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ON THE λ-PROPERTY AND SPACES OF CONVERGENT SEQUENCES

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For X a normed space, we will use the following notation

$$B(X) = \{ x \in X : \|x\| \leq 1 \}$$

$$ext(B(X)) = \{ e \in X : e \text{ is an extreme point of } B(X) \} .$$

We will denote by $c(X)$ (resp. $l_1(X)$) the space of convergent (resp. absolutely summable) sequences of elements of X with its usual norm.

The following concepts were introduced by Aron and Lohman in [2] :

Let x be an element of B(X). If there exist $e \in ext(B(X))$, $y \in B(X)$ and $\lambda \in]0,1]$ such that $x = \lambda e+(1-\lambda)y$, we will say that the triple (e,y,λ) is amenable to x. In this case we can define

$$\lambda(x) = \text{Sup} \{ \lambda \in]0,1] : \text{there exist } (e,y,\lambda) \text{ amenable to } x \}.$$

X is said to have the λ-property if each point in its unit ball admits an amenable triple, and it is defined the λ-function of X as the function $x \longmapsto \lambda(x)$ from B(X) into]0,1].

It is said that X has the uniform λ-property if X has the λ-property and, in addition, satisfies

$$\text{Inf} \{ \lambda(x) : x \in B(X) \} > 0.$$

It can be easily checked that every strictly convex space has the uniform λ-property.

We are interested in the λ-property for spaces of the form c(X). Concerning this, up to date, it is only known that, if X is a strictly convex space, c(X) has the uniform λ-property and for each $x = \{x_n\} \in B(c(X))$ we have

$$\tilde{\lambda}(x) = \text{Inf} \{ \lambda(x_n) : n \in \mathbb{N} \} .$$

where $\tilde{\lambda}(\cdot)$ (resp. $\lambda(\cdot)$) denotes the λ-function of c(X) (resp. X), we will use the same notation throughout the paper. This was proved by Aron - Lohman [2] and Aizpuru [1].

It should be also mentioned that Aron - Lohman proved in [2] that every finite-dimensional normed space has the uniform λ-property.

In order to study the λ-property for $c(X)$ we need the following characterization of the extreme points of its unit ball:

If X is a normed space and $e = \{e_n\} \in B(c(X))$ then

$e \in ext(B(c(X)))$ if, and only if, $e_n \in ext(B(X))$, $\forall\, n \in \mathbb{N}$.

Using this characterization, it is easy to prove the next result.

PROPOSITION. *Let X be a normed space. If $c(X)$ has the λ-property (resp. uniform λ-property), then X has the λ-property (resp. uniform λ-property).*

The converse of the above result is not true as the following example shows:

EXAMPLE. Let C' denote the convex hull of the union of the sets A_1 and A_2 given by

$$A_1 = \{ (x,y,z) \in \mathbb{R}^3 : |x|,|y| \le 1 , z = 0 \}$$

$$A_2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + z^2 = 1 , y = 0 , z \ge 0 \}$$

take $C = (0,0,1) + C'$ and $\|\cdot\|$ the norm on $\mathbb{R}^3$ whose unit ball is $B = co(C \cup (-C))$.

Then $X = (\mathbb{R}^3,\|\cdot\|)$ has the uniform λ-property and $c(X)$ fails to have the λ-property.

<u>Remark</u> : The above space X appears in [2].

As we have said before, up to date, it has been proved that $c(X)$ has the λ-property only when X is a strictly convex space. The next result gives a class of non strictly convex normed spaces X for which $c(X)$ has the uniform λ-property.

THEOREM. *Let X be a finite-dimensional normed space whose unit ball is a polyhedron ($ext(B(X))$ is a finite set), then $c(X)$ has the uniform λ-property and, for each $x = \{x_n\} \in B(c(X))$*

$$\tilde{\lambda}(x) = Inf \{ \lambda(x_n) : n \in \mathbb{N} \} .$$

In the above theorem X has the uniform λ-property. Now we see that this is not necessary in order to get that c(X) has the λ-property. That will be a consequence of the following result.

THEOREM. *Let X be a normed space. The space c(X) has the λ-property if, and only if, $c(l_1(X))$ has the λ-property.*

COROLLARY. *Let X be a normed space satisfying one of the two conditions:*

a) X is strictly convex

b) B(X) is a polyedron,

then $c(l_1(X))$ has the λ-property.

Taking into account that spaces of the form $l_1(X)$ always fail to have the uniform λ-property [3], the comment before the theorems is now clear.

REFERENCES

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