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Quasi-Frobenius Quotient Rings [1]

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Let R be an associative (not necessarily commutative) ring with unit. The study of flat left R–modules permits to achieve homological characterizations for some kind of rings (regular Von Neumann, hereditary). Colby investigated in [1] the rings with the property that every left R–module is embedded in a flat left R–module and called them left *IF* rings. These rings include regular and quasi–Frobenius rings. Another useful tool for the study of non–commutative rings is the classical localization, when it is possible, or the localizations constructed from the most general perspective of torsion theories (mainly, the maximal quotient ring). In a recent paper [3] we try to find a relation between these two approximations to the problem of the determination of the structure for general rings. The suggesting idea is that for commutative domains the class of torsionfree modules is exactly the class of submodules of flat modules.

If $\mathscr{F}_0$ denotes the class of left R–modules that embed in flat left R–modules, we investigate the rings for which this class is the torsionfree class for some hereditary torsion theory τ_0 on the category of left R–modules, R–Mod. These are the left *FTF* ("flat are torsionfree") rings. Analogously, we define right *FTF* rings (with notation $\mathscr{F}_0'$ and τ_0'). If R is a left *FTF* ring, the filter $\mathscr{L}(\tau_0)$ of left ideal associated with τ_0 has an easy description: A left ideal I of R is an element of $\mathscr{L}(\tau_0)$ if and only if I contains a finitely generated left ideal I_0 such that $Hom_R(R/I_0,R) = 0$ [3, Prop. 4.5]. In this note we expose the results obtained in our research on *FTF* rings. The keys for the proofs of our results have been the description of the filter $\mathscr{L}(\tau_0)$ and the well known result due to Lazard that asserts that the flat left R–modules are the direct limits of projective modules. For the undefined concepts the reader is referred to [7]. The following is the main result proved in [3].

THEOREM 1. *The following conditions are equivalents for a ring R:*
i) R is a left FTF ring and satisfies D.C.C. on left annihilators.
ii) R has a twosided maximal quotient ring Q such that Q is a semiprimary left and right QF–3.
iii) R is a right FTF ring and satisfies D.C.C. on right annihilators.
In case i)–iii) hold, R is a τ_0 *and* τ_0'*–artinian.*

Masaike [5, Theorem 2] shows that rings satifying *ii)* are the rings R for which every

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finitely generated $E({}_RR)$ – torsionless left R – module is embedded in a free left R – module and R satifies $D.C.C.$ on left annihilators. We will prove that for a ring with the property that every finitely generated $E({}_RR)$ – torsionless left R – module embeds in a free left R – module, $A.C.C.$ on left annihilators implies $D.C.C.$ on left annihilators. To prove this statement we need to answer to the following question: When is a left FTF ring right FTF? For the definition of coherence relative to a torsion theory, we refers to [4].

PROPOSITION 2. *If R is a ring then R is left FTF and τ_0 – coherent if and only if R is right FTF and τ_0' – coherent.*

Morita proved [6] that a left noetherian ring with $E({}_RR)$ flat has a flat right injective hull too. As a corollary to the proof of the foregoing theorem we improve this result:

COROLLARY 3. *If R is left coherent and $E({}_RR)$ is flat then R is a (left and right) FTF ring.*

The following two results improve Masaike's Theorem before cited.

PROPOSITION 4. *Let R be a ring with the property that every finitely generated $E({}_RR)$ – torsionless left R – module embeds in a free left R – module. If R satisfies A.C.C. on left annihilators then R satisfies D.C.C. on left annihilators.*

THEOREM 5. *For a ring R the following statements are equivalents:*

i) Every finitely generated $E({}_RR)$ – torsionless left R – module embeds in a free left R – module and R satisfies A.C.C. on left annihilators.

ii) Every finitely generated $E({}_RR)$ – torsionless left R – module embeds in a free left R – module and R satisfies D.C.C. on left annihilators.

iii) R is left (or right) FTF and τ_0 – (or τ_0' –) noetherian.

iv) R is left and right FTF and both τ_0 – artinian as τ_0' – artinian.

v) R has a twosided maximal quotient ring Q such that Q is semiprimary left and right QF – 3.

The last results of this note give characterizations of the existence of a quasi – Frobenius left maximal (classical) quotient ring for some rings. These characterizations applies for left $QF-3$ rings (or more generally, for left $QF-3'$ rings) and for left FPF rings with finite left Goldie dimension [2, Corollary 2.9].

THEOREM 6. *For any ring R the following statements are equivalents:*

i) R is a τ_0 – noetherian left FTF ring and τ_0 is an exact torsion theory.

ii) R has a twosided maximal quotient ring Q such that Q is quasi – Frobenius.

THEOREM 7. *Let R be a ring such that every finitely generated submodule of $E({}_RR)$ is torsionless. Then R has a quasi – Frobenius left classical quotient ring if and only if*

1) R satisfies A.C.C. on right annihilators.

2) If I is a finitely generated left ideal of R such that $r(I) = 0$, then I contains a regular element.

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