\alpha'-\alpha'')^2],$$

for $\alpha \in R$; α' and α'' denote the conjugates of α. Denoting by E the subgroup of units with positive norm, Godwin's conjecture is as follows: let $\mu \in E - \{1\}$ be such that $S(\mu)$ is minimun and $\tau \in U - \{\mu^n : n \in \mathbb{Z}\}$ such that $S(\tau)$ is minimun; if $S(\mu) > 9$, then $\{\mu,\tau\}$ is a system of fundamental units of K. In 1980, Gras [4] proved this conjecture in case that K is cyclic.

For three families of cyclic cubic fields we give a system of fundamental units $\{\mu,\mu'\}$. For that we use the already proved Godwin's conjecture [1].

In addition, the unit μ is never totally positive. Hence, every totally positive unit in K is a square in R. We use this fact to obtain a criterion about evenness of the classnumber for fields in these families.

Precisely:

THEOREM 1. *Let* $K = \mathbb{Q}(\theta)$, $Irr(\theta,\mathbb{Q}) = x^3 - px + p$, $p = 3^\delta p_1 \cdots p_r$, $p_i \equiv 1 (mod\, 3)$ *pairwise different primes,* $\delta \in \{0,2\}$, $4p - 27 \in \mathbb{Z}^2$. *The following conditions hold:*

i) $\operatorname{disc}(K) = p^2$.

ii) K is monogenic with integral basis $\{1, \sigma, \sigma^2\}$.

iii) $\{\sigma, \sigma'\}$ *is a system of fundamental units of K, where* $\sigma = (m + \theta_1)/3$, σ' *denotes a conjugate of* σ, $\theta_1 = (4p - 9\theta - 6\theta^2)/(4p-27)^{1/2}$ *and* $m = ((4p-27)^{1/2} - 3)/2$.

EXAMPLES. p = 13, 19, 37, 63, 79, 97, 117, 139, 163, 217, 247, 279, 313, 349, 387, 427, 469, 559, 607, 657, 709, 763, 819, 877, 937, 1063, 1129, 1197, 1267, 1339, 1413, 1489.

THEOREM 2. *Let* $K = \mathbb{Q}(\theta)$, $Irr(\theta,\mathbb{Q}) = x^3 - px + pq$, $p = p_1 \cdots p_r$, $p_i \equiv 1 (mod\, 3)$, *pairwise different primes,* $q > 2$, $4p - 27q^2 = 1$. *The following conditions hold:*

i) $\operatorname{disc}(K) = p^2$.

ii) K is monogenic with integral basis $\{1, \theta, \theta^2\}$; $\{1, \sigma, \tau = (\sigma^2 + ((q+1)/2)\sigma)/q\}$ *is another integral basis of K.*

iii) $\{\mu, \mu'\}$ *is a system of fundamental units of K, where* $\mu = 2 + 3\sigma + 3\tau$, $\sigma = (-1 + \theta_1)/3$,

[1] In all this paper, the rings of integer, rational and natural numbers will be denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{N}$

$\theta_1 = 4p - 9q\theta - 6\theta^2$ *and* $\mu' = -1 - 6\sigma + 3\tau$.
iv) $Irr(\mu, \mathbb{Q}) = x^3 - 3((1+9q)/2)x^2 + ((27q-3)/2)x + 1$.
v) μ *is not totally positive.*

EXAMPLES. $p = 61$, 331, 547, 817, 1141, 1951, 2437, 2977, 3571, 4219, 4921, 5677, 6487, 7351, 8269, 9241, 10267, 11347, 12481, 13669, 14911, 16207, 17557, 18961.

THEOREM 3. *Let* $K = \mathbb{Q}(\theta)$, $Irr(\theta, \mathbb{Q}) = x^3 - px + pq$, $p = 9 \cdot p_1 \cdots p_r$, $p_i \equiv 1 (mod\, 3)$, *pairwise different primes,* $q > 2$, $3 \nmid q$, $4p - 27q^2 = 9$. *The following conditions holds:*
i) $\mathrm{disc}(K) = p^2$.
ii) $\{1, \sigma, \tau = (((q-1)/2) + ((q-1)/2)\sigma + \sigma^2)/q\}$ *is an integral basis of K where* $\sigma = \theta_1/3$ *and* $\theta_1 = 4(p/3) - 3q\theta - 2\theta^2$.
iii) $\{\mu, \mu'\}$ *is a system of fundamental units of K, where* $\mu = \sigma + \tau$ *and* $\mu' = -2\sigma + \tau$.
iv) $Irr(\mu, \mathbb{Q}) = x^3 - ((3+9q)/2)x^2 + ((9q-3)/2)x + 1$.
v) μ *is not totally positive.*

EXAMPLE. $p = 171$, 333, 819, 1143, 2439, 3573, 4221, 5679, 6489, 8271, 9243, 11349, 12483, 14913, 16209.

THEOREM 4. *Let K be like in theorem 2. If* $q \in \mathbb{Z}^2$ *but* $(1+3q)/2 \notin \mathbb{Z}^2_{q_i}$ *for some prime* $q_i \mid q$, *then* h_K *is even.*

EXAMPLE. $p = 547$, $q = 9$; $p = 4219$, $q = 25$.

THEOREM 5. *Let K like in theorem 3. If* $q \in \mathbb{Z}^2$ *but* $(3+3q)/2 \notin \mathbb{Z}^2_{q_i}$ *for some prime* $q_i \mid q$, *then* h_K *is even.*

EXAMPLE. $p = 16209$, $q = 49$.

For the sake of brevity, we omit H. Cohn [2] and M. Watabe's [5], [6] criterion about evenness of classnumber for fields in the first family.

In 1935, Siegel proved that $h_F \longrightarrow \infty$ as the absolute value of the discriminant of the imaginary quadratic fields tends to infinite. Siegel used the formula

$$\lim_{\mathrm{disc}(F) \to \infty} (\log(h_F R_F) / \log(|\mathrm{disc}(F)|^{1/2})) = 1$$

where R_F is the regulator of F.

Since Brauer [1] proved that such a formula remains true for arbitrary number fields (supposed that the degree is fixed), we use it and our previous result to obtain our last two results.

THEOREM 6. *Let K be like in theorem 2, then*

$$\lim_{p \to +\infty} h_K = +\infty.$$

THEOREM 7. *Let K be like in theorem 3, then*

$$\lim_{p \to +\infty} h_K = +\infty.$$

For the first family we have a similar result ([5]).

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