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MULTIPLICATIVE FUNCTIONALS ON ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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Let Ω be an open subset of a real Banach space E and, for $1 \le m \le \infty$, let $C^m(\Omega)$ denote the algebra of all m-times continuously Fréchet differentiable real functions defined on Ω. We are concerned here with the question as to whether every nonzero algebra homomorphism $\varphi: C^m(\Omega) \longrightarrow \mathbb{R}$ is given by evaluation at some point of Ω, i.e., if there exists some $a \in \Omega$ such that $\varphi(f) = f(a)$ for each $f \in C^m(\Omega)$. This problem has been considered in [1,4,5] and [6]. In [6], a positive answer is given in the case that $m < \infty$ and E is a Banach space which admits C^m-partitions of unity and with nonmeasurable cardinal; this result is obtained there as a by-product of the study of two topologies, τ^m_ω and τ^m_δ, introduced on $C^m(\Omega)$. In [1] (respectively, in [4]) a positive answer is given in the case that $\Omega = E$ is a separable Banach space (respectively, the dual of a separable Banach space). In the present note we extend these previous results, and we give an affirmative answer for a wider class of Banach spaces, including super-reflexive spaces with nonmeasurable cardinal. We also provide a direct approach and a unified treatment, since our results here are derived as a consequence of Theorem 1 below, a general result slightly in the spirit of Theorem 12.5 of [7].

Now we introduce some terminology. Let C(X) be the algebra of all continuous real functions defined on the topological space X. If A is a subalgebra of C(X), we denote by $\mathscr{S}(A)$ the set of all nonzero multiplicative

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linear functionals on A. For each $a \in X$, let δ_a denote the functional $f \longrightarrow \delta_a(f) = f(a)$ on A; clearly $\delta_a \in \mathcal{S}(A)$. We shall write $\mathcal{S}(A) = X$ when every $\varphi \in \mathcal{S}(A)$ is of the form $\varphi = \delta_a$ for some $a \in X$. Recall that a subalgebra A of C(X) is said to be inverse-closed if whenever $f \in A$ and $f(x) \neq 0$ for every $x \in X$. then $1/f \in A$. Also, we say that a function $f: X \longrightarrow \mathbb{R}$ belongs locally to A if for each $x \in X$ there exist a neighbourhood V of x and a function $g \in A$ such that $f_{|V} = g_{|V}$.

We have the following:

1. THEOREM. *Let* X *be a realcompact topological space and let* $A \subset C(X)$ *be a subalgebra with identity such that:*

(i) A *is inverse-closed.*

(ii) If $C, D \subset X$ *are (nonempty) disjoint closed subsets, then there exists* $f \in A$ *with* $0 \leq f \leq 1$ $f(C) = \{0\}$ *and* $f(D) = \{1\}$.

(iii) If $f \in C(X)$ *belongs locally to* A, *then* $f \in A$.

Then $\mathcal{S}(A) = X$.

If E is a real Banach space, it is well-known (see [3]) that every open subset of E is realcompact whenever E has nonmeasurable cardinal, i.e., there exists no nontrivial two-valued measure defined on the power set of E. We recall that the requirement for a set to have nonmeasurable cardinal is in fact very mild: it is not known whether measurable cardinals exist, and any that may exist can be regarded as pathological, since they are larger than $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$, and cardinals obtained from them by the standard processes of cardinal arithmetic (cf.[3]). Thus the following is immediate.

2. COROLLARY. *Let* Ω *be an open subset of a real Banach space* E *with nonmeasurable cardinal and which admits* C^m*-partitions of unity* $(1 \leq m \leq \infty)$. *Then* $\mathcal{S}(C^m(\Omega)) = \Omega$.

Remarkable examples of Banach spaces wich admit C^∞-partitions of unity are $c_0(\Gamma)$ for any index set Γ and $L^{2n}(\mu)$ for $n \in \mathbb{N}$ and any measure μ (see e.g. [8]). Now, as a consequence, we obtain:

3. COROLLARY. *Let* Ω *be an open subset of a real Banach space* E *such that there exists a continuous, linear, one-to-one operator from* E *into* $\ell_p(\Gamma)$, *for some* p $(1 < p < \infty)$ *and some index set* Γ *of nonmeasurable cardinal. Then* $\mathcal{S}(C^m(\Omega)) = \Omega$, *for* $1 \leq m \leq \infty$.

Some cases in which Corollary 4 applies are given below. First we recall from [2] that super-reflexive Banach spaces can be defined as those spaces which admit an equivalent uniformly convex norm. In particular, spaces $L^p(\mu)$ are super-reflexive for $1<p<\infty$ and any measure μ.

4. REMARK. The hypotheses of Corollary 4 are satisfied if E is a closed subspace of the following spaces F:

(a) F is any separable Banach space, or F is the dual of any separable Banach space.

(b) F = C(K), where K is a separable, compact space.

(c) F is any super-reflexive Banach space with nonmeasurable cardinal.

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