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CL-SPACES AND NUMERICAL RADIUS ATTAINING OPERATORS

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In 1960, E. Bishop and R. Phelps [4] raised the question of the denseness of the norm attaining (bounded and linear) operators between two Banach spaces X and Y. They gave the first positive answer to this problem taking as Y the scalar field. Afterwards, a lot of papers dealt with this question, such as [7], [16], [18] and many others. Paralelling the investigations on this subject, B. Sims, in his doctoral dissertation, posed the problem of the denseness of the numerical radius attaining operators on an arbitrary Banach space.

Before giving any answer let recall some definitions. Given a Banach space X and a (bounded and linear) operator $T \in L(X)$, define the *numerical range* of T, $V(T)$, in the following way:

$$V(T) = \{f(T(x)) : \|x\| = \|f\| = f(x) = 1\}$$

and the *numerical radius* of T, $v(T)$, is the number

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}.$$

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We say that an operator T *attains its numerical radius* if the above supremum is actually a maximun, that is, there exist $x \in X$, $f \in X^*$, such that

$$\|x\| = \|f\| = f(x) = 1 \quad \text{and} \quad |f(T(x))| = v(T).$$

A complete survey on numerical range can be found in the monographies by F. Bonsall and J. Duncan [5] and [6].

[1]This is part of the author's doctoral dissertation. She wants to thank her superviser Rafael Payá for his help and suggestions.

B. Sims observed that the self-adjoint operators on a Hilbert space could be approximated by self-adjoint operators which attained their numerical radii. Then, I. Berg and B. Sims [3, Theorem] showed that an operator on a uniformly convex space can be perturbed by a compact operator of arbitrarily small norm to obtain an operator which attains its numerical radius. Using this result C. Cardassi [10] proved the denseness of numerical radius attainig operators on a uniformly smooth space. The same author showed that the answer is affirmative for c_0, ℓ_1 [9], $C(K)$ [8] and the spaces $L_1(\mu)$ [11], where μ is a finite and positive measure on a compact and Hausdorff topological space. The unique general answer related to this problem was given by M. Acosta and R. Payá [1], who proved that for an arbitrary Banach space X, the operators on X whose second adjoints attain their numerical radii are dense in the set of all operatos $L(X)$. Afterwards, the same authors improved this result [2], showing that, actually, one has the denseness of the set of the operators whose first adjoints attain their numerical radii.

The answers to this problem seem to be quite similar to the ones given for the denseness of the norm attaining operators. Even more, looking at some examples, such as c_0, ℓ_1 or $C(K)$, one can observe that the operators on these spaces attain their norms if, and only if, attain their numerical radii. For this reason, we looked for a class of Banach spaces that includes the classical Banach spaces for which no answer is known and where the above phenomena happens. The CL-spaces satisfies these two requirements. Let give the definition of the concept of CL-space, introduced by R. Fullerton [12].

Definition 1. A real Banach space is a CL-space if every maximal face L of the unit ball B_X satisfies

$$B_X = \text{co}(L \cup -L),$$

that is, the unit ball is the convex hull of L and $-L$.

In the following all the Banach spaces considered will be real.

For CL-spaces the situation related to numerical radius attaining operators seems to be quite special, as the following result shows:

Theorem 2. *Let X be a CL-space and $T \in L(X)$. The following assertions hold:*

i) $v(T) = \|T\|$.

ii) *T attains its norm if, and only if, T attains its numerical radius.*

Using results by J. Lindenstrauss and Å. Lima, we know that the spaces $C(K)$ (real functions) and $L_1(\mu)$ (any positive measure μ) are CL-spaces (see [17, page 42, Corollary 1 and 2] and [15, Corollary 3.6]). By the above theorem and the fact that we have the norm-denseness of the set of norm attaining operators on those spaces (see [14, Theorem 1] and [13], respectively), we get:

Corollary 3[8]. *The set of numerical radius attaining operators on $C(K)$ is dense in the set of all operators $L(C(K))$.*

Corollary 4. *For any positive measure μ, the set of operators which attain their numerical radii on $L_1(\mu)$ is dense in the space of all operators.*

In this case we improve the previous result by C. Cardassi [11] who proved the same fact when μ is a finite and positive Borel measure on a compact and Hausdorff topological space.

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